Almost Sure Recurrence of the Simple Random Walk Path

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Abstract

It is shown that the simple random walk path on a bounded degree graph, consisting of all vertices visited and edges crossed by the walk, is almost surely a recurrent subgraph.

1 Introduction

Given a graph $G = (V, E)$ with finite degrees, a simple random walk (SRW) on G is a Markov chain on the set of vertices with transition probabilities

$$Prob(w_t = u | w_{t-1} = v) = 1/d_v,$$

provided $\{u, v\} \in E$, were d_v is the number of edges meeting at v.

G is called **recurrent** iff a.s. SRW visits any fixed vertex infinitely often.

Let G be a graph with bounded degrees. Let $PATH$ be the random subgraph of G, consists of all vertices visited and edges crossed by a simple random walk on G, that is, the random walk path.

Theorem 1.1. *$PATH$ is a.s. recurrent.*

- For a recurrent G, the theorem is trivial, since any subgraph of a recurrent graph is recurrent (see [3]).
- The theorem is already known for the Euclidean lattices, since a.s. the SRW paths on three dimensional Euclidean lattice has infinitely many cutpoints, were the past of the path is disjoint from it's future, see [5, 6]. And then recurrence follows by the Nash-Williams criterion [8]. Lyons and Peres (unpublished) constructed an example of a transient walk without cutpoints.

- Morris [7] proved that the components of the Wired Spanning Forest are a.s. recurrent. A result of similar spirit to the theorem but with a different proof. For another a.s. recurrence theorem (for distributional limits of finite planar graphs) see [2].

- Exercise: show that if G is transient and with a uniform bound on the degrees, then a.s. the SRW do not visit all the vertices of G.

- The proof uses the electrical networks interpretation of recurrence. For the connection between SRW and electrical network see [3]. For further reading on recurrence see [10] and the on-line lecture notes [9].

- One can think of Brownian analogue of the theorem. That is a.s. parabolicity of the Wiener sausage, with reflected boundary conditions. It is of interest to formulate similar conjectures and theorems for other generators and other random walks and processes. For background on recurrence in the Riemannian context see e.g. [4].

  For example, consider the range of a branching random walk on a graph G, denoted by $R(BRW)$. Then we conjecture that almost surely $R(BRW)$ is recurrent for BRW with the same branching law. And a similar conjecture should hold for tree indexed random walks. See [1] for definitions and background.

- Is it possible to drop the bounded degree assumption?

Question 1.2. *Given a graph G, denote by $PATH(n)$ the path created by the first n steps of the SRW on G. and by $R(n)$ the maximal electric resistance between pairs of vertices on $PATH(n)$, (when $PATH(n)$ viewed as electrical network were each edge is a one ohm conductor).*

By the theorem, on any bounded degree graph $R(n)$ a.s. increases to infinity (note of course that $R(n)$ increase to infinity do not imply the theorem e.g. balls in the binary tree). Is there a uniform lower bound over all bounded degree graph for the rate at which it grows, that is: Is there a function f,

$$\lim_n f(n) = \infty$$

So that for any infinite graph of bounded degree. a.s.

$$\limsup_n \frac{R(n)}{f(n)} > 0?$$

In particular one can speculate that $f(n) = C\log^2 n$ might work? Were the $\log^2 n$ is a lower bound coming from considering $R(n)$ when G is $\mathbb{Z}^2$, which might be critical.

The proof of the theorem is in the coming two sections. In the next section we consider line graphs with unbounded degrees.

2 Proof of a special case

First, we shall prove the theorem for a very special case. Quite surprisingly, the general case will not be very different. Focusing on this special case will help illustrate the main ideas of the proof.

A graph G is a line graph if $V_G = \mathbb{N}$ and E_G includes only edges connecting successive vertices. Let e_i denote the number of edges connecting i and $i+1$. We place **no** restriction on e_n.

Theorem 2.1. *If G is a line graph then $PATH$ on G is a.s. recurrent.*

Proof. As always, the only interesting case is if G is transient, which is equivalent to $\sum_{i=0}^{\infty} {e_i}^{-1} < \infty$. Let $v(n)$ be the probability that a simple random walk starting at n visits 0. Clearly v is a strictly decreasing function, $v(0) = 1$ and $lim_{n\to\infty} v(n) = 0$. More precisely:

$$v(n) = \frac{\sum_{i=n}^{\infty} {e_i}^{-1}}{\sum_{i=0}^{\infty} {e_i}^{-1}}$$

v is harmonic everywhere except at 0.It follows that the process $v(s_t)$ is "almost" a martingale, i.e. it is a martingale as long as s_t does not reach 0.

Let s_n be the number of times the random walk crossed an edge connecting n and $n+1$, in either direction. Let s'_n be the number of edges connecting n and $n+1$ which belong to $PATH$, i.e. those edges that the random walk has crossed. The resistance of $PATH$ is therefore $\sum_{i=0}^{\infty} {s'_i}^{-1}$. Obviously, $s_n \geq s'_n$' so $\sum_{i=0}^{\infty} {s_i}^{-1} < \sum_{i=0}^{\infty} {s'_i}^{-1}$. We will show that $\sum_{i=0}^{\infty} {s_i}^{-1} = \infty$ almost surely, and therefore $PATH$ is almost surely recurrent.

Lemma 2.2. *$Prob(\sum_{i=0}^{\infty} {s_i}^{-1} = \infty)$ is 0 or 1.*

Proof. Let $\{X_i^j\}_{i,j=0}^{\infty}$ be independent random variables, defined by $Prob(X_i^j = 1) = e_i/(e_{i-1} + e_i)$ and $Prob(X_i^j = -1) = e_{i-1}/(e_{i-1} + e_i)$. Use these variables to construct a simple random walk on G in the obvious manner: $w_{t+1} = w_t + X_{w_t}^t$. Now, s_k is dependant (in the probabilistic sense) only on X_i^j for $i \geq k$, since every time the walk is in $\{0, 1, .., k-1\}$ it will almost surely reach k at some time. Therefore, a change to the values of finitely many of the X_i^js will change only the finitely many s_is and so cannot effect the infiniteness of $\sum_{i=0}^{\infty} {s_i}^{-1}$. By Kolomogorov's zero-one law we get that $Prob(\sum_{i=0}^{\infty} {s_i}^{-1} = \infty)$ is 0 or 1. $\square$

It remains to show that $PATH$ is not almost surely transient. First we shall handle the easy case, where the walk is quickly transient.

Lemma 2.3. *If for infinitely many n, $v(n)/2 > v(n+1)$ then almost surely $\sum_{i=0}^{\infty} {s_i}^{-1} = \infty$.*

Proof. Let $\{n_i\}_{i=0}^{\infty}$ be such an infinite series. Consider $p_i = Prob(s_{n_i} = 1)$, the probability that the random walk crosses an edge from n_i to $n_i + 1$ only once. Let $\tau_i = \min(t | w_t = n_i + 1)$ be the first time the random walk reaches n_i+1. Let $\sigma_i = \min(t | t > \tau_i \cap w_t = n_i)$ be the first time after tau_i the walk reaches n_i or ∞ if it never happens. Since v is harmonic on $\{n_i, n_i + 1..\}$ we get that $\{v(w_t)\}_{t=\tau_i}^{\sigma_i}$ is a bounded martingale. Adopting the convention $v(\infty) = 0$, we get

$$v(n_i + 1) = E(v(\tau_i)) = E(v(\sigma_i)) = 0 \cdot Prob(\sigma_i = \infty) + v(n_i) \cdot Prob(\sigma_i < \infty)$$

Since $v(n_i + 1)/v(n) < 1/2$, the probability of ever reaching n_i after having reached $n_i + 1$ is less than $1/2$. This means that $Prob(s_{n_i} = 1)$ is at least $1/2$. By Fatou's lemma, the probability of $s_{n_i} = 1$ occurring infinitely often is at least $1/2$ and so must be 1 according to the proof of the previous lemma. In particular, $\sum_{i=0}^{\infty} {s_i}^{-1} = \infty$ almost surely. □

Lemma 2.3 shows that if G is quickly transient (in a rather weak sense) then $PATH$ almost surely has infinitely many cut-edges and so must be recurrent.

If the premise of lemma 2.3 is not satisfied then there must exist a sequence of vertices, $\{n_i\}_{i=0}^{\infty}$, such that $n_0 = 0$ and $v(n_i)/2 > v(n_{i+1}) > v(n_i)/4$.

Denote by $PATH_i$ the part of $PATH$ between n_i and n_{i+1}. Let $r_i = \sum_{j=n_i}^{n_{i+1}-1} {s_j}^{-1}$ be the resistance of $PATH_i$.

Let

$$q_i = \sum_{n_i \le w_t, w_{t+1} \le n_{i+1}} (v(w_{t+1}) - v(w_t))^2$$

i.e. sum of $v(w_{t+1}) - v(w_t))^2$ where the sum is taken over the part of the random walk between n_i and n_{i+1}.

Let $\tau_i = \min(t | w_t = n_i)$ be the first time the random walk reaches n_i. Let $\sigma_i = min(t | t > t_i \cap w_t = n_{i-1})$ be the first time after t_i the random walk reaches n_{i-1} or ∞ if it never happens.

Let

$$q'_i = \sum_{\tau_i \le t < \sigma_i} (v(w_{t+1}) - v(w_t))^2$$

i.e. sum of $v(w_{t+1}) - v(w_t))^2$ where the sum is taken over the part of the random walk between times τ_i and σ_i.

Lemma 2.4.

$$E(q_i') < 16v^2(n_i)$$

Proof. For prefixed i, let a_t be equal to $v(w_{t+1}) - v(w_t)$ if $t < \sigma_i$ or 0 if $t \geq \sigma_i$. By definition $v(n_i) + \sum_{t=\tau_i}^{\infty} a_t = v(w_{\sigma_i})$. Take a look at $Var(v(w_{\sigma_i}))$. On the one hand we have

$$Var(v(w_{\sigma_i})) \leq E(v^2(w_{\sigma_i})) \leq v^2(n_{i-1}) < 16v^2(n_i)$$

On the other hand

$$Var(v(w_{\sigma_i})) = \sum_{t=\tau_i}^{\infty} Var(a_t) + 2\sum_{t=\tau_i}^{\infty}\sum_{t'=t+1}^{\infty} Cov(a_t, a_{t'})$$

By harmonicity of v, $E(a_t|w_0, w_1, .., w_t) = 0$. Therefore $Cov(a_t, a_{t'}) = 0$ for all $t \neq t'$. $Var(a_t) = E((v(w_{t+1}) - v(w_t))^2)$. Put together, we get

$$E(\sum_{\tau_i \leq t < \sigma_i} (v(w_{t+1}) - v(w_t))^2 = \sum_{t=\tau_i}^{\infty} Var(a_t) = Var(v(w_{\sigma_i})) < 16v^2(n_i)$$

$\square$

Now we use the connection between q and q' to prove the following lemma.

Lemma 2.5.

$$Prob(q_i < 64v^2(n_i)) > \frac{1}{4}$$

Proof. Using harmonicity of v we get that $Prob(\sigma_i < \infty) = v(n_i)/v(n_{i-1}) < 1/2$. From lemma 2.4 we know that $E(q_i') < 16v^2(n_i)$. q_i' is nonnegative, so by Markov's inequality $Prob(q_i' < 64v^2(n_i)) > 3/4$. This implies

$$Prob(\sigma_i = \infty \cap q_i' < 64v^2(n_i)) > 1/4$$

But if σ_i is ∞ then $q_i' = q_i$ so

$$Prob(q_i < 64v^2(n_i)) > 1/4$$

$\square$

And finally we prove the relation between q_i and R_i, the resistance of $PATH_i$.

Lemma 2.6. *If* $q_i < Cv^2(n_i)$ *then* $R_i > \frac{1}{4C}$

Proof. Recall that s_j is the number of times the walk crossed an edge between j and $j+1$. By definition

$$q_i = \sum_{j=n_i}^{n_{i+1}-1} s_j(v(j)-v(j+1))^2$$

and

$$R_i = \sum_{j=n_i}^{n_{i+1}-1} s_j^{-1}$$

Using Lagrange multipliers method, we try to minimize the value of R_i, under the constraint given by the value of q_i. We get

$$\frac{\partial}{\partial s_j}(R_i + \lambda q_i) = -s_j^{-2} + \lambda(v(j)-v(j+1))^2 = 0$$

Which means that the minimum is achieved when

$$s_j = \lambda^{-\frac{1}{2}}(v(j)-v(j+1))^{-1}$$

substituting s_j in the definition of q_i we get

$$q_i = \lambda^{-\frac{1}{2}} \sum_{j=n_i}^{n_{i+1}-1} (v(j)-v(j+1)) = \lambda^{-\frac{1}{2}}(v(n_i)-v(n_{i+1}))$$

which implies

$$\lambda = (\frac{v(n_i)-v(n_{i+1})}{q_i})^2$$

Turning back to R_i we get

$$R_i = \sum_{j=n_i}^{n_{i+1}-1} s_j^{-1} = \lambda^{\frac{1}{2}} \sum_{j=n_i}^{n_{i+1}-1} (v(j)-v(j+1)) = \frac{(v(n_i)-v(n_{i+1}))^2}{q_i}$$

$$> \frac{(v(n_i)-v(n_{i+1}))^2}{Cv^2(n_i)} > \frac{v^2(n_i)}{4Cv^2(n_i)} = \frac{1}{4C}$$

$\square$

Now our work is nearly done. Combining lemma 2.5 and 2.6 we get that for all i

$$Prob(R_i > \frac{1}{256}) > \frac{1}{4}$$

Using Fatou's lemma again, we get

$$Prob(R_i > \frac{1}{256} \text{ infinitely often}) > \frac{1}{4}$$

From lemma 2.2 we know that the probability of $PATH$ being recurrent is either 0 or 1. We just showed that it cannot be zero and therefore it must be 1.

□

3 Proof of Theorem 1.1

Although the proof of theorem 2.1 seems tailored to the case of line graphs, only minor modifications are needed to adapt it to the general case.

Proof. First, we need to define v. Pick a vertex $g_0 \in G$. Let $v(g)$ be the probability that a simple random walk starting at g visits g_0. For the general case it is not possible to give a simple, closed formula for v, but it is easy to see that the relevant properties of v still hold: v is harmonic except at g_0 and $\lim_{t\to\infty} v(w_t) = 0$ almost surely for w a simple random walk.

Now we shall examine the four lemmas of the special case and prove the corresponding lemmas for the general case.

Lemma 2.2 proves a 0-1 law on the resistance of $PATH$. While the conclusion of the lemma remain true for the general case (we shall prove the resistance to be a.s. infinite), the methods used in the proof are no longer valid. Indeed, it is not true that the resistance of some part of $PATH$, far away from g_0 is a.s. independent of the "decisions" of the random walk made near g_0. Instead of lemma 2.2 we have the following (rather trivial) lemma.

Lemma 3.1. *If*

$$Prob(PATH \text{ is transient}) > 0$$

then for every $C < 1$ there exist a finite sequence of adjacent edges $\overline{w}_0, .., \overline{w}_{t_0}$ such that

$$Prob(PATH \text{ is transient} \mid (w_0, .., w_{t_0}) = (\overline{w}_0, .., \overline{w}_{t_0})) > C$$

Proof. This is standard in measure theory. It follows easily from the regularity of the random walk measure. □

Notice that all the arguments of the special case, as well as the arguments we will use in the general case, can be carried out when the random walk is conditioned to begin with a fixed sequence.

Next, we have lemma 2.3 which handles the simple case where the walk is quickly transient. In the general case, this case cannot happen, since we required the graph be of bounded degree.

Lemma 3.2. *If the degrees of vertices of G are bounded by d, then for g and h adjacent vertices we have*

$$v(h) \leq dv(g)$$

Proof. This follows immediately from harmonicity of v. $\square$

Let $C_i = \{g \in G \mid d^{-2i-1}/d \leq v(g) \leq d^{-2i}\}$ be the set of all vertices whose v values lies between d^{-2i-1} and d^{-2i}. From lemma 3.2 we know that every C_i is a cutset in the sense that it separates C_{i-1} from C_{i+1}. It is not necessarily a cutset in the usual sense, of a set separating g_0 from infinity, nor do these sets need be finite. Indeed, there can be an infinite number of vertices for which v takes value above d^{-2i}. However, since $v(w_t)$ tends to 0 almost surely, the sets C_i are cutset, in the usual sense, in $PATH$ almost surely.

Let $PATH_i$ be all the edges in $PATH$ between C_i and C_{i+1}. More precisely,

$$PATH_i = \{(g,h) \in PATH \mid d^{-2i-2} < v(g) < d^{-2i-1} \cap d^{-2i-2} < v(g) < d^{-2i-1}\}$$

As before, let

$$q_i = \sum_{(w_t,w_{t+1}) \in PATH_i} (v(w_{t+1}) - v(w_t))^2$$

i.e. sum of $v(w_{t+1}) - v(w_t))^2$ where the sum is taken over the part of the random walk between C_i and C_{i+1}.

Let $\tau_i = \min(t|w_t = n_i)$ be the first time the random walk reaches C_i. Let $\sigma_i = min(t|t > t_i \cap w_t = n_{i-1})$ be the first time after t_i the random walk reaches C_{i-1} or ∞ if it never happens.

Let

$$q'_i = \sum_{\tau_i \leq t < \sigma_i} (v(w_{t+1}) - v(w_t))^2$$

i.e. sum of $v(w_{t+1}) - v(w_t))^2$ where the sum is taken over the part of the random walk between times τ_i and σ_i.

Lemma 3.3.

$$E(q_i') < d^4 d^{-4i}$$

Proof. The proof is identical to that of lemma 2.4. This time we get

$$Var(v(w_{\sigma_i})) \leq (d^{-2i+2})^2 = d^{-2i+4}$$

and

$$Var(v(w_{\sigma_i})) = Var(v(w_{\tau_i})) + \sum_{t=\tau_i}^{\infty} Var(v(w_{t+1}) - v(w_t))$$

Since the covariances are, as before, all 0. ☐

Lemma 3.4.

$$Prob(q_i < 4d^4 d^{-4i}) \geq \frac{1}{4}$$

Proof. The proof is (again) identical to the proof of 2.5. Here we have

$$Prob(\sigma_i < \infty) \leq \frac{\sup_{g \in C_i} v(g)}{\inf_{g \in C_{i-1}} v(g)} \leq \frac{1}{d} \leq \frac{1}{2}$$

and

$$Prob(q_i' < 4d^4 d^{-4i}) \geq \frac{3}{4}$$

☐

Next, we define R_i as the resistance of $PATH_i$ when C_i and C_{i+1} are both contracted, each to a single vertex, denoted c_i and c_{i+1}. The contracted $PATH_i$ will be denoted $PATH_i'$.

Lemma 3.5. *If* $q_i < Cd^{-4i}$ *then* $R_i > \frac{1}{4Cd^2}$

Proof. The proof is actually simpler than 2.6. Let $v'(g)$, defined for $g \in PATH_i$ be equal to d^{-2i-1} for $g \in C_i$, to d^{-2i-2} for $g \in C_{i+1}$ and otherwise equal to $v(g)$. By standard abuse of notation we shall refer to v' as defined on $PATH_i'$ too.

Let

$$q_i'' = \sum_{(w_t, w_{t+1}) \in PATH_i} (v'(w_{t+1}) - v'(w_t))^2$$

Obviously, $q_i'' \leq q_i$. Now we use Thompson's Principle (see [3] , page 49) on $PATH_i'$ with the function v'. q_i'' is the "energy dissipation" of v' on $PATH_i'$. By Thompson's Principle the real energy dissipation is lower. Recall that $v'(c_i) = d^{-2i-1}$ and $v'(c_{i+1}) = d^{-2i-2}$.

Put together, we have

$$\frac{(d^{-2i-1} - d^{-2i-2})^2}{R_i} \leq q_i'' \leq q_i < Cd^{-4i}$$

Which yields

$$R_i > \frac{(d^{-2i-1} - d^{-2i-2})^2}{Cd^{-4i}} \geq \frac{1}{4Cd^2}$$

□

Combining lemma 3.4 and 3.5 we get that for all i

$$Prob(R_i > \frac{1}{16d^6}) > \frac{1}{4}$$

Using Fatou's lemma (again) we get

$$Prob(R_i > \frac{1}{16d^6} \textit{ infinitely often}) > \frac{1}{4}$$

By Rayleighs Monotonicity Law (see [3] , page 51) we know that the resistance of $PATH$ is greater than that of the concatenation of $PATH_i'$, which is $\sum_{i=1}^{\infty} R_i$. Therefore, the probability of $PATH$ being recurrent is greater than $\frac{1}{4}$.

As noted earlier, all the arguments we used can be carried out when the random walk is conditioned to begin with a fixed sequence. Using lemma 3.1, we conclude that the probability of $PATH$ not being recurrent must be 0.

□

Remark: a close inspection of the proof reveals that the theorem is also true for a finite union of paths of independent simple random walks. The only difference is that lemma 3.4 applies to each SRW separately, to yield a probability of $\frac{1}{4^k}$ (k being the number of SRWs) for the resistance of the union to be at least $\frac{1}{16kd^6}$.

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