Scoring Guideline
MATH 200/253 Sec 922
Summer 2012
TA: Vincent Chiu
Instructor: Dr. Ori-Gurel

If you have concerns about the quiz, come and see me at the MLC at LSK100 basement. You can find me between 2:00pm to 3:00pm there. If you really can't make it, see the professor

1. Find the absolute minimum and maximum values, and all the points where they occur, of the function $f(x, y, z) = 3xy + z - z^2$ in the unit ball centered around the origin.

Solution

The first step is to find all the critical points of $f(x, y, z)$ that **in** the unit ball $\mathbb{B}_3$.

$$\begin{cases} f_x(x,y,z) = 0 \\ f_y(x,y,z) = 0 \\ f_z(x,y,z) = 0 \end{cases} \iff \begin{cases} 3y = 0 \\ 3x = 0 \\ 1 - 2z = 0 \end{cases} \iff \begin{cases} y = 0 \\ x = 0 \\ z = \frac{1}{2} \end{cases}$$

There is only one critical point of $f(x, y, z)$. Quick inspection tells us that the point $\left(0, 0, \frac{1}{2}\right) \in \mathbb{B}_3$

Now we have to find critical points of $f(x, y, z)$ on the boundary of the constraint. Apply Lagrange Multipliers.

$$\begin{gathered} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ x^2 + y^2 + z^2 = 1 \end{gathered}$$

Thus,

$$\begin{gathered} < 3y, 3x, 1 - 2z > = \lambda < 2x, 2y, 2z > \\ x^2 + y^2 + z^2 = 1 \end{gathered}$$

$$\begin{cases} 3y = \lambda 2x & (1) \\ 3x = \lambda 2y & (2) \\ 1 - 2z = \lambda 2z & (3) \\ x^2 + y^2 + z^2 = 1 & (4) \end{cases}$$

3 marks for correct setup

1 mark was deducted if $g(x, y, z) = 1$ was missing

Often, the true difficulty lies in solving the simutaneous equations. One strategy is to solve for λ first and write everything in terms of the λ since λ appears in all the equations except for the constraints. Of course this isn't the "best" strategy as sometimes it is impossible if the equations are too complicated

If x and y are zero, then (1) and (2) are immediately satisfied and (4) gives us $z = \pm 1$

Plug (1) into (2) to get $9x = 4\lambda x$. This means that either $x = 0$ or $\lambda = \pm\frac{3}{2}$. If $x = 0$ then from (1) we get that $y = 0$ and then $z = \pm 1$, giving points we're already familiar with.

If $\lambda = \frac{3}{2}$, then plugging it into (4) and solving for z will give us $z = \dfrac{1}{5}$.

Plug in $\lambda = \frac{3}{2}$ into either (1) or (2) to get $y = x \implies y^2 = x^2$

Finally, substitute $y^2 = x^2$ and $z = \frac{1}{5}$ into the constraint $x^2 + y^2 + z^2 = 1$ to get

$$x^2 + y^2 + z^2 = 2x^2 + \frac{1}{25} = 1 \iff x^2 = \frac{24}{50} \iff x = \pm\sqrt{\frac{12}{25}} \iff x = \pm\frac{\sqrt{12}}{5}$$

If $\lambda = -\frac{3}{2}$, then plugging it into (3) yields $1 - 2z = -3z$, so $z = -1$ and then by (4), we get $x = 0$ and $y = 0$, again a point we already know.

Summary of Results

Critical Points P_0	$f(P_0)$	Conclusion
$(0, 0, 1)$	0	–
$(0, 0, -1)$	-2	Absolute minimum
$(0, 0, 1/2)$	$1/4$	–
$(\sqrt{12}/5, \sqrt{12}/5, 1/5)$	$8/5$	Absolute maximum
$(-\sqrt{12}/5, -\sqrt{12}/5, 1/5)$	$8/5$	Absolute maximum

1 mark for each the critical points

2 marks for correct conclusion

Remark 1: A lot of people thought the constraint was an equality.

Remark 2: A ball is not the same as a disk. Quite a number of people drew a circle and thought that was a "ball"

2. Find the absolute minimum and maximum values, and all the points where they occur, of the function $f(x,y) = x^2 + xy + x - y$ in the triangle formed by the lines $x = 0$, $y = 0$, and $x + y = 1$

Solution

Here is a plot of the triangle. $D = \{(x,y)|\ 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}$

2 marks for a reasonable plot of D

1 mark was deducted for not coloring it in blue. I am kidding...

Find all the critical points of $f(x,y)$ that may lie in D or on the boundary of D

$$\begin{cases} f_x(x,y) = 0 \\ f_y(x,y) = 0 \end{cases} \iff \begin{cases} 2x + y + 1 = 0 \\ x - 1 = 0 \end{cases} \implies \begin{cases} y = -3 \\ x = 1 \end{cases}$$

However $(1,-3) \notin D$, so reject

1 mark for finding (1,-3)

1 mark for rejecting (1,-3)

Now find points along the boundary of D

Along $y = 0$

$g(x) = f(x,0) = x^2 + x$ where $0 \leq x \leq 1$

This is a parabola, either complete the square or use the first derivative test to find the max and min, $g'(x) = 2x + 1 = 0 \iff x = \frac{-1}{2} \implies (\frac{-1}{2}, g(\frac{-1}{2}))$

Just like before $(\frac{-1}{2}, g(\frac{-1}{2})) \notin D$, so reject

1 mark for finding $(\frac{-1}{2}, g(\frac{-1}{2}))$ **along** $y = 0$

1 mark for rejecting $(\frac{-1}{2}, g(\frac{-1}{2}))$

Along $x = 0$, we have $f(0,y) = -y$ where $0 \leq y \leq 1$

This is a line segment, so the end points will give us the minimum and maximum values along $x = 0$, $f(0,0) = 0$ and $f(0,1) = -1$

1 mark for finding (0,0) and (0,1) **along** $x = 0$

Along $y = 1 - x$ where $0 \leq x \leq 1$

$h(x) = f(x, 1-x) = x^2 + x(1-x) + x - (1-x) = x^2 + x - x^2 + x - 1 + x = 3x - 1$

Another line segment, so the end points will give us the minimum and maximum values along $y = 1 - x$, $h(0) = f(0,1) = -1$ and $h(1) = f(1,0) = 2$

1 mark for finding (1,0) and (0,1) **along** $y = 1 - x$

Summary of results

Critical Points P_0	$f(P_0)$	Conclusion
$(1,-3)$	–	–
$(-\frac{1}{2}, g(-\frac{1}{2}))$	–	–
$(0,0)$	0	–
$(0,1)$	-1	Absolute minimum
$(1,0)$	2	Absolute maximum

1 mark each for stating the absolute maximum and minimum

3. Find the volume of the solid that lies under function $F(x,y) = \dfrac{1}{(1+x+y)^2}$ and above the square $R = \{(x,y)|\ 0 \le x \le 1, 0 \le y \le 0\}$

Solution

This is a very straightforward integration problem since the question saves you the trouble of drawing complicated regions. Instead, it gives you the bounds in abstract Greek symbols, how nice of them?

$$\begin{aligned}
V = \iint_R F(x,y)dA &= \int_0^1 \int_0^1 \frac{1}{(1+x+y)^2} dydx && (1)\\
&= \int_0^1 \frac{(1+x+y)^{-1}}{-1}\Bigg|_{y=0}^{y=1} dx \quad \text{Some of you may want to review on integration techniques} && (2)\\
&= -\int_0^1 (2+x)^{-1} - (1+x)^{-1} dx && (3)\\
&= \ln(1+x) - \ln(2+x)\Big|_0^1 && (4)\\
&= [\ln(2) - \ln(3)] - [\ln(1) - \ln(2)] && (5)\\
&= \ln\left(\frac{2}{3}\right) + \ln(2) && (6)\\
&= \ln\left(\frac{4}{3}\right) && (7)
\end{aligned}$$

4 marks for set up

3 marks for correctly evaluating the indefinite integral

2 marks for substituting the bounds correctly

1 mark for correct simplification