MATH 253 - SEC 104 - W2011T1

1. Consider the figure 8 shape given by the equation $x^2 = y^2 - y^4$ (see figure 1). Write the area bounded by this curve as an iterated integral in Cartesian (rectangular) coordinates in 2 ways and as an iterated integral in polar coordinates. Calculate one of these integrals (bonus points if you do it in more than 1 way).

   Solution: It is easier to consider only the part of the shape in the first quadrant, which, by symmetry, is one quarter of the area. Given some y the limits of integration would be from 0 (the y axis) and the solution for $x^2 = y^2 - y^4$, which is $\sqrt{y^2 - y^4} = y\sqrt{1-y^2}$ (since $y \geq 0$). We see that there is a solution exactly when $0 \leq y \leq 1$. Hence, these are the limits of integration for y and the area is

$$4\int_0^1 \int_0^{y\sqrt{1-y^2}} 1\,dx\,dy = 4\int_0^1 y\sqrt{1-y^2}\,dy = 4/3(-(1-y^2)^{3/2})\Big|_0^1 = 4/3$$

   Similarly, given $0 \leq x$ we find that the limits of integration for y are the two positive solutions for $x^2 = y^2 - y^4$. If we define $a = y^2$ we get that $a^2 - a + x^2 = 0$, so the solutions are $(1 \pm \sqrt{1-4x^2})/2$. There is a solution when $0 \leq x \leq 1/2$ (recall that we only consider nonnegative x), so these are the limits of integration for x. Since $y^2 = a$ we get

$$4\int_0^{\frac{1}{2}} \int_{\sqrt{(1-\sqrt{1-4x^2})/2}}^{\sqrt{(1+\sqrt{1-4x^2})/2}} 1\,dy\,dx$$

   and happily we don't have to evaluate this integral.

   Finally, for polar coordinates, we again consider only the first quadrant, so $0 \leq \theta \leq \pi/2$. Given θ, the limits of integration for r are 0 and the solution for the curve's equation in polar coordinates which is

$$r^2\cos^2\theta = r^2\sin^2\theta r^4 - \sin^4\theta$$

   One solution is $r = 0$ which is the lower limit of integration. The other solution is

$$r = \sqrt{\frac{\sin^2\theta - \cos^2\theta}{\sin^4\theta}}$$

   and it exists whenever $\sin\theta \geq \cos\theta$ which is when $\pi/4 \leq \theta$. Hence the area is

$$4\int_{\pi/4}^{\pi/2} \int_0^{\sqrt{\frac{\sin^2\theta-\cos^2\theta}{\sin^4\theta}}} r\,dr\,d\theta$$

$$= 4\int_{\pi/4}^{\pi/2} (r^2/2)\Big|_0^{\sqrt{\frac{\sin^2\theta-\cos^2\theta}{\sin^4\theta}}} = 2\int_{\pi/4}^{\pi/2} \frac{\sin^2\theta - \cos^2\theta}{\sin^4\theta}\,d\theta$$

   which again, we leave unevaluated.

2. Calculate the volume of the torus (a.k.a. doughnut) given by the equation $(x^2 + y^2 + z^2 + 3)^2 = 16(x^2 + y^2)$ (see figures 2,3). Hint: first write this as a region of type I by checking for which (x, y) is there a z which solves the equation. Then look for the most convenient way to turn this into iterated integral.

   Solution: We need to evaluate $\int\int\int_R 1dV$, where R is the doughnut. We write this in cylindrical coordinates as $(r^2 + z^2 + 3)^2 = 16r^2$. Since both $r^2 + z^2 + 3$ and $4r$ are nonnegative, we can take the square root of both sides and get $r^2 + z^2 + 3 = 4r$. Hence, given r (and θ which doesn't effect anything) the solutions for z are $\pm\sqrt{4r - r^2 - 3}$. Hence, we can write R as $\{(r, \theta, z) \mid 4r - r^2 - 3 \geq 0, -\sqrt{4r - r^2 - 3} \leq z \leq \sqrt{4r - r^2 - 3}\}$. noticing that $4r - r^2 - 3 = 0$ has two solutions $r = 1$ and $r = 3$ and describes a frowning parabola, we see that $4r - r^2 - 3 \geq 0$ exactly when $1 \leq r \leq 3$. Hence, our integral can be written as

$$\int_0^{2\pi} \int_1^3 \int_{-\sqrt{4r-r^2-3}}^{\sqrt{4r-r^2-3}} r\, dz\, dr\, d\theta$$

$$= 2\pi \int_1^3 2r\sqrt{4r - r^2 - 3}\, dr$$

   To evaluate this integral, we substitute $s = r - 2$ (to get something nicer inside the square root) and get

$$= 2\pi \int_{-1}^1 2(s + 2)\sqrt{1 - s^2} ds = 2\pi \int_{-1}^1 2s\sqrt{1 - s^2}\, ds + 2\pi \int_{-1}^1 4\sqrt{1 - s^2}\, ds$$

   The first integral is

$$2\pi(-(2/3)(1 - s^2)^{3/2})\Big|_{-1}^{1} = 0$$

   For the second one we substitute $\sin(t) = s$ and get

$$8\pi \int_{-\pi/2}^{\pi/2} \cos^2(t)\, dt = 4\pi \int_{-\pi/2}^{\pi/2} 1 + \cos(2t)\, dt$$

$$= 4\pi(t + \sin(2t)/2)\Big|_{-\pi/2}^{\pi/2} = 4\pi^2$$

3. Let R be the body bounded by the surfaces $z = 0$, $z = y$ and $y = 1 - x^2$. Write the integral of $F(x, y, z) = y$ over R as an iterated integral in all six possible ways and calculate it.

Solution: First, one has to visualize these three surfaces and realize that the relevant body is defined by the inequalities $z \geq 0$, $z \leq y$ and $y \leq 1 - x^2$. Now, let's find the limits of integration for the $x - y - z$ order. We need to find, given x and y, what are the limits of integration for z. This is easy: $0 \leq z \leq y$. For which (x, y) do we have a solution in z? The condition on z is $0 \leq z \leq y$, so there's a solution for any $0 \leq y$, and of course we have the third inequality $y \leq 1 - x^2$. So the integral can be written as

$$\int\int\int_R y\, dV = \int\int_D \int_0^y y\, dz\, dA$$

where $D = \{(x, y) \mid 0 \leq y \leq 1 - x^2\}$. Next, we do the same thing to D and find out that given x, the limits of integration of y are 0 and $1 - x^2$ and there's a solution exactly when $-1 \leq x \leq 1$, so the integral is equal to

$$\int_{-1}^1 \int_0^{1-x^2} \int_0^y y\, dz\, dy\, dx$$

$$= \int_{-1}^1 \int_0^{1-x^2} y^2\, dy\, dx \;=\; \frac{1}{3}\int_{-1}^1 y^3\Big|_0^{1-x^2} dx$$

$$= \frac{1}{3}\int_{-1}^1 1 - 3x^2 + 3x^4 - x^6\, dx = \frac{1}{3}(x - x^3 + \frac{3x^5}{5} - \frac{x^7}{7})\Big|_{-1}^1$$

$$= \frac{1}{3}(1 - 1 + \frac{3}{5} - \frac{1}{7}) - (-1 + 1 - \frac{3}{5} + \frac{1}{7}) = \frac{32}{105}$$

The limits of integration for the other orders are:

$$\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} \int_0^y y\, dz\, dx\, dy$$

$$= \int_{-1}^1 \int_0^{1-x^2} \int_z^{1-x^2} y\, dy\, dz\, dx = \int_0^1 \int_{-\sqrt{1-z}}^{\sqrt{1-z}} \int_z^{1-x^2} y\, dy\, dx\, dz$$

$$= \int_0^1 \int_0^y \int_{-\sqrt{1-y}}^{\sqrt{1-y}} y\, dx\, dz\, dy = \int_0^1 \int_z^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} y\, dx\, dy\, dz$$

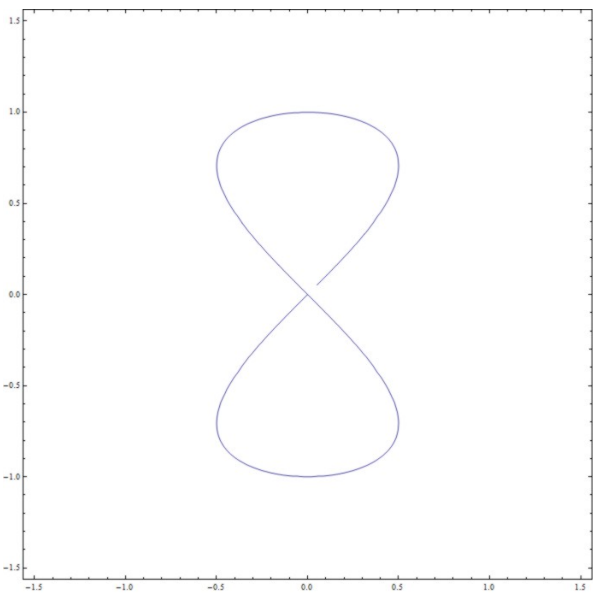

Figure 1: The curve $x^2 = y^2 - y^4$

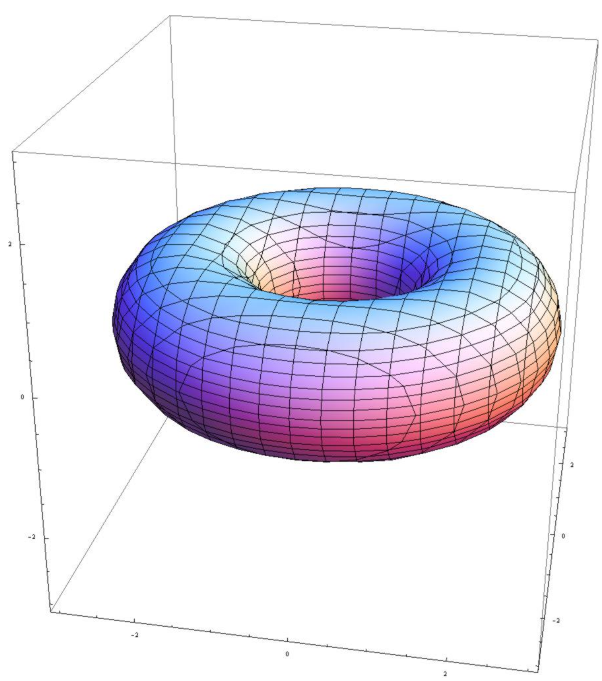

Figure 2: The torus $(x^2 + y^2 + z^2 + 3)^2 = 16(x^2 + y^2)$

Figure 3: Bon appétit!