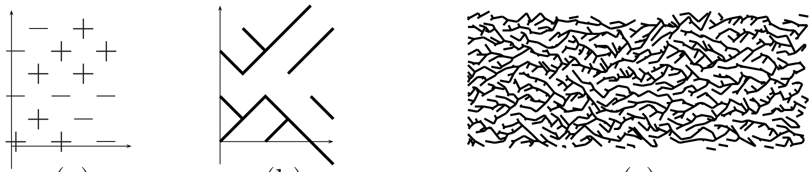
7 Example: the Brownian web as a black noise

7a Convolution semigroup of the Brownian web

A one-dimensional array of random signs can produce some classical and nonclassical noises in the scaling limit, but I still do not know, whether it can produce a black noise, or not. This is why I turn to a two-dimensional array of random signs (a).



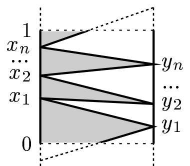
(a) (b) (c)

It produces a system of coalescing random walks (b) that converges to the so-called *Brownian web* (c), consisting of infinitely many coalescing Brownian motions (independent before coalescence).

The Brownian web was investigated by Arratia, Toth, Werner, Soucaliuc, and recently by Fontes, Isopi, Newman and Ravishankar [2] (other references may be found therein). The scaling limit may be interpreted in several ways, depending on the choice of 'observables', and may involve delicate points, because of complicated topological properties of the Brownian web as a random geometric configuration on the plane. However, we avoid these delicate points by treating the Brownian web as a stochastic flow in the sense of Sect. 4, that is, a two-parameter family of random variables in a semigroup.

In order to keep finite everything that can be kept finite, we consider Brownian motions in the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ rather than the line $\mathbb{R}$.

It is well-known that a countable dense set of coalescing 'particles', given at the initial instant, becomes finite, due to coalescence, after any positive time. Moreover, the finite number is of finite expectation. Thus, for any given $t > 0$, the Brownian web on the time interval $(0, t)$ gives us a random map $\mathbb{T} \to \mathbb{T}$ of such an elementary form (a step function):



$$f^{y_1,\dots,y_n}_{x_1,\dots,x_n} : \mathbb{T} \to \mathbb{T}\,,$$
$$x_1 < \dots < x_n < x_1,\ y_1 < \dots < y_n < y_1 \text{ (cyclically)},$$
$$f^{y_1,\dots,y_n}_{x_1,\dots,x_n}(x) = y_{k+1} \text{ for } x \in (x_k, x_{k+1}]\,.$$

Of course, n is random, as well as $x_1, \dots, x_n$ and $y_1, \dots, y_n$. The value at x_k does not matter; we let it be y_k for convenience, but equally well it could be y_{k+1}, or remain undefined. Points $x_1, \dots, x_n$ will be called left critical points of the map, while $y_1, \dots, y_n$ are right critical points.

We introduce the set G_∞ consisting of all step functions $\mathbb{T} \to \mathbb{T}$ and in addition, the identity function. If $f, g \in G_\infty$ then their composition fg belongs to G_∞, thus G_∞ is a semigroup. It consists of pieces of dimensions $2, 4, 6, \dots$ and the identity. Similarly to G_3 (recall (4d2)), G_∞ is not a topological semigroup, since the composition is discontinuous.

The distribution of the random map is a probability measure μ_t on G_∞. These maps form a convolution semigroup, $\mu_s * \mu_t = \mu_{s+t}$. Similarly to 4e, discontinuity of composition does not harm, since the composition is continuous almost everywhere (w.r.t. $\mu_s \otimes \mu_t$). Left and right critical points do not meet.[26]

Having the convolution semigroup, we can construct the stochastic flow, that is, a family of G_∞-valued random variables $(\xi_{s,t})_{s\le t}$ such that

$$\xi_{s,t} \sim \mu_{t-s}\,,$$
$$\xi_{r,s}\xi_{s,t} = \xi_{r,t} \quad \text{a.s.}$$

whenever $-\infty < r < s < t < \infty$, and

$$\xi_{t_1,t_2}, \dots, \xi_{t_{n-1},t_n} \quad \text{are independent}$$

whenever $-\infty < t_1 < \dots < t_n < \infty$.

Indeed, for each i we can take independent $\xi_{k/i,(k+1)/i} : \Omega[i] \to G_\infty$ for $k \in \mathbb{Z}$ and define $\xi_{k/i,l/i} = \xi_{k/i,(k+1)/i} \dots \xi_{(l-1)/i,l/i}$. For any two coarse instants $s \le t$, the distribution of $\xi_{s[i],t[i]}$ converges weakly (for $i \to \infty$) to $\mu_{t[\infty]-s[\infty]}$. The refinement gives us

$$\xi_{s,t} : \Omega \to G_\infty\,, \qquad \xi_{s,t} = f^{y_1(s,t),\dots,y_{n(s,t)}(s,t)}_{x_1(s,t),\dots,x_{n(s,t)}(s,t)}\,;$$

$x_k(\cdot,\cdot)$ and $y_k(\cdot,\cdot)$ are continuous a.s. Also,

$$\mathbb{E}\, n(s,t) < \infty\,. \tag{7a1}$$

We consider the sub-σ-field $\mathcal{F}_{s,t}$ generated by all $\xi_{u,v}$ for $(u,v) \subset (s,t)$ and get a continuous factorization. Time shifts are introduced evidently, and so, we get a noise, — the *noise of coalescence.*

7b Some general arguments

Random variables of the form $\varphi(\xi_{s,t})$ for arbitrary $s \le t$ and arbitrary bounded Borel function $\varphi : G_\infty \to \mathbb{R}$ generate the whole σ-field $\mathcal{F}$. Polynomials of such random variables are dense in L_2, however, we have no reason to think that *linear* combinations of such random variables are dense in L_2.

Denote by Q the orthogonal projection of $L_2(\Omega, \mathcal{F}, P)$ onto the first chaos.

7b1 Lemma. Linear combinations of all $Q\varphi(\xi_{s,t})$ are dense in the first chaos.

Proof: follows easily from the next (quite general) result, or rather, its evident generalization to n factors.

7b2 Lemma. Let $r \le s \le t$, $X \in L_2(\mathcal{F}_{r,s})$, $Y \in L_2(\mathcal{F}_{s,t})$. Then $Q(XY) = Q(X) \cdot \mathbb{E}(Y) + \mathbb{E}(X) \cdot Q(Y)$.

[26]They meet with probability 0, as far as s and t are fixed. Otherwise, delicate points are involved...

Hint to a proof: if $M \in \mathcal{C}$ satisfies $|M \cap (r,t)| = 1$ then either $|M \cap (r,s)| = 1$ and $M \cap (s,t)| = 0$, or $|M \cap (r,s)| = 0$ and $M \cap (s,t)| = 1$.

In order to prove that the noise (of coalescence) is black, it suffices to prove that $Q\varphi(\xi_{s,t}) = 0$ for all s,t,φ. We'll prove that $Q\varphi(\xi_{0,1}) = 0$; the general case is similar. Assuming $\mathbb{E}\,\varphi(\xi_{0,1}) = 0$ we use Prop. 6a2. Note that $\|Q_L\varphi(\xi_{0,1})\|^2 = \|\mathbb{E}\left(\varphi(\xi_{0,1}) \,\middle|\, \mathcal{F}_{s_1,s_2}\right)\|^2 + \dots + \|\mathbb{E}\left(\varphi(\xi_{0,1}) \,\middle|\, \mathcal{F}_{s_{n-1},s_n}\right)\|^2$. It suffices to prove that

$$\|\mathbb{E}\left(\varphi(\xi_{0,1}) \,\middle|\, \mathcal{F}_{t-\varepsilon,t}\right)\| = o(\sqrt{\varepsilon}) \quad \text{for } \varepsilon \to 0\,,$$

uniformly in t. When doing so, we may assume that t is bounded away from 0 and 1. Indeed, $\|\mathbb{E}\left(\varphi(\xi_{0,1}) \,\middle|\, \mathcal{F}_{t,1}\right)\| \to 0$ for $t \to 1-$ due to continuity of the factorization (recall 3d).

7b3 Lemma. $\mathbb{E}\left(\varphi(\xi_{0,1}) \,\middle|\, \mathcal{F}_{t-\varepsilon,t}\right) = \mathbb{E}\left(\varphi(\xi_{0,1}) \,\middle|\, \xi_{t-\varepsilon,t}\right)$.

Hint to a proof:

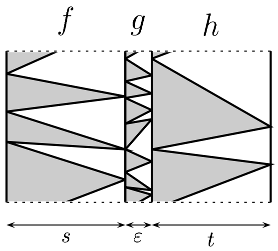
$$\mathbb{E}\left(\varphi(\xi_{t_1,t_5}) \,\middle|\, \xi_{t_2,t_3}, \xi_{t_3,t_4}\right) = \iint \varphi(\xi_{12}\xi_{23}\xi_{34}\xi_{45})\, d\mu_{t_2-t_1}(\xi_{12}) d\mu_{t_5-t_4}(\xi_{45}) = \mathbb{E}\left(\varphi(\xi_{t_1,t_5}) \,\middle|\, \xi_{t_2,t_4}\right) .$$

7c The key argument

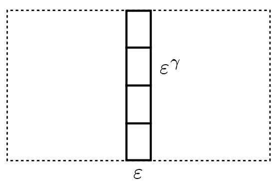
Similarly to 6a4, we consider $X = \varphi(\xi_{0,1}) = \varphi(\xi_{0,t-\varepsilon}\xi_{t-\varepsilon,t}\xi_{t,1})$, $\mathbb{E}\,X = 0$, $|X| \le 1$ a.s. We have to prove that $\|\mathbb{E}\left(X \,\middle|\, \xi_{t-\varepsilon,t}\right)\| = o(\sqrt{\varepsilon})$ for $\varepsilon \to 0$, uniformly in t, when t is bounded away from 0 and 1. Clearly,

$$\mathbb{E}\left(X \,\middle|\, \xi_{t-\varepsilon,t}\right) = \iint \varphi(fgh)\, d\mu_{t-\varepsilon}(f) d\mu_{1-t}(h)\,,$$

where $g = \xi_{t-\varepsilon,t}$.



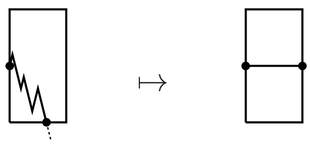
We choose $\gamma \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and divide the strip $(t-\varepsilon, t) \times \mathbb{T}$ into $\sim \varepsilon^{-\gamma}$ 'cells' $(t-\varepsilon, t) \times (z_k, z_{k+1})$ of height $z_{k+1} - z_k \sim \varepsilon^\gamma$.



We want to think of g as consisting of independent cells. Probably it can be done in continuous time, but we have no such technique for now. Instead, we retreat to the discrete-time model. The needed inequality for continuous time results in the scaling limit $i \to \infty$ provided that in discrete time our estimations are uniform in i (for i large enough).

So, random signs that produce g are divided into cells. Cells are independent and, taken together, they determine g uniquely.

However, a path may cross many cells. That is rather improbable, since $\gamma < 1/2$, but it may happen. We enforce locality by a forgery! Namely, if the path starting at the middle of a cell reaches the bottom or the top edge of the cell, we replace the whole cell with some other cell (it may be chosen once and for all) where it does not happen.



Now cells are 'local'; a path cannot cross more than two cells; but of course, the stochastic flow is changed. Namely, g is changed with an exponentially small (for $\varepsilon \to 0$) probability, which changes $\mathbb{E}\left(X \,\middle|\, \xi_{t-\varepsilon,t} \right)$ by $o(\sqrt{\varepsilon})$ (much less, in fact). Still, cells are independent.

Does a cell (of g) influence the composition, fgh? It depends on f and h. If the left edge $\{t-\varepsilon\} \times [z_k, z_{k+1}]$ of the cell contains no right critical point of f, the cell can influence, since a path starting in an adjacent cell can cross the boundary between cells. However, if the enlarged left edge $\{t-\varepsilon\} \times [z_k - \varepsilon^\gamma, z_{k+1} + \varepsilon^\gamma]$ contains no right critical point of f (in which case we say 'the cell is blocked by f'), then the cell cannot influence, because of the enforced locality. Similarly, if the enlarged right edge $\{t\} \times [z_k - \varepsilon^\gamma, z_{k+1} + \varepsilon^\gamma]$ contains no left critical point of h (in which case we say 'the cell is blocked by h'), the cell cannot influence.

The probability of being not blocked by f is the same for all cells, since the distribution of f is invariant under rotations of $\mathbb{T}$ (discretized...). The sum of these probabilities does not exceed $3\mathbb{E}n(0, t-\varepsilon)$ (recall (7a1)), which is $O(1)$ when $\varepsilon \to 0$. (Here we need t to be bounded away from 0.) Thus,

$$\begin{gathered}\mathbb{P}\left(\text{a given cell is not blocked by } f \right) = O(\varepsilon^\gamma)\,;\\ \mathbb{P}\left(\text{a given cell is not blocked by } h \right) = O(\varepsilon^\gamma)\,;\\ \mathbb{P}\left(\text{a given cell is not blocked} \right) = O(\varepsilon^{2\gamma})\,;\\ \mathbb{P}\left(\text{at least one cell is not blocked} \right) = O(\varepsilon^\gamma)\,.\end{gathered}$$

In the latter case we may say that g is not blocked (by f, h).

Denote by A the event "g is not blocked by f, h" (it is determined by f and h, not g). We have

$$\begin{gathered}\mathbb{P}\left(A \right) = O(\varepsilon^\gamma)\,;\\ X = X - \mathbb{E}X = \big(X \cdot \mathbf{1}_A - \mathbb{E}(X \cdot \mathbf{1}_A) \big) + \big(X \cdot (\mathbf{1} - \mathbf{1}_A) - \mathbb{E}(X \cdot (\mathbf{1} - \mathbf{1}_A)) \big)\,;\\ \mathbb{E}\left(X \cdot (\mathbf{1} - \mathbf{1}_A) \,\middle|\, g \right) = \mathbb{E}(X \cdot (\mathbf{1} - \mathbf{1}_A))\,;\\ \mathbb{E}\left(X \,\middle|\, g \right) = \mathbb{E}\left(X \cdot \mathbf{1}_A \,\middle|\, g \right) - \mathbb{E}(X \cdot \mathbf{1}_A)\,;\end{gathered}$$

we have to prove that $\|\mathbb{E}\left(X \cdot \mathbf{1}_A \,\middle|\, g \right) - \mathbb{E}(X \cdot \mathbf{1}_A)\| = o(\sqrt{\varepsilon})$. Note that it does not result from the trivial estimation $\|X \cdot \mathbf{1}_A\| \le \|\mathbf{1}_A\| = \sqrt{\mathbb{P}\left(A \right)} = O(\varepsilon^{\gamma/2})$, $\gamma \in \left(\frac13, \frac12 \right)$. Note also that, when g influences, its influence is usually not small (irrespective of ε) because of the stepwise nature of f and h.

We have

$$\| \mathbb{E} \left(X \cdot \mathbf{1}_A \,\middle|\, g \right) - \mathbb{E} (X \cdot \mathbf{1}_A) \| = \sup_\psi \operatorname{Cov} \big(X \cdot \mathbf{1}_A, \psi(g) \big) \,,$$

where the supremum is taken over all Borel functions $\psi : G_\infty \to \mathbb{R}$ such that $\operatorname{Var}\big(\psi(g)\big) \le 1$. Using the correlation coefficient

$$\operatorname{Corr}\big(X \cdot \mathbf{1}_A, \psi(g) \big) = \frac{ \operatorname{Cov}\big(X \cdot \mathbf{1}_A, \psi(g) \big) }{ \sqrt{\operatorname{Var}(X \cdot \mathbf{1}_A)} \sqrt{\operatorname{Var}(\psi(g))} }$$

we may prove that

$$\operatorname{Corr}\big(X \cdot \mathbf{1}_A, \psi(g) \big) = o(\varepsilon^{(1-\gamma)/2}) \,,$$

since it implies $\operatorname{Cov}(\dots) = o(\varepsilon^{(1-\gamma)/2}) \cdot \| X \cdot \mathbf{1}_A \| = o(\varepsilon^{(1-\gamma)/2} \varepsilon^{\gamma/2}) = o(\sqrt{\varepsilon})$. Instead of $o(\varepsilon^{(1-\gamma)/2})$ we'll get $O(\varepsilon^\gamma)$, which is also enough, since $\gamma > 1/3$.

It remains to apply a quite general lemma given below, interpreting its Y_k as the whole k-th cell (of g), X_k as the indicator of the event "the k-th cell is not blocked" ($k = 1, \dots, n$), X_0 as the pair (f, h), and $\varphi(\dots)$ as $X \cdot \mathbf{1}_A$. The lemma is formulated for real-valued random variables Y_k, but it does not matter; the same holds evidently for arbitrary spaces, and in fact, we need only finite spaces. The product $X_k Y_k$ is a trick for 'blocking' Y_k when $X_k = 0$. Note that dependence between $X_0, X_1, \dots, X_n$ is allowed.

7c1 Lemma. Let $(X_0, X_1, \dots, X_n)$ and $(Y_1, \dots, Y_n)$ be two independent random vectors, $Y_k : \Omega \to \mathbb{R}$, $X_k : \Omega \to \{0, 1\}$ for $k = 1, \dots, n$, $X_0 : \Omega \to \mathbb{R}$, and random variables $Y_1, \dots, Y_n$ are independent. Then

$$\operatorname{Corr}\Big(\varphi(X_0, X_1, \dots, X_n; X_1 Y_1, \dots, X_n Y_n),\, \psi(Y_1, \dots, Y_n) \Big) \le \sqrt{ \max_{k=1,\dots,n} \mathbb{P} \left(X_k = 1 \right) }$$

for every Borel functions $\varphi : \mathbb{R} \times \{0,1\}^n \times \mathbb{R}^n \to \mathbb{R}$, $\psi : \mathbb{R}^n \to \mathbb{R}$ such that the correlation is well-defined (that is, $0 < \operatorname{Var} \varphi(\dots) < \infty$, $0 < \operatorname{Var} \psi(\dots) < \infty$).

Proof. Consider the orthogonal (in $L_2(\Omega)$) projection Q from the space of all random variables of the form $\psi(Y_1, \dots, Y_n)$ to the space of all random variables of the form $\varphi(X_0, X_1, \dots, X_n; X_1 Y_1, \dots, X_n Y_n)$, that is, $Q\psi(Y_1, \dots, Y_n) = \mathbb{E} \left(\psi(Y_1, \dots, Y_n) \,\middle|\, X_0, X_1, \dots, X_n; X_1 Y_1, \dots, X_n Y_n \right)$. The space of all $\psi(Y_1, \dots, Y_n)$ is spanned by *factorizable* random variables $\psi(Y_1, \dots, Y_n) = \psi_1(Y_1) \dots \psi_n(Y_n)$. For such a ψ we have

$$\begin{aligned} Q\psi(Y_1, \dots, Y_n) = \mathbb{E} \left(\psi_1(Y_1) \dots \psi_n(Y_n) \,\middle|\, X_0, X_1, \dots, X_n; X_1 Y_1, \dots, X_n Y_n \right) = \\ = \Big(\prod_{k : X_k = 0} \mathbb{E}\, \psi_k(Y_k) \Big) \Big(\prod_{k : X_k = 1} \psi_k(Y_k) \Big) \,; \end{aligned}$$

$$\|Q\psi(Y_1,\dots,Y_n)\|^2 = \mathbb{E}\Big(\mathbb{E}\big(\,|Q\psi(Y_1,\dots,Y_n)|^2\,\big|\,X_0,\dots,X_n\,\big)\Big) =$$
$$= \mathbb{E}\left(\Big(\prod_{k:X_k=0}|\mathbb{E}\,\psi_k(Y_k)|^2\Big)\Big(\prod_{k:X_k=1}\mathbb{E}\,|\psi_k(Y_k)|^2\Big)\right).$$

If, in addition, $\mathbb{E}\,\psi_1(Y_1) = 0$ then $\|Q\psi(Y_1,\dots,Y_n)\|^2 \le \mathbb{P}\big(\,X_1 = 1\,\big)\cdot\|\psi(Y_1,\dots,Y_n)\|^2$. Similarly,

$$\|Q\psi(Y_1,\dots,Y_n)\|^2 \le \Big(\max_k \mathbb{P}\big(\,X_k = 1\,\big)\Big)\cdot\|\psi(Y_1,\dots,Y_n)\|^2$$

if $\mathbb{E}\,\psi(Y_1,\dots,Y_n) = 0$ and, of course, ψ is factorizable, that is, $\psi(Y_1,\dots,Y_n) = \psi_1(Y_1)\dots\psi_n(Y_n)$. The latter assumption cannot be eliminated just by saying that factorizable random variables of zero mean span all random variables of zero mean. Instead, we use two facts.

The first fact: the space of all random variables $\psi(\dots)$ has an *orthogonal* basis consisting of factorizable random variables satisfying an additional condition: each factor $\psi_k(Y_k)$ is either of zero mean, or equal to 1. (For a proof, start with an orthogonal basis for functions of Y_1 only, the first basis function being constant; do the same for Y_2; take all products; and so on.)

The second fact: the operator Q maps *orthogonal* factorizable random variables, satisfying the additional condition, into *orthogonal* random variables. Indeed, let $\psi(Y_1,\dots,Y_n) = \psi_1(Y_1)\dots\psi_n(Y_n)$, $\psi'(Y_1,\dots,Y_n) = \psi'_1(Y_1)\dots\psi'_n(Y_n)$, and each $\psi_k(Y_k)$ either is of zero mean, or equals 1; the same for each $\psi'_k(Y_k)$. If $\mathbb{E}\big(\psi(Y_1,\dots,Y_n)\psi'(Y_1,\dots,Y_n)\big) = 0$ then $\mathbb{E}\big(\psi_k(Y_k)\psi'_k(Y_k)\big) = 0$ for at least one k; let it happen for $k = 1$. We have not only $\mathbb{E}\big(\psi_1(Y_1)\psi'_1(Y_1)\big) = 0$ but also $\big(\mathbb{E}\,\psi_1(Y_1)\big)\big(\mathbb{E}\,\psi'_1(Y_1)\big) = 0$, since ψ_1 and ψ'_1 cannot both be equal to 1. Therefore

$$\mathbb{E}\,\big(Q\psi(Y_1,\dots,Y_n)\big)\big(Q\psi'(Y_1,\dots,Y_n)\big) =$$
$$= \mathbb{E}\left(\Big(\prod_{k:X_k=0}\big(\mathbb{E}\,\psi_k(Y_k)\big)\big(\mathbb{E}\,\psi'_k(Y_k)\big)\Big)\Big(\prod_{k:X_k=1}\psi_k(Y_k)\psi'_k(Y_k)\Big)\right) = 0\,,$$

since the first term vanishes whenever $X_1 = 0$, and the second term vanishes whenever $X_1 = 1$.

$\square$

Combining all together, we get the conclusion.

7c2 Theorem. The noise of coalescence is black.

7d Remarks

Another proof of Theorem 7c2 should be possible, by showing that all random variables are sensitive (recall 5b5). To this end, we divide the time axis $\mathbb{R}$ into intervals of small length ε, choose a random subset of intervals such that each interval is chosen with a small probability $1-\rho = 1-e^{-\lambda} \sim \lambda$, independently of others. On each chosen interval we replace local random data with fresh (independent) data.

Consider the path $X(\cdot)$ of the Brownian web, starting at the origin, $X(t) = \xi_{0,t}(0)$ for $t \in [0, \infty)$; it behaves like a Brownian motion. After the replacement we get another path $Y(\cdot)$. Their difference, $\bigl(X(t) - Y(t)\bigr)/\sqrt{2}$, behaves like another Brownian motion when outside 0, but is somewhat sticky at 0. Namely, during each chosen (to the random set) time interval, the point 0 has nothing special; however, outside these time intervals, the point 0 is absorbing. In this sense, chosen time intervals act like factors f_* in the random product of factors f_-, f_+, f_* studied in Sect. 4. There, f_* occurs with a small probability $1/(2\sqrt{i}) \to 0$ (recall 4e4), which produces a non-degenerate stickiness in the scaling limit. Here, in contrast, a time interval is chosen with probability $1 - \rho \sim \lambda$ that does not tend to 0 when the interval length ε tends to 0. Naturally, stickiness disappears in the limit $\varepsilon \to 0$ (a proof uses the idea of (4c9)). That is, interaction between $X(\cdot)$ and $Y(\cdot)$ disappears in the limit. They become independent.

Probably, the same argument works for any finite number of paths $X_k(t) = \xi_{0,t}(x_k)$; they should be asymptotically independent of $Y_k(\cdot)$ for $\varepsilon \to 0$, but I did not prove it.

The spectral measure μ_X of the random variable $X = \xi_{0,1}(0)$ is written down explicitly in [8]. Or rather, its discrete counterpart is found; the scaling limit follows by (a generalization of) Theorem 3c5 (see also [9]). The measure μ_X is a probability measure (since $\|X\| = 1$), it may be thought of as the distribution of a random perfect subset of $(0, 1)$. Note that the random subset is not at all a function on the probability space $(\Omega, \mathcal{F}, P)$ that carries the Brownian web. There is no sense to speak about 'the joint distribution of the random set and the Brownian web'. In fact, they may be treated as incompatible (non-commuting) measurements in the framework of quantum probability, see [7].

A wonder: μ_X is the distribution of $(\theta - M) \cap (0, 1)$, where M is the set of zeros of the usual Brownian motion, and θ is independent of M and distributed uniformly on $(0, 1)$.

Moreover, the corresponding equality holds exactly (not only asymptotically) in the discrete-time model. Strangely enough, the Brownian motion (or rather, random walk) does not appear in the calculation of the spectral measure. The relation to Brownian motion is observed at the end, as a surprise!

7d1 Question. Can μ_X (for $X = \xi_{0,1}(0)$) be found via some natural construction of a Brownian motion whose zeros form the spectral set (after the transformation $x \mapsto \theta - x$)? (See [8, Problem 1.5].)

We see that μ_X (for $X = \xi_{0,1}(0)$) is concentrated on sets of Hausdorff dimension 1/2.

7d2 Question. Is μ_X concentrated on sets of Hausdorff dimension 1/2 for an arbitrary random variable X (over the noise of coalescence)?

A positive answer would probably give us another proof that the noise is black. A stronger conjecture may be made.

7d3 Question. Is μ_X for an arbitrary $\mathcal{F}_{0,1}$-measurable X (over the noise of coalescence) absolutely continuous w.r.t. $\mu_{\xi_{0,1}(0)}$?