# 2 Abstract nonsense of scaling limit

## 2a More on our limiting procedures

The joint compactification $K$ of $\Omega_1 \uplus \Omega_2 \uplus \dots$, used in 1c, is not quite satisfactory. Return to Example 1d3:

$$f_u(\omega) = \frac{1}{\sqrt{n}} \sum_{k<un} \tau_k(\omega) \quad \text{for } u \in [0,1] \cap \mathbb{Q} \tag{2a1}$$

($\mathbb{Q}$ being the set of rational numbers). The limiting model is the Brownian motion, restricted to $[0,1] \cap \mathbb{Q}$. What about an irrational point, $v \in [0,1] \setminus \mathbb{Q}$? The random variable $f_v$ may be defined on $\Omega$ as the limit (say, in $L_2$) of $f_u$ for $u \to v$, $u \in [0,1] \cap \mathbb{Q}$. On the other hand, $f_v$ is naturally defined on $\Omega_1 \uplus \Omega_2 \uplus \dots$ (by the same formula (2a1)). However, $f_v$ is not a continuous function on the compact space $K$.[4] Thus, the weak convergence $P_n \to P$ is relevant to $f_u$ but not $f_v$. Something is wrong!

The wrong thing is the uniform topology used in (1c4)–(1c7). A right topology should take measures $P_n$ into account. We have two ways, 'moderate' and 'radical'.

Here is the 'moderate' way. We choose some appropriate subsets $B_n \subset (\Omega_1 \uplus \Omega_2 \uplus \dots)$, $B_1 \subset B_2 \subset \dots$, such that

$$\inf_i P_i(B_n \cap \Omega_i) \uparrow 1 \quad \text{for } n \to \infty$$

and replace in (1c5)–(1c7) the assumption "$f_n \in C$, $f_n \to f$ uniformly $\Longrightarrow f \in C$" with

$$f_n \in C,\ f_n \to f \text{ uniformly on each } B_n \quad \Longrightarrow \quad f \in C\,. \tag{2a2}$$

**2a3 Example.** Continuing (2a1) we define $B_n$ by

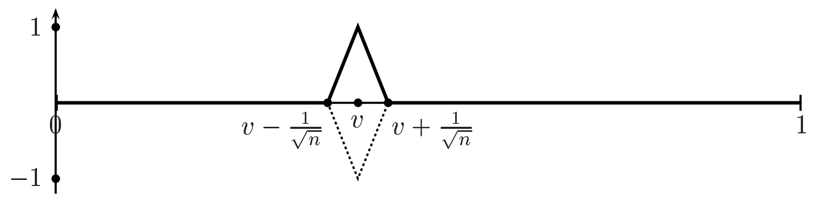
$$B_n \cap \Omega_i = \left\{ \omega \in \Omega_i : \sup_{0 \le k < l \le i} \frac{\left| \frac{1}{\sqrt{i}} \sum_{j=k}^{l} \tau_j(\omega) \right|}{\left( \frac{l-k}{i} \right)^{1/3}} \le n \right\},$$

then[5]

$$|f_u(\omega) - f_v(\omega)| \le n \cdot |u-v|^{1/3} \quad \text{for } \omega \in B_n \cap \Omega_i$$

if $i$ is large enough (namely, $2/i < |u-v|$). The set $C$ (satisfying (2a2)) generated by $f_u$ for all rational $u$, contains also $f_u$ for all irrational $u$.

[4] There exist $\omega_n \in \Omega_n$ such that $\lim_n f_u(\omega_n)$ exists for all $u \in [0,1] \cap \mathbb{Q}$, but $\lim_n f_v(\omega_n)$ does not exist.




[5] Of course, $|u-v|^\alpha$ for any $\alpha \in (0,1/2)$ may be used, not only $|u-v|^{1/3}$.

Similarly to 1c, we may translate (2a2) into the topological language. For each $n$, the restriction of $C$ to $B_n$ corresponds to a joint compactification $(K_n, \alpha_n)$ of $B_n \cap \Omega_i$. Clearly, $K_{n_1} \subset K_{n_2}$ for $n_1 < n_2$, and $\alpha_{n_1} = \alpha_{n_2}|_{K_{n_1}}$. Thus, we get a *joint $\sigma$-compactification*

$$\alpha : (\Omega_1 \uplus \Omega_2 \uplus \dots) \to K_\infty = K_1 \cup K_2 \cup \dots$$

We do not need a topology on the union $K_\infty$ of metrizable compact spaces $K_1 \subset K_2 \subset \dots$ [6] We just define $C(K_\infty)$ as the set of all functions $g : K_\infty \to \mathbb{R}$ such that $g|_{K_n}$ is continuous (on $K_n$) for each $n$. We have

$$C = \alpha^{-1}\big(C(K_\infty)\big) \,,$$

that is, observables $f \in C$ are functions of the form

$$f = g \circ \alpha \,, \quad \text{that is, } f(\omega) = g(\alpha(\omega)), \quad g \in C(K_\infty) \,.$$

If measures $\alpha(P_n)$ weakly converge (w.r.t. bounded functions of $C(K_\infty)$, recall (1c8), (1c9)), we get the limiting model $(\Omega, P)$ by taking $\Omega = K_\infty$ and $P = \lim_{n\to\infty} \alpha(P_n)$.

**2a4 Example.** Continuing 2a3 we see that the limiting measure $P$ exists, and the joint distribution of all $f_u$ (extended to $K_\infty$ by continuity) w.r.t. $P$ is the Wiener measure. The 'uniform' metric on $K_\infty$,

$$\operatorname{dist}(x, y) = \sup_{0 \le u \le 1} |f_u(x) - f_u(y)| \,,$$

is continuous on each $K_n$. Therefore, every function continuous in the 'uniform' metric belongs to $C(K_\infty)$. Our joint $\sigma$-compactification is another form of the usual weak convergence of random walks to the Brownian motion.

That was the 'moderate way'. It requires special subsets $B_n \subset (\Omega_1 \uplus \Omega_2 \uplus \dots)$, in contrast to the 'radical way'; basically, the latter allows the sequence of sets $B_n$ to depend on a sequence of functions $f_n$, see (2a2). In other words, instead of uniform (or 'locally uniform') convergence, we introduce a weaker topology by the metric[7]

$$\operatorname{dist}(f, g) = \sup_n \int \frac{|f(\omega) - g(\omega)|}{1 + |f(\omega) - g(\omega)|} \, dP_n(\omega) \,. \tag{2a5}$$

If $f_n \in C(K)$ and $\operatorname{dist}(f_n, f) \to 0$ then $f_n$ converge in probability w.r.t. $P$; thus, $f$ is naturally defined $P$-almost everywhere.[8]

[6] But if you want, $K_\infty$ may be equipped with the inductive limit topology; that is, $U \subset K_\infty$ is open if and only if for every $n$, $U \cap K_n$ is open (in $K_n$). However, the topology usually is not metrizable.

[7] Alternatively, we may restrict ourselves to bounded functions $\Omega_1 \uplus \Omega_2 \uplus \dots \to [-1, +1]$ (applying a transformation like arctan) and use, say,

$$\operatorname{dist}(f, g) = \sup_n \int |f(\omega) - g(\omega)| \, dP_n(\omega) \,.$$

[8] In fact, every (equivalence class of) $P$-measurable function can be obtained in that way provided that, for each $n$, supports of $P_n$ and $P$ do not intersect. It means that every random variable on the limiting probability space is the scaling limit of some function on $\Omega_1 \uplus \Omega_2 \uplus \dots$ (see also 2c8).

Let $C$ be the closure of $C(K)$ in the metric (2a5), then

$$\int \varphi(f_1,\dots,f_d)\,dP_n \xrightarrow[n\to\infty]{} \int \varphi(f_1,\dots,f_d)\,dP$$

for every $d$, every bounded continuous function $\varphi : \mathbb{R}^d \to \mathbb{R}$, and every $f_1,\dots,f_d \in C$. The joint distribution of $f_1,\dots,f_d$ w.r.t. $P_n$ converges (weakly) to that w.r.t. $P$. So, the weak convergence $P_n \to P$ is relevant for the whole $C$ (not only $C(K)$). That is the idea of the 'radical way', presented systematically in next subsections (2b, 2c).

Returning again to Example 1d3 we see that $f_v$ (for $v \in [0,1]$) is the limit of $f_u$ (for $u \in [0,1] \cap \mathbb{Q}$) in the metric (2a5); thus, $f_v \in C$ for all $v \in [0,1]$.

However, much more can be said. Not only

$$\operatorname{Lim}\Big( \frac{1}{\sqrt{n}} \sum_{an<k<bn} \tau_k(\omega) \Big) = \int_a^b dB(t)\,,$$

where 'Lim' means the scaling limit (as explained above), but also

$$\operatorname{Lim}\Big( n^{-d/2} \sum_{an<k_1<\dots<k_d<bn} \tau_{k_1}(\omega)\dots\tau_{k_d}(\omega) \Big) = \\ = \int\dots\int_{a<t_1<\dots<t_d<b} dB(t_1)\dots dB(t_d) = \frac{1}{n!} H_d\big(B(b)-B(a), b-a\big)$$

where $H_d$ is the Hermite polynomial (see for instance [5, IV.3.8]). Taking finite linear combinations and their closure in the metric (2a5) we get

$$\text{(2a6)}\quad \operatorname{Lim}\Big( \sum_{d=0}^\infty n^{-d/2} \sum_{0<k_1<\dots<k_d<n} \psi_d\big(\tfrac{k_1}{n},\dots,\tfrac{k_d}{n}\big)\tau_{k_1}(\omega)\dots\tau_{k_d}(\omega) \Big) = \\ = \sum_{d=0}^\infty \int\dots\int_{0<t_1<\dots<t_d<1} \psi_d(t_1,\dots,t_d)dB(t_1)\dots dB(t_d)$$

provided that functions $\psi_d$ are Riemann integrable, and vanish for $d$ large enough. The right-hand side is well-defined for all $\psi_d \in L_2$ such that $\sum_d \|\psi_d\|_2^2 < \infty$; the scaling limit may be kept by replacing $\psi_d\big(\frac{k_1}{n},\dots,\frac{k_d}{n}\big)$ with the mean value of $\psi_d$ on the $1/n$-cube centered at $\big(\frac{k_1}{n},\dots,\frac{k_d}{n}\big)$. Now, $(0,1)$ may be replaced with the whole $\mathbb{R}$; $\psi_d$ is defined on $\Delta_d = \{(x_1,\dots,x_d) \in \mathbb{R}^d : x_1 < \dots < x_d\}$. The right-hand side of (2a6) gives us an isometric linear correspondence between $L_2(\Delta_0 \uplus \Delta_1 \uplus \Delta_2 \uplus \dots)$ and $L_2(\Omega,\mathcal{F},P)$, where $(\Omega,\mathcal{F},P)$ is the probability space describing the Brownian motion (on the whole $\mathbb{R}$).

## 2b Coarse probability space: definition and simple example

**2b1 Definition.** A *coarse probability space* $\big((\Omega[i],\mathcal{F}[i],P[i])_{i=1}^\infty, \mathcal{A}\big)$ consists of a sequence of probability spaces $(\Omega[i],\mathcal{F}[i],P[i])$ and a set $\mathcal{A}$ of subsets of the disjoint union $\Omega[\text{all}] = \Omega(1) \uplus \Omega(2) \uplus \dots$, satisfying the following conditions.

(a) $\forall A \in \mathcal{A}\ \forall i\ (A \cap \Omega[i]) \in \mathcal{F}[i]$.

(b) $\forall A, B \in \mathcal{A}\ \big(A \cap B \in \mathcal{A},\ A \cup B \in \mathcal{A},\ \Omega[\text{all}] \setminus A \in \mathcal{A}\big)$.

(c) $\mathcal{A}$ contains every $A \subset \Omega[\text{all}]$ such that $\forall i\ (A \cap \Omega[i]) \in \mathcal{F}[i]$ and $P[i]\big(A \cap \Omega[i]\big) \to 0$ for $i \to \infty$.

(d) $\big(\cup_{k=1}^\infty A_k\big) \in \mathcal{A}$ for every pairwise disjoint $A_1, A_2, \dots \in \mathcal{A}$ such that $\sum_k \sup_i P[i]\big(A_k \cap \Omega[i]\big) < \infty$.

(e) $\lim_i P[i]\big(A \cap \Omega[i]\big)$ exists for every $A \in \mathcal{A}$.

(f) There exists a finite or countable subset $\mathcal{A}_1 \subset \mathcal{A}$ that *generates* $\mathcal{A}$ in the sense that the least subset of $\mathcal{A}$ satisfying (b)–(d) and containing $\mathcal{A}_1$ is the whole $\mathcal{A}$.

A set $\mathcal{A}$ satisfying (a)–(f) will be called a *coarse $\sigma$-field*[9] (on the *coarse sample space* $(\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty$). Each set $A$ belonging to the coarse $\sigma$-field $\mathcal{A}$ will be called *coarsely measurable* (w.r.t. $\mathcal{A}$), or a *coarse event.*

**2b2 Note.** Condition 2b1(c) is equivalent to

(c1) $\forall i\ \mathcal{F}[i] \subset \mathcal{A}$. That is, if a set $A \subset \Omega[\text{all}]$ is contained in some $\Omega[i]$, and is $\mathcal{F}[i]$-measurable, then $A \in \mathcal{A}$.

Also, Condition 2b1(d) is equivalent to each of the following conditions (d1)–(d4). There, we assume that $A \subset \Omega[\text{all}]$, $\forall i\ \big(A \cap \Omega[i]\big) \in \mathcal{F}[i]$, and $\forall k\ A_k \in \mathcal{A}$.

(d1) If $A_k \uparrow A$ (that is, $A_1 \subset A_2 \subset \dots$ and $A = \cup_k A_k$) and $\sup_i P[i]\big((A \setminus A_k) \cap \Omega[i]\big) \to 0$ for $k \to \infty$, then $A \in \mathcal{A}$.

(d2) If $\sup_i P[i]\big((A \triangle A_k) \cap \Omega[i]\big) \to 0$ for $k \to \infty$, then $A \in \mathcal{A}$. (Here $A \triangle A_k = (A \setminus A_k) \cup (A_k \setminus A)$.)

(d3) If $A_k \uparrow A$ and $\limsup_i P[i]\big((A \setminus A_k) \cap \Omega[i]\big) \to 0$ for $k \to \infty$, then $A \in \mathcal{A}$.

(d4) If $\limsup_i P[i]\big((A \triangle A_k) \cap \Omega[i]\big) \to 0$ for $k \to \infty$, then $A \in \mathcal{A}$.

So, we have 10 equivalent combinations: (c)&(d), (c1)&(d), (c)&(d1), (c1)&(d1), (c)&(d2), ..., (c1)&(d4). (I omit the proof.)

However, "$\sup_i$" in (d) cannot be replaced with "$\limsup_i$".

**2b3 Lemma.** Let $\mathcal{A}_1$ be a finite or countable set satisfying 2b1(a,e) and

(b1) $\forall A, B \in \mathcal{A}\ \big(A \cap B \in \mathcal{A}\big)$.

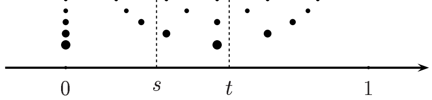
Then the least set $\mathcal{A}$ containing $\mathcal{A}_1$ and satisfying 2b1(b,c,d) is a coarse $\sigma$-field.

The proof is left to the reader.

In such a case we'll say that the coarse $\sigma$-field $\mathcal{A}$ is *generated* by the set $\mathcal{A}_1$.

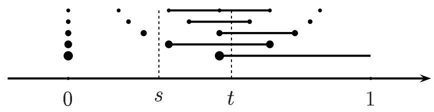
**2b4 Example.** Let $\Omega[i] = \{0, \frac{1}{i}, \dots, \frac{i-1}{i}\}$, and $P[i]$ be the uniform distribution on $\Omega[i]$. Every interval $(s,t) \subset (0,1)$ gives us a set $A_{s,t} \subset \Omega[\text{all}]$,

$$A_{s,t} \cap \Omega[i] = (s,t) \cap \Omega[i]\,.$$




[9] It is not a $\sigma$-field, unless $\mathcal{A}$ contains all sets satisfying 2b1(a).

We take a dense countable set of pairs $(s,t)$ (say, rational $s,t$) and consider the set $\mathcal{A}_1$ of the corresponding $A_{s,t}$. The set $\mathcal{A}_1$ satisfies the conditions of Lemma 2b3, therefore it generates a coarse $\sigma$-field $\mathcal{A}$. In fact, $\mathcal{A}$ consists of all $A = A[1] \uplus A[2] \uplus \dots$ such that sets $A[i] + (0, 1/i) \subset (0,1)$ converge in probability to some $A[\infty] \subset (0,1)$; that is, $\operatorname{mes}\big(A[\infty] \triangle (A[i] + (0,1/i))\big) \to 0$ for $i \to \infty$.




If $A = A_{s,t}$ then, of course, $A[\infty] = (s,t)$.

**2b5 Example.** Continuing Example 1c1, we take $\Omega[i] = \{-1,+1\}^i$ with the uniform distribution $P[i]$. Given $n$ and $a = (a_1,\dots,a_n) \in \{-1,+1\}^n$, we consider $A_a \subset \Omega[\text{all}]$,

$$A_a \cap \Omega[i] = \{(\tau_1,\dots,\tau_i) : \tau_1 = a_1, \dots, \tau_n = a_n\} \quad \text{for } i \ge n\,.$$

Such sets $A_a$ (for all $a$ and $n$) are a countable collection $\mathcal{A}_1$ satisfying the conditions of Lemma 2b3, therefore it generates a coarse $\sigma$-field $\mathcal{A}$. In fact, $\mathcal{A}$ consists of all $A = A[1] \uplus A[2] \uplus \dots$ such that sets $\beta_i^{-1}(A) \subset (0,1)$ converge in probability to some $A[\infty] \subset (0,1)$; here $\beta_i : (0,1) \to \Omega[i]$ is such a measure preserving map:

$$\beta_i(x) = \big((-1)^{c_1},\dots,(-1)^{c_i}\big) \quad \text{when } x - \Big(\frac{c_1}{2} + \dots + \frac{c_i}{2^i}\Big) \in \Big(0, \frac{1}{2^i}\Big)\,,$$

for any $c_1,\dots,c_i \in \{0,1\}$.

You may guess that some limiting procedure produces a ('true', not coarse) probability space out of any given coarse probability space. Indeed, such a procedure (called 'refinement') is described in the next subsection.

## 2c Good use of joint compactification

Having a coarse probability space $\big((\Omega[i],\mathcal{F}[i],P[i])_{i=1}^\infty, \mathcal{A}\big)$ and its refinement $(\Omega,\mathcal{F},P)$ (to be defined later), we may hope that the Hilbert space $L_2[\infty] = L_2(\Omega,\mathcal{F},P)$ is in some sense the limit of Hilbert spaces $L_2[i] = L_2\big(\Omega[i],\mathcal{F}[i],P[i]\big)$. That is indeed the case in the framework of joint compactification, as we'll see. A bad use of the framework, tried in 1c, is a joint compactification of given probability spaces. A good use, considered here, is a joint compactification of metric (Hilbert, …) spaces built over the given probability spaces.

**2c1 Definition.** A *coarse Polish space* is $\big((S[i],\rho[i])_{i=1}^\infty, c\big)$, where each $(S[i],\rho[i])$ is a Polish space (that is, a complete separable metric space[10]), and $c \subset S[1] \times S[2] \times \dots$ is a set of sequences $x = \big(x[1],x[2],\dots\big)$ satisfying the following conditions.

(a) If $x_1,x_2 \in S[1] \times S[2] \times \dots$ are such that $\rho[i]\big(x_1[i],x_2[i]\big) \to 0$ (for $i \to \infty$), then $(x_1 \in c) \iff (x_2 \in c)$.

(b) If $x,x_1,x_2,\dots \in S[1] \times S[2] \times \dots$ are such that $\sup_i \rho[i]\big(x_k[i],x[i]\big) \to 0$ (for $k \to \infty$), then $\big(\forall k \;\; x_k \in c\big) \implies \big(x \in c\big)$.

[10] Many authors define a Polish space as a metrizable topological space admitting a complete separable metric. However, I assume that a metric is given.

(c) $\lim_i \rho[i]\big(x_1[i], x_2[i]\big)$ exists for every $x_1, x_2 \in c$.

(d) There exists a finite or countable subset $c_1 \subset c$ that *generates* $c$ in the sense that the least subset of $c$ satisfying (a), (b) and containing $c_1$ is the whole $c$.

**2c2 Note.** Condition 2c1(d) does not change if 'satisfying (a), (b)' is replaced with 'satisfying (b)'. That is, 2c1(d) is just separability of $c$ in the metric $x_1, x_2 \mapsto \sup_i \rho[i]\big(x_1[i], x_2[i]\big)$.

The refinement of a coarse Polish space $\big((S[i], \rho[i])_{i=1}^\infty, c\big)$ is basically the metric space $\big(c, \tilde\rho\big)$, where

$$\tilde\rho(x_1, x_2) = \lim_i \rho[i]\big(x_1[i], x_2[i]\big) .$$

Though, $\tilde\rho$ is a pseudometric (semimetric), it may vanish for some $x_1 \ne x_2$. The equivalence class, denote it $x[\infty]$, of a sequence $x \in c$ consists of all $x_1 \in c$ such that $\rho[i]\big(x_1[i], x[i]\big) \to 0$. On the set $S[\infty]$ of all equivalence classes we introduce a metric $\rho[\infty]$,

$$\rho[\infty]\big(x_1[\infty], x_2[\infty]\big) = \lim_{i\to\infty} \rho[i]\big(x_1[i], x_2[i]\big) ;$$

thus, $\big(S[\infty], \rho[\infty]\big)$ is a metric space. We write

$$\big(S[\infty], \rho[\infty]\big) = \operatorname{Lim}_{i\to\infty,c}\big(S[i], \rho[i]\big)$$

and call $\big(S[\infty], \rho[\infty]\big)$ the *refinement* of the coarse Polish space $\big((S[i], \rho[i])_{i=1}^\infty, c\big)$. Also, for every $x = (x[1], x[2], \dots) \in c$ we denote its equivalence class $x[\infty] \in S[\infty]$ by

$$x[\infty] = \operatorname{Lim}_{i\to\infty,c} x[i] ,$$

and call it the refinement of $x$.

**2c3 Lemma.** For every coarse Polish space, its refinement is a Polish space.

*Proof.* Separability follows from (d); completeness is to be proven. Let $x_1, x_2, \dots$ be a Cauchy sequence in $(S, \rho)$; we have to find $x \in S$ such that $\rho(x_k, x) \to 0$. We may assume that $\sum_k \rho(x_k, x_{k+1}) < \infty$. Each $x_k$ is an equivalence class; using (a) we choose for each $k = 1, 2, 3, \dots$ a representative $s_k \in S[1] \times S[2] \times \dots$ of $x_k$ such that $\sup_i \rho[i]\big(s_k[i], s_{k+1}[i]\big) \le 2\rho(x_k, x_{k+1})$. Completeness of $\big(S[i], \rho[i]\big)$ ensures existence of $s_\infty[i] = \lim_k s_k[i]$. Condition (b) ensures $s_\infty \in c$. The equivalence class $x \in S$ of $s_\infty$ satisfies $\rho(x_k, x) \le \sup_i \rho[i]\big(s_k[i], s_\infty[i]\big) \to 0$ for $k \to \infty$. $\square$

Let $\big(S[i], \rho[i])_{i=1}^\infty, c\big)$ be a coarse Polish space, and $(S, \rho)$ its refinement. On the disjoint union $\big(S[1] \uplus S[2] \uplus \dots\big) \uplus S$ we introduce a topology, namely, the weakest topology making continuous the following functions $f_s : \big(S[1] \uplus S[2] \uplus \dots\big) \uplus S \to [0, \infty)$ for $s \in c$,

$$\begin{aligned} f_s(x) &= \rho[i]\big(x, s[i]\big) \quad \text{for } x \in S[i] , \\ f_s(x) &= \rho(x, s[\infty]) \quad \text{for } x \in S , \end{aligned}$$

and an additional function $f_0 : \big(S[1] \uplus S[2] \uplus \dots\big) \uplus S \to [0, \infty)$, $f_0(x) = 1/i$ for $x \in S[i]$, $f_0(x) = 0$ for $x \in S$. On every $S[i]$ separately (and also on $S$), the new topology coincides with the old topology, given by $\rho[i]$ (or $\rho$).

We may choose a sequence $(s_k)$ dense in $c$; the topology is generated by functions $f_{s_k}$ (and $f_0$), therefore it is a metrizable topology. Moreover the sequence of functions $\big(\frac{f_{s_k}(\cdot)}{1+f_{s_k}(\cdot)}\big)_{k=1}^\infty$ (and $f_0$) maps the disjoint union into the metrizable compact space $[0,1]^\infty$, and is a homeomorphic embedding. Thus, we have a joint compactification of all $S[i]$ and $S$; and so, we treat them as subsets of a compact metrizable space $K$;

$$S[i] \subset K\,, \quad S \subset K\,.$$

**2c4 Lemma.** Let $s_\infty \in S$, $s_1 \in S[1], s_2 \in S[2], \dots$ Then $s_k \to s_\infty$ in $K$ if and only if $s = (s_1, s_2, \dots) \in c$ and $\operatorname{Lim}_{k\to\infty,c} s_k = s_\infty$.

The proof is left to the reader.

The assumption '$s_\infty \in S$' is essential. Other limiting points (not belonging to $S$) may exist; corresponding sequences converge in $K$ but do not belong to $c$. And, of course, sets $S, S[1], S[2], \dots$ are not closed in $K$, unless they are compact.

**2c5 Lemma.** A set $c_1 \subset c$ generates $c$ if and only if the set of refinements $\{x[\infty] : x \in c_1\}$ is dense in $S[\infty]$.

The proof is left to the reader.

Given continuous functions $f[i] : S[i] \to \mathbb{R}$, $f[\infty] : S[\infty] \to \mathbb{R}$, we write $f[\infty] = \operatorname{Lim}_{i\to\infty,c} f[i]$ if $f[i](x[i]) \to f[\infty](x[\infty])$ whenever $x[\infty] = \operatorname{Lim}_{i\to\infty,c} x[i]$. If functions $f[i]$ are equicontinuous (say, $|f[i](x) - f[i](y)| \le \rho[i](x,y)$ for all $i$ and $x, y \in S[i]$), then it is enough to check that $f[i](x_k[i]) \to f[\infty](x_k[\infty])$ for some sequence $(x_k)_{k=1}^\infty$, $x_k \in c$, such that the sequence $(x_k[\infty])_{k=1}^\infty$ is dense in $S[\infty]$.

Given continuous maps $f[i] : S[i] \to S[i]$, $f[\infty] : S \to S$, we write $f[\infty] = \operatorname{Lim}_{i\to\infty,c} f[i]$ if $\operatorname{Lim}_{i\to\infty,c} f[i](x[i]) = f[\infty](x[\infty])$ whenever $x[\infty] = \operatorname{Lim}_{i\to\infty,c} x[i]$. That is, $\operatorname{Lim}\big(f[i](x[i])\big) = \big(\operatorname{Lim} f[i]\big)\big(\operatorname{Lim} x[i]\big)$. If maps $f[i]$ are equicontinuous then, again, convergence may be checked on $x_k$ such that $x_k[\infty]$ are dense.

Given continuous maps $f[i] : S[\infty] \to S[i]$, we may ask, whether $\operatorname{Lim}_{i\to\infty,c} f[i](x) = x$ for all $x \in S[\infty]$, or not. If maps $f[i]$ are equicontinuous then, still, convergence may be checked for a dense subset of $S[\infty]$.

If every $S[i]$ is not only a metric space but also a Hilbert (or Banach) space, and $c$ is linear (that is, closed under linear operations), then the refinement $S$ is also a Hilbert (or Banach) space, and linear operations are continuous on $\big(S[1] \cup S[2] \cup \dots\big) \cup S \subset K$ in the sense that

$$\operatorname{Lim}_{i\to\infty,c}(as_1[i] + bs_2[i]) = a \operatorname{Lim}_{i\to\infty,c} s_1[i] + b \operatorname{Lim}_{i\to\infty,c} s_2[i]$$

for all $s_1, s_2 \in c$.

Consider the case of Hilbert spaces $S[i] = H[i]$, $S = H$. Given linear[11] operators $R[i] : H[i] \to H[i]$, we may ask about $\operatorname{Lim} R[i]$. If it exists, we get

$$\operatorname{Lim}\big(R[i]x[i]\big) = \big(\operatorname{Lim} R[i]\big)\big(\operatorname{Lim} x[i]\big)\,.$$

If $\sup_i \|R[i]\| < \infty$, then $R[i]$ are equicontinuous, and convergence may be checked on a sequence $x_k$ such that vectors $x_k[\infty]$ span $H$ (that is, their linear combinations are dense in

[11] Continuous, of course.

$H$). For example, one-dimensional orthogonal projections; if $x[\infty] = \operatorname{Lim} x[i]$ then $\operatorname{Proj}_{x[\infty]} = \operatorname{Lim} \operatorname{Proj}_{x[i]}$.

Given linear operators $R[i] : H \to H[i]$, we may ask whether $\operatorname{Lim} R[i](x) = x$ for all $x \in H$, or not. If $\sup_i \|R[i]\| < \infty$ then convergence may be checked on a sequence that spans $H$. Note that such $R[i]$ always exist; moreover, $\|R[i]\| \le 1$ may be ensured. Indeed, we take $x_k$ such that $x_k[\infty]$ are an orthonormal basis of $H$. After some correction, $x_k[i]$ become orthogonal (for each $i$), and $\|x_k(i)\| \le 1$.[12] Now we let $R[i]x_k[\infty] = x_k[i]$.

We return to coarse *probability* spaces.

Let $\big( (\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty, \mathcal{A} \big)$ be a coarse probabilty space. For each $i$ the pseudometric $A, B \mapsto P[i](A \triangle B)$ on $\mathcal{F}[i]$ gives us the metric space $\mathrm{MALG}[i] = \mathrm{MALG}\big(\Omega[i], \mathcal{F}[i], P[i]\big)$ of all equivalence classes of measurable sets. It is not only a metric space but also a Boolean algebra, and moreover, a measure algebra (as defined in [3, 17.44]). Treating every coarse event $A \in \mathcal{A}$ as a sequence of $A[1] \in \mathrm{MALG}[1], A[2] \in \mathrm{MALG}[2], \dots$ we get a coarse Polish space $\big( (\mathrm{MALG}[i])_{i=1}^\infty, \mathcal{A} \big)$. Its refinement is a Polish space $\mathrm{MALG}[\infty]$. The set $\mathcal{A}$ is closed under Boolean operations (union, intersection, complement). Therefore $\mathrm{MALG}[\infty]$ is not only a metric space but also a Boolean algebra. Using Lemma 2c3 it is easy to check that $\mathrm{MALG}[\infty]$ is a measure algebra. Therefore it is (up to isomorphism) of the form

$$\mathrm{MALG}[\infty] = \mathrm{MALG}(\Omega, \mathcal{F}, P)$$

for some probability space $(\Omega, \mathcal{F}, P)$. In the nonatomic case we may take $(\Omega, \mathcal{F}, P) = (0, 1)$ with Lebesgue measure; in general, we may take a shorter interval plus a finite or countable set of atoms. Such a probability space $(\Omega.\mathcal{F}, P)$ (unique up to isomorphism) will be called the refinement of the coarse probability space $\big( (\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty, \mathcal{A} \big)$, and we write

$$(\Omega, \mathcal{F}, P) = \operatorname{Lim}_{i\to\infty,\mathcal{A}} \big( \Omega[i], \mathcal{F}[i], P[i] \big)$$

(in practice, sometimes I omit "$i \to \infty$" or "$\mathcal{A}$" (or both) under the "Lim").

Every sequence $A = (A[1], A[2], \dots) \in \mathcal{A}$ has its refinement

$$\operatorname{Lim}_{i\to\infty,\mathcal{A}} A[i] = A[\infty] \in \mathrm{MALG}(\Omega, \mathcal{F}, P) \,.$$

**2c6 Lemma.** A subset $\mathcal{A}_1$ of a coarse $\sigma$-field $\mathcal{A}$ generates $\mathcal{A}$ is and only if the refinement $\mathcal{F}$ of $\mathcal{A}$ is generated (mod 0) by refinements $A[\infty]$ of all $A \in \mathcal{A}_1$.

*Proof.* We apply Lemma 2c5 to the algebra generated by $\mathcal{A}_1$. □

In order to define $L_2(\mathcal{A})$ as a set of functions on $\Omega[\mathrm{all}]$, we start with indicators $\mathbf{1}_A$ for $A \in \mathcal{A}$, form their linear combinations, and take their completion in the metric

$$\|f\|_{L_2(\mathcal{A})} = \sup_i \|f[i]\|_{L_2[i]} \,,$$

where $L_2[i] = L_2\big(\Omega[i], \mathcal{F}[i], P[i]\big)$; the completion is a Banach (not Hilbert) space $L_2(\mathcal{A})$. Each element $f$ of the completion is evidently identified with a sequence of $f[i] \in L_2[i]$, or a function on $\Omega[\mathrm{all}]$. We have a coarse Polish space $\big( (L_2[i])_{i=1}^\infty, L_2(\mathcal{A}) \big)$. It has its refinement, $L_2[\infty]$.

[12] Of course, $\|x_k[i]\| \to 1$, but in general we cannot ensure $\|x_k[i]\| = 1$. It may happen that $\dim H[i] < \infty$ but $\dim H = \infty$.

**2c7 Lemma.** The refinement $L_2[\infty]$ of $\big((L_2[i])_{i=1}^\infty, L_2(\mathcal{A})\big)$ is (canonically isomorphic to) $L_2(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is the refinement of $\big((\Omega[i], \mathcal{F}[i], P[i])_{i=1}^\infty, \mathcal{A}\big)$.

*Proof.* We define the canonical map $L_2(\mathcal{A}) \to L_2(\Omega, \mathcal{F}, P)$ first on indicators by $\mathbf{1}_A \mapsto \mathbf{1}_{A[\infty]}$, and extend it by linearity and continuity to the whole $L_2(\mathcal{A})$. We note that the image of $f \in L_2(\mathcal{A})$ in $L_2(\Omega, \mathcal{F}, P)$ depends only on the refinement $f[\infty] \in L_2[\infty]$ of $f$, and their norms are equal (both are equal to $\lim_i \|f[i]\|$). We have a linear isometric embedding $L_2[\infty] \to L_2(\Omega, \mathcal{F}, P)$. Its image is closed (since $L_2[\infty]$ is complete by Lemma 2c3), and contains indicators $\mathbf{1}_B$ for all $B \in \mathrm{MALG}(\Omega, \mathcal{F}, P)$; therefore the image is the whole $L_2(\Omega, \mathcal{F}, P)$. $\square$

**2c8 Note.** The same holds for $L_p$ for each $p \in (0, \infty)$, and for the space $L_0$ of all random variables (equipped with the topology of convergence in probability). Elements of $L_0(\mathcal{A})$ will be called coarsely measurable (w.r.t. $\mathcal{A}$) functions (on $\Omega[\mathrm{all}]$), or *coarse random variables;* elements of $L_2(\mathcal{A})$ — square integrable coarse random variables.

Let $f$ be a coarse random variable, then (usual) random variables $f[i] : \Omega[i] \to \mathbb{R}$ converge in distribution (for $i \to \infty$) to the refinement $f[\infty] : \Omega \to \mathbb{R}$. The distribution of $f[\infty]$ will be called the *limiting distribution* of $f$.

It may happen that $f \in L_2(\mathcal{A})$ but $(\operatorname{sgn} f) \notin L_2(\mathcal{A})$. An example: $f(\omega) = \frac{(-1)^i}{i}$ for all $\omega \in \Omega[i]$. Here, the limiting distribution is an atom at 0, and the function 'sgn' is discontinuous at 0.

**2c9 Lemma.** (a) Let $f : \Omega[\mathrm{all}] \to \mathbb{R}$ be a coarse random variable, and $\varphi : \mathbb{R} \to \mathbb{R}$ a continuous function, then $\varphi \circ f : \Omega[\mathrm{all}] \to \mathbb{R}$ is a coarse random variable.

(b) The same as (a) but $\varphi$ may be discontinuous at points of a set $Z \subset \mathbb{R}$ negligible w.r.t. the limiting distribution of $f$.

*Proof.* If $f$ is a linear combination of indicators, then $\varphi \circ f$ is another linear combination of the same indicators. A straightforward approximation gives (a) for uniformly continuous $\varphi$. In general, for every $\varepsilon$ there exists a compact set $K \subset \mathbb{R} \setminus Z$ of probability $\ge 1 - \varepsilon$ w.r.t. the limiting distribution, and also w.r.t. the distribution of $f[i]$ for all $i$ (since all these distributions are a compact set of distributions). The restriction of $f$ to $K$ is uniformly continuous. The limit for $\varepsilon \to 0$ is uniform in $i$. $\square$

For a given Polish space $S$ we may define a coarse $S$-valued random variable as a map $f : \Omega[\mathrm{all}] \to S$ such that (usual) random variables $f[i] : \Omega[i] \to S$ converge in distribution (for $i \to \infty$), and $f^{-1}(B) \in \mathcal{A}$ for every $B \subset S$ such that the boundary of $B$ is negligible w.r.t. the limiting distribution of $f$.

For $S = \mathbb{R}$ the new definition conforms with the old one.

A coarse $\sigma$-field generated by a given sequence of sets (coarse events) was defined after Lemma 2b3. Often it is convenient to generate a coarse $\sigma$-field by a sequence of functions (coarse random variables). A function $f : \Omega[\mathrm{all}] \to \mathbb{R}$ is coarsely $\mathcal{A}$-measurable if and only if $\mathcal{A}$ contains sets $f^{-1}\big((-\infty, x)\big)$ for all $x \in \mathbb{R}$ except for atoms (if any) of the limiting distribution of $f$. A dense countable subset of these $x$ is enough. So, a coarse $\sigma$-field generated by a finite or countable set of functions $f$ is nothing but the coarse $\sigma$-field generated by a countable set of sets of the form $f^{-1}\big((-\infty, x)\big)$. More generally, $S$-valued (coarse) random variables may be used; they are reduced to the real-valued case by composing with appropriate continuous functions $S \to \mathbb{R}$.

**2c10 Lemma.** A sequence of functions $f_k : \Omega[\text{all}] \to \mathbb{R}$ generates a coarse $\sigma$-field if and only if for every $n$, $n$-dimensional random variables $\big(f_1[i], \dots, f_n[i]\big) : \Omega[i] \to \mathbb{R}^n$ converge in distribution (for $i \to \infty$).

**2c11 Note.** The same holds for an arbitrary Polish space instead of $\mathbb{R}$.

**2c12 Note.** Comparing 2c10 and (1c9) we see that every joint compactification of $\Omega_1 \uplus \Omega_2 \uplus \dots$ (in the sense of 1c, assuming (1c8)) may be upgraded (or downgraded?) to a coarse probability space. Namely, we take a sequence of functions $f_k$ that generates $C$ and consider the coarse $\sigma$-field $\mathcal{A}$ generated by $(f_k)$. Every $f \in C$ is a coarse random variable, since $L_0(\mathcal{A})$ is closed under all operations used in (1c5), or (1c6), or (1c7).[13] Therefore $\mathcal{A}$ does not depend on the choice of $(f_k)$.

[13]Of course, $L_0(\mathcal{A})$ usually contains no sequence dense in the *uniform* topology.