1 Basic notions

1a Why be normal?

1a1 Definition. Functions $f, g \in L_2(0,1)$ are *identically distributed,* if they satisfy the following equivalent conditions:

(a) $\int_0^1 \alpha\big(f(\omega)\big)\,\mathrm{d}\omega = \int_0^1 \alpha\big(g(\omega)\big)\,\mathrm{d}\omega$ for all bounded continuous functions $\alpha : \mathbb{R} \to \mathbb{R}$;

(b) $\int_0^1 \exp\big(\mathrm{i}\lambda f(\omega)\big)\,\mathrm{d}\omega = \int_0^1 \exp\big(\mathrm{i}\lambda g(\omega)\big)\,\mathrm{d}\omega$ for all $\lambda \in \mathbb{R}$;

(c) $\operatorname{mes}\{\omega \in (0,1) : f(\omega) \le a\} = \operatorname{mes}\{\omega \in (0,1) : g(\omega) \le a\}$ for all $a \in \mathbb{R}$.

By 'mes' I denote Lebesgue measure.

1a2 Exercise. Functions $f(\omega) = \cos(2\pi\omega)$ and $g(\omega) = \sin(2\pi \cdot 2005\omega)$ are identically distributed. Prove it by checking each of 1a1(a), (b), (c) separately.

Hint: $\cos(x + 2\pi) = \cos x$; $\sin(\frac{\pi}{2} + x) = \cos x$.

1a3 Exercise. Prove that 1a1(a)$\Longleftrightarrow$(b).

Hint. Show that it is sufficient to prove (a) for periodic α. Use the fact that every periodic function is the uniform limit of some trigonometric polynomials.

1a4 Exercise. Prove that 1a1(a)$\Longleftrightarrow$(c).

Hint. Recall the bounded convergence theorem. Approximate continuous functions by step functions, and the other way round.

1a5 Exercise. Consider two functions

$$f(\omega) = \cos 2\pi\omega, \quad g(\omega) = \tfrac{1}{\sqrt{2}} \sin 2\pi\omega + \tfrac{1}{\sqrt{2}} \sin(4\pi\omega)\,.$$

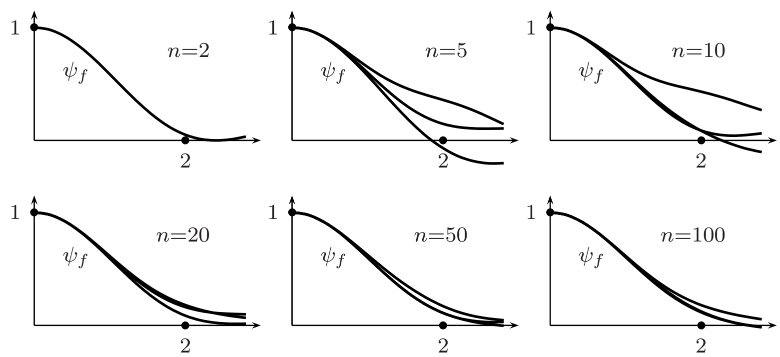
(a) Check that the equality $\int \alpha(f) = \int \alpha(g)$ holds for all quadratic polynomials α, however, f and g are not identically distributed.

(b) Show that the functions $\psi_f(\lambda) = \int \mathrm{e}^{\mathrm{i}\lambda f}$, $\psi_g(\lambda) = \int \mathrm{e}^{\mathrm{i}\lambda g}$ are different, however, $\psi_f(0) = \psi_g(0)$, $\psi'_f(0) = \psi'_g(0)$ and $\psi''_f(0) = \psi''_g(0)$.

We see that different functions $f \in L_2(0,1)$ satisfying $\|f\| = 1$ and $\int f = 0$ lead to different functions $\psi_f(\lambda) = \int \mathrm{e}^{\mathrm{i}\lambda f}$. What about a typical$f$? Let us try

$$\text{(1a6)}\quad f(x) = \frac{1}{\sqrt{n}}\bigl(\pm\cos(2\pi\omega) \pm \sin(2\pi\omega) \pm \dots \pm \cos(2\pi n\omega) \pm \sin(2\pi n\omega)\bigr)\,;$$

we have a finite set of 2^{2n} functions. Here are some examples; the signs are chosen at random.



For large n the typical ψ_f becomes close to $\exp(-\frac12\lambda^2)$. Only the real part of ψ_f is plotted above, but the imaginary part is taken into account in the table of $|\psi_f(\lambda) - \exp(-\frac12\lambda^2)|$ given below; as before, f is chosen at random.

λ	0	0.4	0.8	1.2	1.6	2
$n = 2$	0	0.006	0.04	0.1	0.2	0.3
$n = 100$	0	0.0003	0.003	0.01	0.02	0.03

We feel that the most typical distribution of a function $f \in L_2(0,1)$ satisfying $\|f\| = 1$ and $\int f = 0$ should be as follows.

1a7 Definition. A function $f \in L_2(0,1)$ has the *standard normal distribution* (symbolically, $f \sim \mathrm{N}(0,1)$), if

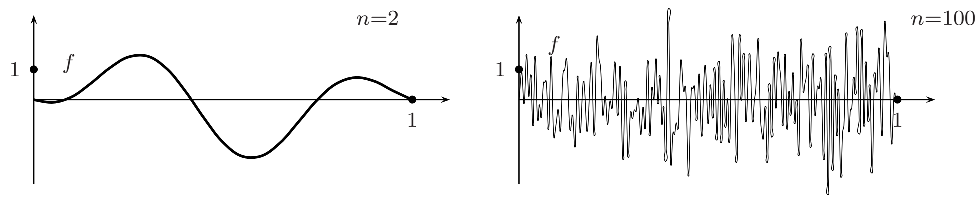
$$\int_0^1 \exp\bigl(\mathrm{i}\lambda f(\omega)\bigr)\,\mathrm{d}\omega = \exp\bigl(-\tfrac12\lambda^2\bigr)$$

for all $\lambda \in \mathbb{R}$.

All such f must be identically distributed (recall 1a1(b)).

1a8 Exercise. If $f \sim \mathrm{N}(0,1)$ then $(-f) \sim \mathrm{N}(0,1)$. Prove it.

We see that the limiting distribution is symmetric, unlike the distribution of a typical f of the form (1a6).



What about existence of $f \sim \mathrm{N}(0,1)$? A naive hope to get such f by taking the limit of (1a6) as $n \to \infty$ is dashed by the following observation.

1a9 Exercise. $\langle f_n, g \rangle \to 0$ as $n \to \infty$ for all $g \in L_2$, if each f_n is of the form (1a6), no matter how the signs are chosen. Prove it.

Hint: first, check it for all trigonometric polynomials g.

A function cannot oscillate infinitely fast. However, why oscillate? The intuition tells us that every function is distributed like some increasing function (just because every array of real numbers may be sorted). Let us try to find an *increasing* function f with the standard normal distribution.

1a10 Exercise. Let $f : (0,1) \to \mathbb{R}$ be a strictly increasing function that satisfies $f(0+) = -\infty$, $f(1-) = +\infty$ and has a derivative f' continuous on $(0,1)$. Then

$$\int_0^1 \exp\bigl(\mathrm{i}\lambda f(\omega)\bigr)\,\mathrm{d}\omega = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\lambda x} g'(x)\,\mathrm{d}x\,,$$

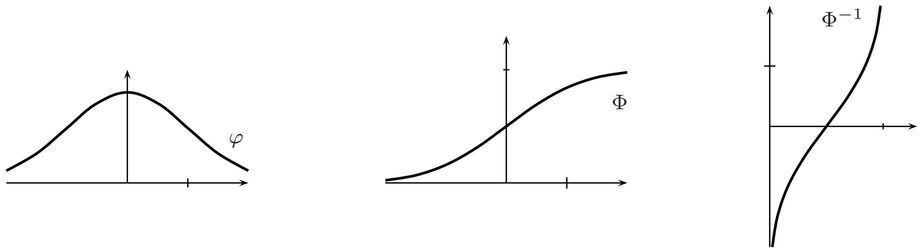
where $g : \mathbb{R} \to (0,1)$ is the inverse function to f. Prove it.

In order to get $f \sim \mathrm{N}(0,1)$, the inverse Fourier transform of g' should be $\exp(-\frac12\lambda^2)$. However, the latter is well-known to be the Fourier transform of itself (up to a constant),

$$\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\lambda x}\mathrm{e}^{-x^2/2}\,\mathrm{d}x = \sqrt{2\pi}\,\mathrm{e}^{-\lambda^2/2} \quad \text{for all } \lambda \in \mathbb{R}\,.$$

Thus, we take

$$g'(x) = \underbrace{\frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}\,\mathrm{d}x\,;}_{\varphi(x)} \qquad g(x) = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-u^2/2}\,\mathrm{d}x\,;}_{\Phi(x)} \qquad f(\omega) = \Phi^{-1}(\omega)$$



and get $f \sim \mathrm{N}(0,1)$.

The function φ is called the standard normal density, and Φ is the standard normal c.d.f. (cumulative distribution function).

If $f \sim \mathrm{N}(0,1)$ then for any $a, b \in \mathbb{R}$ the function $af + b$ (that is, $af + b \cdot \mathbf{1}$) is said to be distributed normally with the mean b and the variance a^2 (or, the standard deviation $|a|$); symbolically, $af + b \sim \mathrm{N}(b, a^2)$.

1b The two-dimensional normal distribution

A pair of functions $f_1, f_2 \in L_2(0,1)$ may be treated as a single vector-function $f \in L_2\big((0,1) \to \mathbb{R}^2\big)$.

1b1 Definition. Vector-functions $f, g \in L_2\big((0,1) \to \mathbb{R}^2\big)$ are *identically distributed,* if they satisfy the following equivalent conditions:

(a) $\int_0^1 \alpha\big(f(\omega)\big)\,\mathrm{d}\omega = \int_0^1 \alpha\big(g(\omega)\big)\,\mathrm{d}\omega$ for all bounded continuous functions $\alpha : \mathbb{R}^2 \to \mathbb{R}$;

(b) $\int_0^1 \exp\big(\mathrm{i}\langle \lambda, f(\omega)\rangle\big)\,\mathrm{d}\omega = \int_0^1 \exp\big(\mathrm{i}\langle \lambda, g(\omega)\rangle\big)\,\mathrm{d}\omega$ for all $\lambda \in \mathbb{R}^2$;

(c) $\operatorname{mes}\{\omega \in (0,1) : f(\omega) \le a\} = \operatorname{mes}\{\omega \in (0,1) : g(\omega) \le a\}$ for all $a \in \mathbb{R}^2$.

Here $\langle \lambda, f(\omega)\rangle = \lambda_1 f_1(\omega) + \lambda_2 f_2(\omega)$, and the inequality $f(\omega) \le a$ is treated coordinate-wise, $f_1(\omega) \le a_1$ and $f_2(\omega) \le a_2$.

Equivalence of 1b1(a),(b),(c) can be proven similarly to 1a3, 1a4.

1b2 Exercise. Vector-functions $f(\omega) = (\cos 2\pi\omega, \cos 4\pi\omega)$ and $g(\omega) = (\sin 2\pi\omega, \sin 4\pi\omega)$ are not identically distributed, even though f_1, g_1 are identically distributed and f_2, g_2 are identically distributed, too. However, $h(\omega) = \big(\cos(2\pi \cdot 2005\omega), \cos(4\pi \cdot 2005\omega)\big)$ is distributed like f. Prove it all.

Let us try vector-functions $f = (f_1, f_2)$, where f_1, f_2 are of the form (1a6) each; there are 2^{4n} such f. Each leads to $\psi_f : \mathbb{R}^2 \to \mathbb{C}$,

$$\psi_f(\lambda) = \int_0^1 \exp\bigl(\mathrm{i}\langle \lambda, f(\omega)\rangle\bigr)\,\mathrm{d}\omega\,.$$

(Do not confuse $\psi_f(\lambda)$ and $\psi_{f_1}(\lambda_1)\psi_{f_2}(\lambda_2)$.) For large n the typical ψ_f becomes close to $\exp\bigl(-\frac12|\lambda|^2\bigr) = \exp\bigl(-\frac12\lambda_1^2 - \frac12\lambda_2^2\bigr)$. Here are typical values of $|\psi_f(\lambda) - \exp\bigl(-\frac12|\lambda|^2\bigr)$.

λ_1	0	0	0.4	0	0.4	0.8	0.8	0.8	1.6	1.6
λ_2	0	0.4	0	0.8	0.4	0	0.4	0.8	0	1.6
$n = 3$	0	0.01	0.003	0.06	0.02	0.03	0.06	0.1	0.2	0.4
$n = 100$	0	0.002	0.0005	0.01	0.02	0.006	0.02	0.05	0.04	0.07

1b3 Definition. A vector-function $f \in L_2\bigl((0,1) \to \mathbb{R}^2\bigr)$ has the 2-*dimensional standard normal distribution*, if

$$\int_0^1 \exp\bigl(\mathrm{i}\langle \lambda, f(\omega)\rangle\bigr)\,\mathrm{d}\omega = \exp\bigl(-\tfrac12|\lambda|^2\bigr)$$

for all $\lambda \in \mathbb{R}^2$.

All such f must be identically distributed.

1b4 Exercise. If (f_1, f_2) has the 2-dimensional standard normal distribution, then

(a) $f_1 \sim \mathrm{N}(0,1)$ and $f_2 \sim \mathrm{N}(0,1)$;

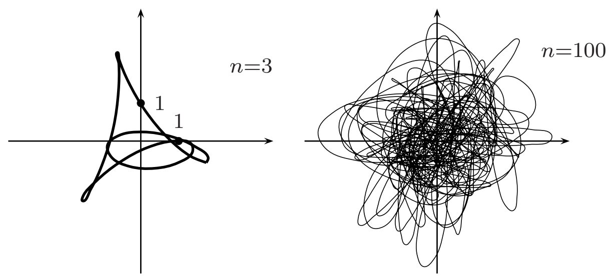
(b) $\frac{1}{\sqrt2}(f_1 + f_2) \sim \mathrm{N}(0,1)$;

(c) moreover, $f_1 \cos\alpha + f_2 \sin\alpha \sim \mathrm{N}(0,1)$ for every $\alpha \in \mathbb{R}$;

(d) for every $\alpha \in \mathbb{R}$ the vector-function $\bigl(f_1\cos\alpha + f_2\sin\alpha,\ -f_1\sin\alpha + f_2\cos\alpha\bigr)$ has the 2-dimensional standard normal distribution.

Prove it.

We see that the limiting distribution is symmetric under rotations, unlike the distribution of a typical f. It is difficult to plot the distribution of f, but we can plot its support, the image, $f\bigl((0,1)\bigr) = \{(f_1(\omega), f_2(\omega)) : \omega \in (0,1)\}$:



What about existence of f satisfying 1b3? The condition $f_1 \sim \mathrm{N}(0,1)$, $f_2 \sim \mathrm{N}(0,1)$ is necessary but not sufficient (just try $f_1 = f_2$).

1b5 Definition. Functions $f, g \in L_2(0,1)$ are *independent,* if they satisfy the following equivalent conditions:

(a) $\int_0^1 \alpha\big(f(\omega)\big)\beta\big(g(\omega)\big)\,\mathrm{d}\omega = \big(\int_0^1 \alpha\big(f(\omega)\big)\,\mathrm{d}\omega\big)\big(\int_0^1 \beta\big(g(\omega)\big)\,\mathrm{d}\omega\big)$ for all bounded continuous functions $\alpha, \beta : \mathbb{R} \to \mathbb{R}$;

(b) $\int_0^1 \exp\big(\mathrm{i}\lambda f(\omega) + \mathrm{i}\mu g(\omega)\big)\,\mathrm{d}\omega = \big(\int_0^1 \exp\big(\mathrm{i}\lambda f(\omega)\big)\,\mathrm{d}\omega\big)\big(\int_0^1 \exp\big(\mathrm{i}\mu g(\omega)\big)\,\mathrm{d}\omega\big)$ for all $\lambda, \mu \in \mathbb{R}$;

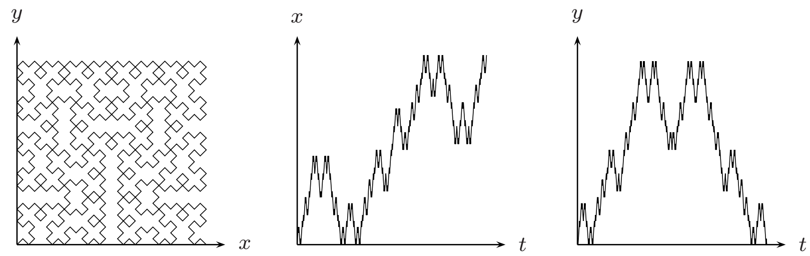
(c) $\operatorname{mes}\{\omega \in (0,1) : f(\omega) \le a,\ g(\omega) \le b\} = \operatorname{mes}\{\omega \in (0,1) : f(\omega) \le a\} \cdot \operatorname{mes}\{\omega \in (0,1) : g(\omega) \le b\}$ for all $a \in \mathbb{R}$.

Equivalence of 1b5(a),(b),(c) can be proven similarly to 1a3, 1a4.

1b6 Exercise. Functions f, g of the form (1a6) cannot be independent. Prove it.

Hint. $0 = \operatorname{mes}\{\omega : a < f(\omega) < b,\ c < g(\omega) < d\} \ne \operatorname{mes}\{\omega : a < f(\omega) < b\} \cdot \operatorname{mes}\{\omega : c < g(\omega) < d\}$ for some a, b, c, d.

In fact, continuously differentiable functions on $(0,1)$ cannot be independent, unless one (or both) of them is constant. However, two non-constant continuous functions can be independent! The famous Peano curve gives an example.



$$x\big(\tfrac{t}{4}\big) = \tfrac{y(t)}{2};\quad x\big(\tfrac{1+t}{4}\big) = \tfrac{x(t)}{2};\quad x\big(\tfrac{2+t}{4}\big) = \tfrac{1+x(t)}{2};\quad x\big(\tfrac{3+t}{4}\big) = \tfrac{2-y(t)}{2};$$
$$y\big(\tfrac{t}{4}\big) = \tfrac{x(t)}{2};\quad y\big(\tfrac{1+t}{4}\big) = \tfrac{1+y(t)}{2};\quad y\big(\tfrac{2+t}{4}\big) = \tfrac{1+y(t)}{2};\quad y\big(\tfrac{3+t}{4}\big) = \tfrac{1-x(t)}{2}$$
for $0 \le t \le 1$.

However, continuity of these functions is of no interest to us; discontinuous independent functions are easier to construct:

$$u\Big(\frac{\beta_1}{2} + \frac{\beta_2}{4} + \frac{\beta_3}{8} + \dots\Big) = \frac{\beta_1}{2} + \frac{\beta_3}{4} + \frac{\beta_5}{8} + \dots\,,$$
$$v\Big(\frac{\beta_1}{2} + \frac{\beta_2}{4} + \frac{\beta_3}{8} + \dots\Big) = \frac{\beta_2}{2} + \frac{\beta_4}{4} + \frac{\beta_6}{8} + \dots\,,$$

where $\beta_1, \beta_2, \dots \in \{0,1\}$ are binary digits. These functions u, v are independent. Each of the two is uniformly distributed on $(0,1)$, that is,

$\operatorname{mes}\{\omega : u(\omega) \le x\} = x$ for $0 \le x \le 1$, and the same for v. Therefore functions $f = \Phi^{-1} \circ u$ and $g = \Phi^{-1} \circ v$ are distributed $\mathrm{N}(0,1)$ each, and independent; thus, the vector-function (f,g) has the 2-dimensional standard normal distribution.

The vector-function constructed above looks rather clumsy, since its two-dimensional values are ascribed to one-dimensional points ω in a tricky way. However, the distribution of its values is very nice; in some sense, it is the most typical distribution of a pair of functions $f,g \in L_2(0,1)$ such that $\|f\| = 1$, $\|g\| = 1$. (The restrictions $\int f = 0$, $\int g = 0$ may be dropped; these properties emerge naturally in the limit, as well as $\int fg = 0$ and many others.)

1c Gaussian spaces

1c1 Definition. A (closed linear) subspace $G \subset L_2(0,1)$ is *Gaussian,* if $g \sim \mathrm{N}(0,1)$ for every $g \in G$ such that $\|g\| = 1$.

1c2 Exercise. There exists a 2-dimensional Gaussian subspace of $L_2(0,1)$. Prove it.

Hint: use 1b4(c).

1c3 Definition. (a) A finite sequence $f_1, \dots, f_n$ of functions $f_k \in L_2(0,1)$ is *orthogaussian,* if

$$\int_0^1 \exp\bigl(\mathrm{i}\lambda_1 f_1(\omega) + \dots + \mathrm{i}\lambda_n f_n(\omega)\bigr)\,\mathrm{d}\omega = \exp\bigl(-\tfrac12\lambda_1^2 - \dots - \tfrac12\lambda_n^2\bigr)$$

for all $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

(b) An infinite sequence $f_1, f_2, \dots$ of functions $f_k \in L_2(0,1)$ is *orthogaussian,* if

$$\int_0^1 \exp\bigl(\mathrm{i}\lambda_1 f_1(\omega) + \dots + \mathrm{i}\lambda_n f_n(\omega)\bigr)\,\mathrm{d}\omega = \exp\bigl(-\tfrac12\lambda_1^2 - \dots - \tfrac12\lambda_n^2\bigr)$$

for all $n = 1, 2, \dots$ and $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

1c4 Exercise. A pair f_1, f_2 is orthogaussian if and only if $f_1 \sim \mathrm{N}(0,1)$, $f_2 \sim \mathrm{N}(0,1)$ and f_1, f_2 are independent. Prove it.

1c5 Exercise. If a pair f_1, f_2 is orthogaussian then it is orthonormal, that is,

$$\|f_1\| = 1, \quad \|f_2\| = 1, \quad \langle f_1, f_2 \rangle = 0\,.$$

Prove it.

Similarly to 1b5 we define independence of $f_1, \dots, f_n$. Similarly to 1c4, a finite sequence $f_1, \dots, f_n$ is orthogaussian if and only if $f_1 \sim \mathrm{N}(0,1), \dots, f_n \sim \mathrm{N}(0,1)$ and $f_1, \dots, f_n$ are independent.

Pairwise independence does not suffice! A hint toward a counterexample: take f_1, f_2, f_3 orthogaussian and restrict them to the set $\{\omega : f_1(\omega)f_2(\omega)f_3(\omega) > 0\}$.

1c6 Exercise. The following three conditions are equivalent for every n-dimensional subspace $G \subset L_2(0,1)$:

(a) some orthonormal basis $f_1, \dots, f_n$ of G is orthogaussian;

(b) every orthonormal basis $f_1, \dots, f_n$ of G is orthogaussian;

(c) G is Gaussian.

Prove it.
Hint: (a)$\Longrightarrow$(c)$\Longrightarrow$(b).

For every n we may decompose the countable set of binary digits into n countable subsets, construct n independent functions distributed uniformly on $(0,1)$ each, and transform them into an orthogaussian sequence (via Φ^{-1}). Thus, an n-dimensional Gaussian subspace of $L_2(0,1)$ exists for every n.

1c7 Exercise. The set $\{f \in L_2(0,1) : f \sim \mathrm{N}(0,1)\}$ is a closed subset of $L_2(0,1)$.

Prove it.

Hint: $|\mathrm{e}^{\mathrm{i}\lambda a} - \mathrm{e}^{\mathrm{i}\lambda b}| \le |\lambda||a-b|$.

1c8 Exercise. Generalize 1c6 to infinite sequences.

Hint: use 1c7 for proving (a)$\Longrightarrow$(c).

Using the fact that countable union of countable sets is countable, we get the following result.

1c9 Proposition. There exists a infinite-dimensional Gaussian subspace of $L_2(0,1)$.

A Gaussian subspace is a paradise! Here,

- functions of equal norms are always identically distributed;
- orthogonal functions are always independent;
- independence is equivalent to pairwise independence.

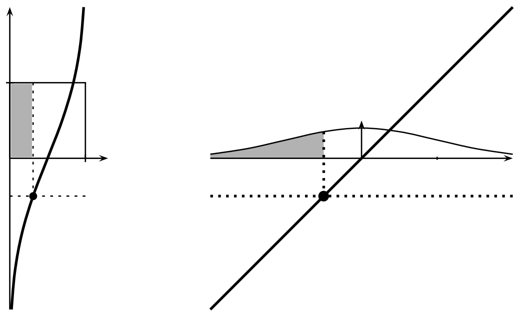
These are the simplest manifestations of an amusing harmony between geometry and probability, inherent to Gaussian measures and processes.

1d The standard n-dimensional Gaussian measure

If $f \sim \mathrm{N}(0,1)$ then

$$\int_0^1 \exp\bigl(\mathrm{i}\lambda f(\omega)\bigr)\,\mathrm{d}\omega = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\lambda x}\varphi(x)\,\mathrm{d}x \quad \text{for all } \lambda \in \mathbb{R}\,, \qquad \text{where } \varphi(x) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\,\mathrm{d}x \tag{1d1}$$

(recall 1a10 and the argument after it). We may say that two functions are identically distributed, one function being $f : (0,1) \to \mathbb{R}$, the other function being just $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x$ (the identical map); however, the domain $\mathbb{R}$ of g is equipped with the measure $\varphi(x)\,\mathrm{d}x$ rather than Lebesgue measure $\mathrm{d}x$.



In this sense, φ is the density of (the distribution of) f.

More formally, we may consider the Hilbert space $L_2(\mathbb{R}, \varphi)$ of all (equivalence classes of) measurable functions $g : \mathbb{R} \to \mathbb{R}$ with

$$\|g\|^2 = \int_{-\infty}^{+\infty} g^2(x)\varphi(x)\,\mathrm{d}x < \infty\,,$$

and generalize 1a1 accordingly.

1d2 Definition. Functions $f \in L_2(0,1)$, $g \in L_2(\mathbb{R}, \varphi)$ are *identically distributed,* if they satisfy the following equivalent conditions:

(a) $\int_0^1 \alpha\bigl(f(\omega)\bigr)\,\mathrm{d}\omega = \int_{-\infty}^{+\infty} \alpha\bigl(g(x)\bigr)\varphi(x)\,\mathrm{d}x$ for all bounded continuous functions $\alpha : \mathbb{R} \to \mathbb{R}$;

(b) $\int_0^1 \exp\bigl(\mathrm{i}\lambda f(\omega)\bigr)\,\mathrm{d}\omega = \int_{-\infty}^{+\infty} \exp\bigl(\mathrm{i}\lambda g(x)\bigr)\varphi(x)\,\mathrm{d}x$ for all $\lambda \in \mathbb{R}$;

(c) $\operatorname{mes}\{\omega \in (0,1) : f(\omega) \le a\} = \gamma^1\{x \in \mathbb{R} : g(x) \le a\}$ for all $a \in \mathbb{R}$.

Here

$$\gamma^1(A) = \int_A \varphi(x)\,\mathrm{d}x \quad \text{for any measurable } A \subset \mathbb{R}\,. \tag{1d3}$$

Equivalence of 1d2(a),(b),(c) can be proven similarly to 1a3, 1a4. We may write $L_2(\mathbb{R}, \varphi)$ or $L_2(\mathbb{R}, \gamma^1)$, it is the same.

1d4 Exercise. (a) Similarly to 1a7 define the notion '$g \sim \mathrm{N}(0,1)$' for $g \in L_2(\mathbb{R}, \varphi)$.

(b) Check that $g \sim \mathrm{N}(0,1)$ for the identical function $g(x) = x$ treated as an element of $L_2(\mathbb{R}, \varphi)$

1d5 Exercise. (a) Whether $L_2(\mathbb{R}, \varphi)$ contains two independent f, g distributed $\mathrm{N}(0,1)$ each, or not?

(b) The same but $g(x) = x$ for all x.

Hints. (a): pass from $L_2(\mathbb{R}, \varphi)$ to $L_2(0,1)$ using $\Phi : \mathbb{R} \to (0,1)$. (b): f should be orthogonal to all bounded continuous functions.

1d6 Exercise. If $f \in L_2\big((0,1) \to \mathbb{R}^2\big)$ has the 2-dimensional standard normal distribution, then

$$\int_0^1 \exp\big(\mathrm{i}\langle \lambda, f(\omega) \rangle\big) \, \mathrm{d}\omega = \int_{\mathbb{R}^2} \exp\big(\mathrm{i}\langle \lambda, x \rangle\big) \varphi_2(x) \, \mathrm{d}x \, ,$$

$$\text{where } \varphi_2(x) = \frac{1}{2\pi} \exp\big(-\tfrac12 |x|^2\big) \quad \text{for } x \in \mathbb{R}^2 \, .$$

Prove it.

Hint: factor the integral.

The corresponding measure γ^2 is rotation-invariant (since its density φ_2 is).

1d7 Exercise. Define a Gaussian subspace of $L_2(\mathbb{R}^2, \varphi_2)$ and prove that the space of linear functions $\mathbb{R}^2 \to \mathbb{R}$ is a 2-dimensional Gaussian subspace of $L_2(\mathbb{R}^2, \varphi_2)$.

For every $n = 1, 2, \dots$ we have a probability measure γ^n on $\mathbb{R}^n$,

$$\gamma^n(\mathrm{d}x) = \underbrace{(2\pi)^{-n/2} \exp\big(-\tfrac12 |x|^2\big)}_{\varphi_n(x)} \, \mathrm{d}x \, ,$$

$$\gamma^n(\mathrm{d}x_1 \dots \mathrm{d}x_n) = \gamma^1(\mathrm{d}x_1) \dots \gamma^1(\mathrm{d}x_n) \, .$$

Linear functions $\mathbb{R}^n \to \mathbb{R}$ are an n-dimensional Gaussian subspace of $L_2(\mathbb{R}^n, \gamma^n)$. Coordinate functions $f_k(x_1, \dots, x_n) = x_k$ are an orthogonal, therefore orthogaussian, basis of this subspace. The measure γ^n is rotation-invariant (since its density φ_n is).

1e Gaussian random vectors

Various definitions are in use of Gaussian random vectors in Hilbert, Banach, linear topological, linear measurable and other spaces. As far as I understand, all 'good' definitions are special cases of the following one.

1e1 Definition. Let E be a linear space. A function $\xi : (0,1) \to E$ is a *Gaussian random vector,*[1] if

(a) all measurable functions of the form $f \circ \xi$, where $f : E \to \mathbb{R}$ is linear, are contained in a Gaussian subspace of $L_2(0,1)$;[2]

(b) there exist linear functions $f_1, f_2, \dots : E \to \mathbb{R}$ such that $f_k \circ \xi$ are measurable, and f_k separate points of E, that is, $\forall e \in E \setminus \{0\} \;\; \exists k \;\; f_k(e) \ne 0$.

The dimension (finite, or ∞) of the least possible Gaussian subspace used in (a) is called the dimension of the random vector.

1e2 Exercise. The general form of a one-dimensional Gaussian random vector is

$$\xi(\omega) = \zeta(\omega) e$$

for $\zeta \in L_2(0,1)$, $\zeta \sim \mathrm{N}(0,1)$ and $e \in E$, $e \ne 0$.

Prove it.

Hint: $f_k(\xi(\omega)) = c_k \zeta(\omega)$ for some c_k, except for a negligible set of ω. Therefore $\frac{1}{\zeta(\omega)} \xi(\omega)$ does not depend on ω.

1e3 Exercise. The general form of a n-dimensional Gaussian random vector is

$$\xi(\omega) = \zeta_1(\omega) e_1 + \dots + \zeta_n(\omega) e_n$$

for orthogaussian sequence $\zeta_1, \dots, \zeta_n$ and linearly independent vectors $e_1, \dots, e_n \in E$.

Prove it.

Hint: $f_k(\xi(\omega)) = c_{k,1}\zeta_1(\omega) + \dots + c_{k,n}\zeta_n(\omega)$; therefore any $\xi(\omega_1), \dots, \xi(\omega_{n+1})$ must be linearly dependent.

1f Gaussian measures

1f1 Definition. (a) The *distribution* of a Gaussian random vector $\xi : (0,1) \to E$ is the measure γ defined by

$$\gamma(A) = \operatorname{mes}\{\omega : \xi(\omega) \in A\} = \operatorname{mes} \xi^{-1}(A)$$

for all $A \in E$ such that $\xi^{-1}(A)$ is a measurable subset of $(0,1)$. These A are called *γ-measurable.*

(b) Two Gaussian random vectors $(0,1) \to E$ are *identically distributed,* if their distributions are equal.[3]

[1] Or rather, a *centered* Gaussian random vector. However, I assume them all to be centered, unless otherwise stated.

[2] That is, the equivalence class of the function $f \circ \xi$ belongs to the Gaussian subspace.

[3] In particular, defined on the same class of sets.

(c) Distributions of (all possible) Gaussian random vectors $\xi : (0,1) \to E$ are called *Gaussian measures*[1] on E.

Clearly, all γ-measurable sets are a σ-field, and every subset of a measurable negligible set is a measurable negligible set (that is, of γ measure 0).

1f2 Exercise. Let $\xi_1, \xi_2 : (0,1) \to E$ be identically distributed Gaussian random vectors. Then ξ_1, ξ_2 are of the same dimension (finite or infinite).

Prove it.

Hint: $\|f \circ \xi_1\| = \|f \circ \xi_2\|$.

Now we may define the dimension (finite or infinite) of a Gaussian measure as the dimension of any corresponding Gaussian random vector.

Especially, γ^n is an n-dimensional Gaussian measure on $\mathbb{R}^n$. The proof is straightforward, except for one point explained below. We take $\xi : (0,1) \to \mathbb{R}^n$ as in 1e3 ($e_1, \dots, e_n$ being the standard basis). Every Lebesgue measurable set $A \subset \mathbb{R}^n$ is γ-measurable (think, why), but we need also the converse: every γ-measurable set $A \subset \mathbb{R}^n$ is Lebesgue measurable. This is a special case of a general theorem well-known in the theory of *standard* measure spaces. Sketch of a proof: for every $\varepsilon > 0$ there exists a *compact* set $C \subset \xi^{-1}(A)$ such that $\operatorname{mes} C \ge \operatorname{mes} \xi^{-1}(A) - \varepsilon$ and the restriction $\xi|_C$ is continuous (Lusin's theorem). Then the set $\xi(C) \subset A$ is compact, therefore, Lebesgue measurable. It remains to take $\varepsilon_n \to 0$ and get $\xi(C_1) \cup \xi(C_2) \cup \dots = A$ up to a γ-negligible set.

In other words, a finite orthogaussian sequence leads to an *isomorphism* between probability spaces $\big((0,1), \operatorname{mes}\big)$ and $(\mathbb{R}^n, \gamma^n)$.

1f3 Exercise. An n-dimensional Gaussian measure on a linear space E may be defined equivalently as the image of γ^n under a linear embedding $\mathbb{R}^n \to E$.

Prove it.

In particular, an n-dimensional Gaussian measure γ on $\mathbb{R}^n$ is the image of γ^n under an invertible linear transformation $\mathbb{R}^n \to \mathbb{R}^n$; such γ has a density p_γ (thus, $\gamma(A) = \int_A p_\gamma(x)\,\mathrm{d}x$), and $p_\gamma(x) = \mathrm{const} \cdot \exp\big(-Q(x)\big)$ for a strictly positive quadratic form Q on $\mathbb{R}^n$.

For $m < n$, an m-dimensional Gaussian measure on $\mathbb{R}^n$ is concentrated on an m-dimensional linear subspace of $\mathbb{R}^n$.

[1] Or rather, *centered* Gaussian measures. However, I assume them all to be centered, unless otherwise stated.

1f4 Exercise. A measure γ on $\mathbb{R}^n$ is Gaussian if and only if the function

$$\psi_\gamma(\lambda) = \int_{\mathbb{R}^n} \exp\bigl(\mathrm{i}\langle\lambda, x\rangle\bigr)\,\gamma(\mathrm{d}x)$$

satisfies the condition: $(-\ln\psi_\gamma)$ is a positive (maybe, not strictly positive) quadratic form on $\mathbb{R}^n$.

Prove it.

Hint: every positive quadratic form on $\mathbb{R}^n$ can be written as $x \mapsto \langle x, y_1\rangle^2 + \dots + \langle x, y_m\rangle^2$ for some linearly independent $y_1, \dots, y_m \in \mathbb{R}^n$.

1f5 Exercise. Describe explicitly all 1-dimensional Gaussian measures γ on $\mathbb{R}^2$ and their quadratic forms $(-\ln\psi_\gamma)$.

1g Gaussian processes

1g1 Definition. (a) A *Gaussian* (random) *process*[1] on a set T is a map $\Xi : T \to L_2(0,1)$ such that all $\Xi(t)$ for $t \in T$ are contained in a Gaussian subspace of $L_2(0,1)$;

(b) two Gaussian processes $\Xi_1, \Xi_2 : T \to L_2(0,1)$ are *identically distributed,* if for every n and $t_1, \dots, t_n \in T$, two vector-functions $\bigl(\Xi_k(t_1), \dots, \Xi_k(t_n)\bigr) : (0,1) \to \mathbb{R}^n$ (for $k = 1, 2$) are identically distributed.

A Gaussian process Ξ on a single-point set $T = \{t_1\}$ is just a function (or rather, equivalence class) $\Xi(t_1) \in L_2(0,1)$, $\Xi(t_1) \sim \mathrm{N}(0,1)$. A Gaussian process Ξ on a finite set $T = \{t_1, \dots, t_n\}$ is the same as a Gaussian random vector $\bigl(\Xi(t_1), \dots, \Xi(t_n)\bigr) : (0,1) \to \mathbb{R}^T$. The same holds for a countable T. (However, for uncountable T the situation is more complicated.)

The *covariance function* of a Gaussian process $\Xi : T \to L_2(0,1)$ is the function $T \times T \to \mathbb{R}$,

$$(s,t) \mapsto \operatorname{Cov}\bigl(\Xi(s), \Xi(t)\bigr) = \langle \Xi(s), \Xi(t)\rangle = \int_0^1 \Xi(s)(\omega) \cdot \Xi(t)(\omega)\,\mathrm{d}\omega\,.$$

1g2 Exercise. Two Gaussian processes on T are identically distributed if and only if their covariance functions are equal.

Prove it.

In other words, the distribution of a Gaussian process is uniquely determined by its covariance function. Especially, if $\|\Xi(t)\| = 1$ for all $t \in T$ (that

[1] Or rather, *centered* Gaussian process. However, I assume them all to be centered, unless otherwise stated.

is, Ξ maps T into the unit sphere of $L_2(0,1)$) then the distribution of Ξ is uniquely determined by the corresponding metric on T,

$$(s,t) \mapsto \|\Xi(s) - \Xi(t)\| .$$

In general, it is determined by the metric together with the function $t \mapsto \|\Xi(t)\|$.

1g3 Exercise. Whether there exists a Gaussian process Ξ on the four-element set $\{1,2,3,4\}$ such that

$$\|\Xi(1) - \Xi(2)\| = 1, \; \|\Xi(2) - \Xi(3)\| = 1, \; \|\Xi(3) - \Xi(4)\| = 1, \; \|\Xi(4) - \Xi(1)\| = 1,$$
$$\|\Xi(1) - \Xi(3)\| = 2, \quad \|\Xi(2) - \Xi(4)\| = 2,$$

or not?

Hint: think about $\langle \Xi(1) - \Xi(2), \Xi(2) - \Xi(3) \rangle$.

Stationary examples

As the first example let us consider the Gaussian random vector $\zeta_1 \cos t + \zeta_2 \sin t$ in the space of 2π-periodic continuous functions $\mathbb{R} \to \mathbb{R}$, and the corresponding Gaussian process. More exactly, the former is $\omega \mapsto (t \mapsto \zeta_1(\omega) \cos t + \zeta_2(\omega) \sin t)$, while the latter is $t \mapsto (\omega \mapsto \dots)$.

1g4 Definition. A Gaussian process $\Xi : \mathbb{R} \to L_2(0,1)$ is *stationary* (in other words, homogeneous, or shift-invariant) if for every $s \in \mathbb{R}$ the two processes $t \mapsto \Xi(t)$ and $t \mapsto \Xi(s+t)$ are identically distributed.

1g5 Exercise. The process $\zeta_1 \cos t + \zeta_2 \sin t$ is stationary.

Prove it.

According to 1g2, a Gaussian process is stationary if and only if its covariance function depends only on the time interval,

$$\operatorname{Cov}\big(\Xi(s), \Xi(t)\big) = \operatorname{Cov}\big(\Xi(0), \Xi(t-s)\big) .$$

More generally, the process $\sum_{k=1}^n c_k(\zeta_{2k-1} \cos kt + \zeta_{2k} \sin kt)$ is stationary for any n and $c_1, \dots, c_n$. Still more generally, we may try

$$\Xi(t) = \sum_{k=1}^\infty c_k(\zeta_{2k-1} \cos kt + \zeta_{2k} \sin kt) \tag{1g6}$$

for some $c_1, c_2, \dots \in \mathbb{R}$.

1g7 Exercise. (a) If $\sum c_k^2 < \infty$ then the series (1g6) converges in $L_2(0,1)$ for every $t \in \mathbb{R}$.

(b) If $\sum c_k^2 = \infty$ then the series (1g6) diverges in $L_2(0,1)$ for every $t \in \mathbb{R}$. Prove it.
Hint: $\zeta_1, \zeta_2, \dots$ are orthonormal.

We see that (1g6) defines a Gaussian process on $\mathbb{R}$, provided that $\sum c_k^2 < \infty$.

1g8 Exercise. If $\sum c_k^2 < \infty$ then Ξ is a continuous map $\mathbb{R} \to L_2(0,1)$.

Prove it.

Hint: $\sum_1^\infty \|f_k - g_k\|^2 \le \sum_1^n \|f_k - g_k\|^2 + \Big(\sqrt{\sum_{n+1}^\infty \|f_k\|^2} + \sqrt{\sum_{n+1}^\infty \|g_k\|^2} \Big)^2$.

We see that (1g6) defines a continuous curve in a Gaussian space, provided that $\sum c_k^2 < \infty$. What about the corresponding Gaussian random vector,

$$\xi(\omega) = \Big(t \mapsto \sum_{k=1}^\infty c_k \big(\zeta_{2k-1}(\omega) \cos kt + \zeta_{2k}(\omega) \sin kt \big) , \tag{1g9}$$

when is it well-defined?

1g10 Exercise. If $\sum c_k^2 < \infty$ then the series (1g9) converges in $L_2(0,2\pi)$ for almost every $\omega \in (0,1)$.

Prove it.

Hint: $\sum c_k^2 \big(\zeta_{2k-1}^2(\omega) + \zeta_{2k}^2(\omega) \big) < \infty$ for almost all ω, since $\sum c_k^2 \int_0^1 \big(\zeta_{2k-1}^2(\omega) + \zeta_{2k}^2(\omega) \big) \, \mathrm{d}\omega < \infty$.

We see that (1g9) defines a Gaussian random vector in $L_2(0,2\pi)$, provided that $\sum c_k^2 < \infty$.

1g11 Exercise. If $\sum |c_k| < \infty$ then the series (1g9) converges in the space of 2π-periodic continuous functions $\mathbb{R} \to \mathbb{R}$ (that is, uniformly in t).

Prove it.

Hint: $\sum |c_k| \big(|\zeta_{2k-1}| + |\zeta_{2k}| \big) < \infty$ almost everywhere on $(0,1)$, since $\sum |c_k| \int_0^1 \big(|\zeta_{2k-1}(\omega)| + |\zeta_{2k}(\omega)| \big) \, \mathrm{d}\omega < \infty$.

We see that (1g9) defines a Gaussian random vector in $C[0,2\pi]$, provided that $\sum |c_k| < \infty$. It does not follow from 1g8! Do not think that $\sum c_k^2 < \infty$ is enough!

What happens if $\sum |c_k| = \infty$ but $\sum c_k^2 < \infty$ (for instance, $c_k = 1/k$)? In fact, the condition $c_k^2 = O(1/k^{1+\varepsilon})$ is sufficient (and not necessary) for convergence of (1g9) in $C[0,2\pi]$, which is far from being evident. However, the condition $\sum c_k^2 < \infty$ is not sufficient for uniform, and even pointwise convergence of (1g9) (which also is far from being evident); for some coefficients we have a random equivalence class of functions, but not a random function.

Examples with independent increments

In discrete time we may produce a Gaussian random walk

$$\Xi(n) = \zeta_1 + \dots + \zeta_n$$

from an orthogaussian sequence $\zeta_1, \zeta_2, \dots$ We get a Gaussian random process on $T = \{0, 1, 2, \dots\}$, or equivalently, a Gaussian random vector in the linear space $\mathbb{R}^\infty = \mathbb{R}^{\{0,1,2,\dots\}}$ of all (infinite) sequences. The increments $\Xi(n) - \Xi(n-1)$ are independent. But what about continuous time?

1g12 Definition. A Gaussian random process Ξ on $[0, \infty)$ such that $\Xi(0) = 0$ has *independent increments,* if

$$\Xi(t_2) - \Xi(t_1), \dots, \Xi(t_n) - \Xi(t_{n-1}) \quad \text{are independent}$$

whenever $0 \le t_1 < \dots < t_n < \infty$.

Here is a boring example: $\Xi(t) = \zeta_1 + \dots + \zeta_n$ for $n \le t < n+1$. How to construct an interesting example?

1g13 Exercise. A Gaussian process Ξ on $[0, \infty)$ such that $\Xi(0) = 0$ has independent increments if and only if

$$\operatorname{Cov}\big(\Xi(s) - \Xi(r),\, \Xi(u) - \Xi(t)\big) = 0$$

whenever $0 \le r < s \le t < u < \infty$.

Prove it.

1g14 Exercise. The distribution of a Gaussian process Ξ on $[0, \infty)$ with independent increments (such that $\Xi(0) = 0$) is uniquely determined by the increasing function $t \mapsto \|\Xi(t)\|^2$.

Prove it.

The increasing function is a step function for the boring example mentioned above. How to make it strictly increasing? According to 1g13, we need a curve in a Gaussian space G, that satisfies a geometric property (namely, has orthogonal increments). Being geometric, the property is unrelated to G. Having such a curve in some Hilbert space, we may transfer it to G using the fact that all Hilbert spaces (of the same dimension) are geometrically the same.

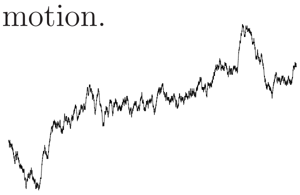
It is easy to find the needed curve in $L_2(0, \infty)$; just indicators,

$$t \mapsto \mathbf{1}_{(0,t)} \,.$$

Now, any linear isometric embedding $\tilde{\Xi} : L_2(0,\infty) \to G$ (called *isonormal process* on $L_2(0,\infty)$) gives us

$$\Xi(t) = \tilde{\Xi}\big(\mathbf{1}_{(0,t)}\big) ,$$

a Gaussian process with independent increments, satisfying $\|\Xi(t)\| = \sqrt{t}$. Such a process is called *Brownian motion*[1] (or the Wiener process) and denoted by B (or W) rather than Ξ. It is unique in distribution. It corresponds to a Gaussian random continuous function $[0,\infty) \to \mathbb{R}$ (which is not evident), also called Brownian motion.



1g15 Exercise. (a) *'scaling'* For any $c \in (0,\infty)$, the Gaussian process

$$t \mapsto \frac{1}{c} B(c^2 t)$$

is also a Brownian motion.

(b) *'time reversal'* For any $T \in (0,\infty)$, the following two Gaussian processes on $[0,T]$ are identically distributed:

$$t \mapsto B(t) \quad \text{and} \quad t \mapsto B(T) - B(t) .$$

Prove it.

1g16 Exercise. *'Ornstein-Uhlenbeck process'* Prove that the Gaussian process

$$t \mapsto \mathrm{e}^{-t} B(\mathrm{e}^{2t})$$

on $\mathbb{R}$ is stationary, and its covariance function is $(s,t) \mapsto \mathrm{e}^{-|s-t|}$.

More examples: large T

On the set $T = \{-1,+1\}^n$ consider the Gaussian process

$$\Xi(\sigma_1, \dots, \sigma_n) = -\frac{1}{\sqrt{n}} \sum_{k<l} \zeta_{k,l} \sigma_k \sigma_l ,$$

where $(\zeta_{k,l})_{k<l}$ is a family of $n(n-1)/2$ orthogaussian functions. They model disorder in a system of n spins $\sigma_1, \dots, \sigma_n = \pm 1$; namely, $\Xi(\sigma_1, \dots, \sigma_n)$ is

[1] By probabilists, not by physicists.

the energy of the spin configuration $\sigma_1, \dots, \sigma_n$. That is the Sherrington-Kirkpatrick model for spin glasses, well-known in statistical physics. It ignores the geometric location of atoms, assuming that all pairs interact in the same way ('mean field approximation').[1]

Consider now the random analytic function on the complex plane $\mathbb{C}$, defined by

$$\xi(z) = \sum_{k=0}^{\infty} \frac{1}{\sqrt{k!}} \frac{\zeta_{2k} + \mathrm{i}\zeta_{2k+1}}{\sqrt{2}} z^k$$

where $\zeta_0, \zeta_1, \dots$ are orthogaussian. Points z such that $\xi(z) = 0$ are a random discrete set in $\mathbb{C}$, well-known to physicists as the flat CAZP (chaotic analytic zero points) model. It is invariant under shifts and rotations of the plane (even though the Gaussian process is not stationary).

Various stationary isotropic (that is, rotation-invariant in distribution) Gaussian random process (called also random fields) on $\mathbb{R}^n$ may be constructed as

$$\Xi(x) = \tilde{\Xi}\big(y \mapsto \varphi(\|x + y\|)\big) \,,$$

where φ is a (good enough) function on $[0, \infty)$ and $\tilde{\Xi}$ is the isonormal process on $L_2(\mathbb{R}^n)$ (that is, a linear isometric map from $L_2(\mathbb{R}^n)$ to a Gaussian space); it is applied to the rotation-invariant function $y \mapsto \varphi(\|y\|)$, shifted by x.

By the *white noise* on $\mathbb{R}^n$ some people mean just the isonormal process on $L_2(\mathbb{R}^n)$; others mean its restriction to (the set of all) indicators of measurable subsets of $\mathbb{R}^n$,

$$\Xi(A) = \tilde{\Xi}(\mathbf{1}_A) \,;$$

note that $\Xi(A \uplus B) = \Xi(A) + \Xi(B)$, the summands being independent.

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[1]This is one of several interesting models discussed in:
M. Talagrand, "Huge random structures", Proc. Intern. Congress Math. 1998, Vol. 1, 507–536.