6 Harris flows as Brownian rotations

6a Some diffeomorphisms

On the circe $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{e^{i\alpha} : \alpha \in \mathbb{R}\}$ we consider the differential equation (for a particle that moves on $\mathbb{T}$)

(6a1)
$$\frac{d}{dt} z = \frac{z + z^{-1}}{2} iz \,, \quad \text{that is,} \quad \frac{d}{dt}\alpha = \cos\alpha \,.$$

It can be solved explicitly,

$$\frac{z + z^{-1}}{2} iz = \frac{z^2 + 1}{2} i = \frac{(z-i)(z+i)}{2} i \,;$$
$$\frac{2}{i} \frac{dz}{(z-i)(z+i)} = dt \,; \quad \frac{1}{z+i} - \frac{1}{z-i} = dt \,; \quad \ln \frac{z+i}{z-i} = t + \text{const} \,.$$

Or, in terms of α,

$$\frac{e^{i\alpha} + i}{e^{i\alpha} - i} = i \frac{\cos\alpha}{1 - \sin\alpha} = i \frac{1 + \sin\alpha}{\cos\alpha} = i \frac{1 + \tan\frac{\alpha}{2}}{1 - \tan\frac{\alpha}{2}} = i \tan\Big(\frac{\alpha}{2} + \frac{\pi}{4}\Big) \,;$$
$$\ln \Big| \tan\Big(\frac{\alpha}{2} + \frac{\pi}{4}\Big) \Big| = t + \text{const} \,.$$

The solution,

(6a2)
$$z_t = i \frac{e^t (z_0 + i) + (z_0 - i)}{e^t (z_0 + i) - (z_0 - i)} \,,$$

is well-defined for all $t \in \mathbb{R}$. However, its power series $z_t = \sum c_k t^k$ has only a finite radius of convergence, since the solution has poles for $t \in \mathbb{C}$.

Another differential equation,

(6a3)
$$\frac{d}{dt} z = \frac{z - z^{-1}}{2i} iz \,, \quad \text{that is,} \quad \frac{d}{dt}\alpha = \sin\alpha \,,$$

may be written as

(6a4)
$$\frac{d}{dt}(-iz) = \frac{(-iz) + (-iz)^{-1}}{2} i(-iz) \,,$$

which is (6a1) for $(-iz)$.

We have two one-parameter semigroups (in fact, groups) of diffeomorphisms of $\mathbb{T}$, and they do not commute with each other. We want to combine these two evolutions with independent random coefficients. The idea is to get infinitesimally (for small t)

$$z_t = z_0 + \frac{z_0 + z_0^{-1}}{2} iz_0 B_1(t) + \frac{z_0 - z_0^{-1}}{2i} iz_0 B_2(t) \,;$$
$$\alpha_t = \alpha_0 + B_1(t) \cos\alpha_0 + B_2(t) \sin\alpha_0 \,;$$

the motion starting at α_0 is driven by the Brownian motion $B_1 \cos\alpha_0 + B_2 \sin\alpha_0$ just as the motion starting at 0 is driven by B_1. The model should be homogeneous, that is, invariant (in distribution) under rotations of $\mathbb{T}$.

In order to use the technique of Sections 3, 4 we treat the group of diffeomorphisms $\mathbb{T} \to \mathbb{T}$ as embedded into the group of rotations (invertible linear isometries) $L_2(\mathbb{T}) \to L_2(\mathbb{T})$. I mean $L_2(\mathbb{T}, \mathbb{R})$ (but $L_2(\mathbb{T}, \mathbb{C})$ can be used as well). For each t the diffeomorphism φ_t defined by (6a2),

$$\varphi_t(z) = i\frac{e^t(z+i) + (z-i)}{e^t(z+i) - (z-i)}\,,$$

gives us a rotation U_t of $L_2(\mathbb{T})$,

(6a5)
$$U_t f(z) = \sqrt{|\varphi_t'(z)|}\, f\big(\varphi_t(z)\big)\,;$$

$$\|U_t f\|^2 = \int |U_t f(e^{i\alpha})|^2\,d\alpha = \int |f\big(\varphi_t(e^{i\alpha})\big)|^2 |\varphi_t'(e^{i\alpha})|\,d\alpha = \int |f(e^{i\beta})|^2\,d\beta = \|f\|^2\,.$$

According to (6a1),

$$\frac{\partial}{\partial t}\varphi_t(z) = \frac{\varphi_t(z) + (\varphi_t(z))^{-1}}{2} i\varphi_t(z)\,.$$

We have a one-parameter semigroup (in fact, group) $(U_t)_t$, and we calculate its generator $A = \frac{d}{dt}\big|_{t=0} U_t$,

$$\begin{aligned} Af(z) &= \frac{d}{dt}\bigg|_{t=0} |\varphi_t'(z)|^{1/2} f\big(\varphi_t(z)\big) = \\ &= \frac12 |\varphi_0'(z)|^{-1/2} \bigg(\frac{d}{dt}\bigg|_{t=0} |\varphi_t'(z)| \bigg) f\big(\varphi_0(z)\big) + |\varphi_0'(z)|^{1/2} f'\big(\varphi_0(z)\big) \frac{d}{dt}\bigg|_{t=0} \varphi_t(z) = \\ &\qquad\qquad = -\frac12 \frac{z - z^{-1}}{2i} f(z) + f'z) \frac{z + z^{-1}}{2} iz\,, \end{aligned}$$

which is easier to understand in terms of α:

$$\varphi_t(e^{i\alpha}) = e^{i\psi_t(\alpha)}\,; \quad |\varphi_t'(e^{i\alpha})| = \psi_t'(\alpha)\,;$$
$$\frac{\partial}{\partial t}\psi_t(\alpha) = \cos\psi_t(\alpha)\,;$$
$$\frac{d}{dt}\psi_t'(\alpha) = \frac{\partial^2}{\partial t\,\partial\alpha}\psi_t(\alpha) = \frac{\partial}{\partial\alpha}\frac{\partial}{\partial t}\psi_t(\alpha)\,;$$
$$\frac{d}{dt}\bigg|_{t=0} |\varphi_t'(e^{i\alpha})| = \frac{d}{dt}\bigg|_{t=0} \psi_t'(\alpha) = \frac{\partial}{\partial\alpha}\frac{\partial}{\partial t}\bigg|_{t=0} \psi_t(\alpha) = \frac{d}{d\alpha}\cos\alpha = -\sin\alpha\,.$$

Of course, A is not defined on the whole $L_2(\mathbb{T})$ (and is not a bounded operator). Our calculation makes sense only for f analytic in a neighborhood of $\mathbb{T}$.

We introduce the differential operator $d_\alpha = \frac{d}{d\alpha}$, that is,

$$d_\alpha f(e^{i\alpha}) = \frac{d}{d\alpha} f(e^{i\alpha})$$

(thus, $d_\alpha f(z) = izf'(z)$ for analytic f), and a multiplication operator $\frac{z+z^{-1}}{2}$, that is,

$$\Big(\frac{z+z^{-1}}{2} f \Big)(z) = \frac{z+z^{-1}}{2} f(z) \, ,$$

then

$$A = \frac{z+z^{-1}}{2} \circ d_\alpha = \frac12 \Big(\frac{z+z^{-1}}{2} d_\alpha + d_\alpha \frac{z+z^{-1}}{2} \Big) \, . \tag{6a6}$$

Indeed, applying d_α to the function $z \mapsto \frac{z+z^{-1}}{2}$ we get the function $z \mapsto -\frac{z-z^{-1}}{2i}$. The formula (6a6) holds on $C^1(\mathbb{T})$.

The other differential equation (6a3) leads similarly to another generator, $\frac{z-z^{-1}}{2i} \circ d_\alpha$.

6b Analytic vectors

In a finite dimension, if A is the generator of a one-parameter semigroup $(U_t)_t$, then necessarily $U_t = e^{tA} = \sum_k \frac{t^k}{k!} A^k$ for all t. In the infinite dimension the situation is more complicated (since a generator is not bounded, in general).

A function $f \in C^\infty(\mathbb{T})$ is called an *analytic vector* for the operator $A = \frac{z+z^{-1}}{2} \circ d_\alpha$ (or another differential operator...), if the (vector-valued) power series[1]

$$\sum_k \frac{t^k}{k!} A^k f \tag{6b1}$$

has a non-zero radius of convergence. In other words: if

$$\sqrt[k]{\|A^k f\|} = O(k) \quad \text{for } k \to \infty \, . \tag{6b2}$$

6b3 Exercise. For every $n \in \mathbb{Z}$, the function $f(z) = z^n$ is an analytic vector for the operator $d_\alpha z$.[2]

Prove it.

Hint: $d_\alpha z^n = inz^n$; $\|(d_\alpha z)^k z^n\| = (n+1)\dots(n+k)\|z^{n+k}\|$ for $n \ge 0$. (Do not forget negative n.)

6b4 Exercise. The same (as 6b3) for the operator $A = \frac{z+z^{-1}}{2} \circ d_\alpha$.

Hint: $\|A^k z^n\| \le (n+1)\dots(n+k)\|z^{n+k}\|$ for $n \ge 0$.

We see that the series

$$e^{tA} f = \sum_k \frac{t^k}{k!} A^k f$$

converges for small t, if f is one of z^n or their linear combination, thus, for a set dense in $L_2(\mathbb{T})$. In fact, any function analytic in a neighborhood of $\mathbb{T}$ is an analytic vector.

[1] Here (in 6b) I write operators on the left: $(BA)f = B(Af)$ etc. But afterwards (in 6c) I will return to the opposite notation: $f(AB) = (fA)B$ etc.

[2] Here we consider complex-valued functions (just for convenience).

6b5 Exercise. $\langle e^{tA}f, e^{tA}g\rangle = \langle f, g\rangle$ if f, g are analytic vectors and t is small enough.

Prove it.

Hint: $\langle Af, g\rangle = -\langle f, Ag\rangle$, since $\langle d_\alpha f, g\rangle = -\langle f, d_\alpha g\rangle$.

Extending by continuity, we get a rotation e^{tA} of $L_2(\mathbb{T})$, and $e^{sA}e^{tA} = e^{(s+t)A}$ for $|s|, |t|$ small enough. Extending by multiplicativity we get rotations e^{tA} for all $t \in \mathbb{R}$ (a one-parameter group). In fact, $U_t = e^{tA}$ for all $t \in [0, \infty)$.

Dealing with two operators, $A_1 = \frac{z+z^{-1}}{2} \circ d_\alpha$ and $A_2 = \frac{z-z^{-1}}{2i} \circ d_\alpha$, we call $f \in C^\infty(\mathbb{T})$ an analytic vector for the pair (A_1, A_2), if

$$\sqrt[k]{\max_{j_1,\dots,j_k=1,2} \|A_{j_1} \dots A_{j_k} f\|} = O(k) \quad \text{for } k \to \infty\,.$$

Similarly to 6b4, each z^n is an analytic vector for the pair. Any analytic vector for (A_1, A_2) is analytic for every linear combination $c_1A_1 + c_2A_2$.

6c Stochastic integrals: does the series converge?

As was said in 3b (for a finite dimension), a stochastic integral of a vector-function is treated coordinate-wise. Now, given a function $f_1 \in L_2\big([0,\infty), H\big)$ where H is a Hilbert space, we define $\int_0^\infty f_1(t)\, dB(t)$ by

$$\Big\langle \int_0^\infty f_1(t)\, dB(t), g \Big\rangle = \int_0^\infty \langle f_1(t), g\rangle\, dB(t) \quad \text{for all } g \in H\,. \tag{6c1}$$

Clearly,

$$\Big\| \int f_1\, dB \Big\|^2_{L_2(\Omega, H)} = \|f_1\|^2 = \int_0^\infty \|f_1(t)\|^2\, dt\,. \tag{6c2}$$

The same for $\iint f_2(s,t)\, dB(s)dB(t)$ etc. A generalization to $B_1(\cdot), \dots, B_m(\cdot)$ is straightforward.

We take

$$H = L_2(\mathbb{T})\,, \qquad i\sigma_1 = \frac{z+z^{-1}}{2} \circ d_\alpha\,, \qquad i\sigma_2 = \frac{z-z^{-1}}{2i} \circ d_\alpha\,. \tag{6c3}$$

6c4 Exercise. $\sigma_1^2 + \sigma_2^2 = -d_\alpha^2 + \frac14$.

Prove it.

Hint: $\frac{z+z^{-1}}{2} d_\alpha = d_\alpha \frac{z+z^{-1}}{2} - \frac{z-z^{-1}}{2i}$, etc.

We want to define $Y_t = \operatorname{Texp}\big(i\int_0^t dX_s\big)$, where $X_t = \sigma_1 B_1(t) + \sigma_2 B_2(t)$, following the formulas of 3c (with $v = 0$):

$$fY_t = \underbrace{f \exp\Big(-\frac12(\sigma_1^2+\sigma_2^2)t\Big)}_{fI_0(t)} + \tag{6c5}$$
$$+ \underbrace{\int_0^t f \exp\Big(-\frac12(\sigma_1^2+\sigma_2^2)s\Big)\big(i\sigma_1\, dB_1(s) + i\sigma_2\, dB_2(s)\big) \exp\Big(-\frac12(\sigma_1^2+\sigma_2^2)(t-s)\Big)}_{fI_1(t)} + \dots$$

However, convergence of the series $\sum_k fI_k(t)$ is not evident (since operators σ_1, σ_2 are not bounded). We take (for now) $f(z) = z^n$. By 6c4, $f(\sigma_1^2 + \sigma_2^2) = (n^2 + \frac14)f$, therefore $f\exp\big(-\frac12(\sigma_1^2+\sigma_2^2)t\big) = \exp\big(-\frac12(n^2+\frac14)t\big)f$ and $\|fI_0(t)\| \le \|f\| = \sqrt{2\pi}$. Further, $\|f\exp(\dots)i\sigma_1\exp(\dots)\| \le \|f\sigma_1\| \le (n+1)\|f\|$ and the same for σ_2; so,

$$\|fI_1(t)\|_{L_2(\Omega,H)} \le \sqrt{2}(n+1)\sqrt{t}\|f\|\,.$$

Similarly to 6b4,

$$\|fI_k(t)\|_{L_2(\Omega,H)} \le \sqrt{2^k}\,(n+1)\dots(n+k)\sqrt{\frac{t^k}{k!}}\,\|f\|\,,$$

but it is useless; the right-hand side tends to ∞ (when $k\to\infty$) for every $t>0$. We should not refuse a help from $\exp\big(-\frac12(\sigma_1^2+\sigma_2^2)t\big)$.

For a while we consider a simplified version $fI_k^0(t)$ of the integral $fI_k(t)$; namely, we replace each $\big(i\sigma_1\,dB_1(s) + i\sigma_2\,dB_2(s)\big)$ with just $i\sigma_1\,dB_1(s)$. We have for $f(z)=z^n$,

$$\frac{\|fI_k^0(t)\|^2_{L_2(\Omega,H)}}{\|f\|^2} = \int\dots\int_{0<s_1<\dots<s_k<t} \sum_{n_k} \Bigg| \frac{1}{2^k} \sum_{n_1,\dots,n_{k-1}} i\frac{n_0+n_1}{2}\dots i\frac{n_{k-1}+n_k}{2} \cdot$$
$$\cdot\exp\bigg(-\frac{4n_0^2+1}{8}s_1 - \frac{4n_1^2+1}{8}(s_2-s_1) - \dots - \frac{4n_k^2+1}{8}(t-s_k)\bigg)\Bigg|^2 ds_1\dots ds_k\,;$$

here $n_0, n_1, \dots, n_{k-1}, n_k$ are integers satisfying $n_0 = n$, $n_1 = n_0 \pm 1$, $n_2 = n_1 \pm 1$, $\dots$, $n_k = n_{k-1} \pm 1$.

We take into account that

$$\sum_{n_k} \bigg| \frac{1}{2^k} \sum_{n_1,\dots,n_{k-1}} \dots \bigg|^2 \le \frac{1}{2^k} \sum_{n_1,\dots,n_k} |\dots|^2 p_k(n_k-n)\,,$$

where $p_k(n_k-n) = \frac{1}{2^k}\sum_{n_1,\dots,n_{k-1}} 1$ is the (binomial) probability of the random walk to come to n_k. Return from I_k^0 to I_k costs a factor 2^k, and we get

$$\frac{1}{2^k}\frac{(n+\frac12)^2}{n^2+\frac14}\dots\frac{(n+k-\frac12)^2}{(n+k)^2+\frac14}\operatorname{Density}\big(t;\,n^2+\tfrac14,\dots,(n+k)^2+\tfrac14\big) \le \frac{\|fI_k(t)\|^2_{L_2(\Omega,H)}}{\|f\|^2} \le$$
$$\le \sum_{n_1,\dots,n_k} \frac{(n_0+n_1)^2}{4n_0^2+1}\dots\frac{(n_{k-1}+n_k)^2}{4n_k^2+1}\operatorname{Density}\big(t;\,n_0^2+\tfrac14,\dots,n_k^2+\tfrac14\big)p_k(n_k-n)\,;$$

here $\operatorname{Density}(t;\,c_0,\dots,c_k)$ denotes the density at t of the (distribution of the) random variable $\frac{\xi_0}{c_0}+\dots+\frac{\xi_k}{c_k}$, where $\xi_0,\dots,\xi_k$ are independent random variables distributed $\operatorname{Exp}(1)$ each.

The lower bound is about $1/2^k$ (for large k), since the series $\sum_k \frac{\xi_k}{k^2}$ converges.

The upper bound is large. Indeed, let the walk $n_0, n_1, \dots, n_k$ go up to $n+\frac{k}{2}$ and return back to n. Then $\operatorname{Density}(\dots)$ is not small, and $p_k(n_k-n) = p_k(0)$ is of order $1/\sqrt{k}$. However, the number of relatively close paths (of the walk) is much larger than $\sqrt{k}$.

Does the series (6c5) converge, or not?

6d Finite-dimensional approximation

In the space $H = L_2(\mathbb{T}, \mathbb{R})$ we consider the orthogonal projection P_n onto the $(2n+1)$-dimensional subspace $P_n H$ spanned by real and imaginery parts of z^k, $k = 0, 1, \dots, n$; that is, by $1, \cos\alpha, \dots, \cos n\alpha$ and $\sin\alpha, \dots, \sin n\alpha$. (Alternatively, in $H = L_2(\mathbb{T}, \mathbb{C})$ the $(2n+1)$-dimensional subspace $P_n H$ is spanned by z^k, $k = -n, -n+1, \dots, n$.)

Operators $i\sigma_1^{(n)}$, $i\sigma_2^{(n)}$ on $P_n H$ are defined by

(6d1) $$f i\sigma_1^{(n)} = f i\sigma_1 P_n \,, \quad f i\sigma_2^{(n)} = f i\sigma_2 P_n \quad \text{for } f \in P_n H \,.$$

Note that $\big(\sigma_1^{(n)}\big)^* = \sigma_1^{(n)}$ and $\big(\sigma_2^{(n)}\big)^* = \sigma_2^{(n)}$. We consider the Brownian rotation $Y_t^{(n)} = \operatorname{Texp}\big(i\int_0^t (\sigma_1^{(n)}\, dB_1(s) + \sigma_2^{(n)}\, dB_2(s))\big)$ on $P_n H$. For $n \to \infty$ we may hope for convergence of $fY_t^{(n)}$ in $L_2(\Omega, H)$, at least for $f(z) = z^k$. To this end we investigate the pair $\big(fY_t^{(m)}, fY_t^{(n)}\big)$, striving to estimate $\mathbb{E}\,|fY_t^{(m)} - fY_t^{(n)}|^2$. But first we need two quite general digressions.

The first digression. In the algebra $M_{m+n}(\mathbb{R})$ we consider the subalgebra $M_m(\mathbb{R}) \oplus M_n(\mathbb{R})$ of matrices of the form $A \oplus B = \big(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\big)$, $A \in \mathrm{M}_m(\mathbb{R})$, $B \in \mathrm{M}_n(\mathbb{R})$. Similarly to 4b, the two algebras $\mathrm{M}_m(\mathbb{R})$ and $\mathrm{M}_n(\mathbb{R})$ are embedded as *commuting* subalgebras. In contrast to 4b, embeddings do not conserve units, and commutativity is trivial: $(A \oplus 0)(0 \oplus B) = 0$. Similarly to 4b4 (but simpler),

$$(A \oplus B)(C \oplus D) = AC \oplus BD \,.$$

In contrast to 4b, every vector of $\mathbb{R}^{m+n}$ is of the form $x \oplus y$, $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$. Of course, $|x \oplus y|^2 = |x|^2 + |y|^2$. We have

$$(x \oplus y)(A \oplus B) = xA \oplus yB \,.$$

If $A \in \mathrm{SO}(m)$ and $B \in \mathrm{SO}(n)$ then $A \oplus B \in \mathrm{SO}(m+n)$.

Similarly to 4c, having a morphism $(B_t, Y_t)_t$, $Y_t : \Omega \to \mathrm{SO}(m)$, and a morphism $(B_t, Z_t)_t$, $Z_t : \Omega \to \mathrm{SO}(n)$, we may form a morphism $(B_t, Y_t \oplus Z_t)_t$. The same holds for several driving Brownian motions.

6d2 Exercise. (a) Let $Y_t = \operatorname{Texp}\big(i\int_0^t (\sigma^Y dB_s + v^Y ds)\big)$ and $Z_t = \operatorname{Texp}\big(i\int_0^t (\sigma^Z dB_s + v^Z ds)\big)$, then

$$Y_t \oplus Z_t = \operatorname{Texp}\Big(i\int_0^t ((\sigma^Y \oplus \sigma^Z)\, dB_s + (v^Y \oplus v^Z)\, ds)\Big) \,.$$

Prove it.

(b) Generalize it for several driving Brownian motions.

Hint. If $Y_t = \operatorname{Texp}\big(i\int_0^t (\sigma_1\, dB_1(s) + \dots + \sigma_k\, dB_k(s) + v\, ds)\big)$ then (for $t \to 0$)

$$Y_t = \mathbf{1} + i\sum_k \sigma_k B_k(t) + o(\sqrt{t}) \,,$$
$$\mathbb{E}\, Y_t - \mathbb{E}\, Y_t^* = 2ivt + o(t) \,.$$

The second digression. A matrix $U \in \mathrm{M}_n(\mathbb{R})$ may be treated as a linear map $\mathbb{R}^n \to \mathbb{R}^n$, namely $x \mapsto xU$. Similarly, a matrix $U \in \mathrm{M}_{n^2}(\mathbb{R})$ may be treated as a linear map $\mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$,

$$\mathrm{M}_n(\mathbb{R}) \ni A \mapsto AU \in \mathrm{M}_n(\mathbb{R})\,;$$

note that AU is not a usual product of matrices. There are several reasonable actions of $\mathrm{M}_{n^2}(\mathbb{R})$ on $\mathrm{M}_n(\mathbb{R})$; I choose this one:

$$A(B \otimes C) = B^* AC \quad \text{for all } A, B, C \in \mathrm{M}_n(\mathbb{R})\,;$$

the right-hand side is the usual product of matrices, and B^* is the transpose of B. In terms of indices,

$$(xU)_\alpha = \sum_\beta x_\beta U^\beta_\alpha\,, \qquad (AU)^\alpha_\beta = \sum_{\gamma,\delta} A^\gamma_\delta U^{\gamma,\delta}_{\alpha,\beta}\,;$$

indeed, $(B^*AC)^\alpha_\beta = \sum_{\gamma,\delta} (B^*)^\alpha_\gamma A^\gamma_\delta C^\delta_\beta = \sum_{\gamma,\delta} B^\gamma_\alpha A^\gamma_\delta C^\delta_\beta = \sum_{\gamma,\delta} A^\gamma_\delta (B \otimes C)^{\gamma,\delta}_{\alpha,\beta}$. Note that

$$A(UV) = (AU)V \quad \text{for } A \in \mathrm{M}_n(\mathbb{R}),\ U, V \in \mathrm{M}_{n^2}(\mathbb{R})\,.$$

Let $(Y_t)_t$ be a Brownian motion in $\mathrm{SO}(n)$ and $B \in \mathrm{M}_n(\mathbb{R})$, then

$$\mathbb{E}\, {Y_t}^* B Y_t = B e^{tA_2}\,;$$

indeed, $\mathbb{E}\, {Y_t}^* B Y_t = \mathbb{E}\, B(Y_t \otimes Y_t) = B \mathbb{E}\,(Y_t \otimes Y_t) = Be^{tA_2}$. Similarly,

$$\mathbb{E}\, Y_t B Y_t^* = B e^{tA_2^*}\,.$$

Thus,

$$\mathbb{E}\, \langle \psi Y_t B, \psi Y_t \rangle = \langle \psi B e^{tA_2^*}, \psi \rangle\,;$$

in this sense, A_2^* is the generator of the dynamics on quadratic forms on $\mathbb{R}^n$.[3] Note that

$$\big(B e^{tA_2^*} \big)^* = B^* e^{tA_2^*}\,, \qquad \big(BA_2^* \big)^* = B^* A_2^*\,,$$

since $(Y_t B Y_t^*)^* = Y_t B^* Y_t^*$. If $B = B^*$ then $(BA_2^*)^* = BA_2^*$. It is important that

$$B \ge 0 \quad \text{implies} \quad B e^{tA_2^*} \ge 0\,; \tag{6d3}$$

here $B \ge 0$ means that $B^* = B$ and $\langle \psi B, \psi \rangle \ge 0$ for all ψ. We have $|\psi|^2 = \mathbb{E}\,|\psi Y_t|^2 = \langle \psi \mathbf{1} e^{tA_2^*}, \psi \rangle$, which means that

$$\mathbf{1} e^{tA_2^*} = \mathbf{1}\,; \qquad \mathbf{1} A_2^* = 0\,.$$

However, $(\mathbf{1}A_2^*)^\alpha_\beta = \sum_{\gamma,\delta} \mathbf{1}^\gamma_\delta (A_2^*)^{\gamma,\delta}_{\alpha,\beta} = \sum_\gamma (A_2)^{\alpha,\beta}_{\gamma,\gamma}$; the equality $\mathbf{1}A_2^* = 0$ becomes $\sum_\gamma (A_2)^{\alpha,\beta}_{\gamma,\gamma} = 0$, just (4g14). *(The end of the second digression.)*

[3]It is also the generator of the dynamics on quadratic functions on $\mathrm{SO}(n)$, see 4f14.

6d4 Exercise. Let $(Y_t)_t$, $(Z_t)_t$ be as in 6d2, and A_2^Y, A_2^Z, A_2 correspond to $(Y_t)_t$, $(Z_t)_t$, $(Y_t \oplus Z_t)_t$ respectively. Then

$$\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} e^{tA_2^*} = \begin{pmatrix} Be^{tA_2^{Y*}} & 0 \\ 0 & Ce^{tA_2^{Z*}} \end{pmatrix}, \qquad \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix} A_2^* = \begin{pmatrix} BA_2^{Y*} & 0 \\ 0 & CA_2^{Z*} \end{pmatrix}$$

for all B, C.

Prove it.

Hint: $\begin{pmatrix} Y_t & 0 \\ 0 & Z_t \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}\begin{pmatrix} Y_t^* & 0 \\ 0 & Z_t^* \end{pmatrix} = \begin{pmatrix} Y_tBY_t^* & 0 \\ 0 & Z_tCZ_t^* \end{pmatrix}$.

The same holds for several driving Brownian motions.

We return to the problem of estimating $\mathbb{E}\,|fY_t^{(m)} - fY_t^{(n)}|^2$. Of course, $fY_t^{(m)} : \Omega \to P_mH$, $fY_t^{(n)} : \Omega \to P_nH$. We may identify P_mH with $\mathbb{R}^{2m+1}$, P_nH with $\mathbb{R}^{2n+1}$, assume that $m < n$ and treat $\mathbb{R}^{2m+1}$ as a subspace of $\mathbb{R}^{2n+1}$; then

$$fY_t^{(m)} \oplus fY_t^{(n)} : \Omega \to \mathbb{R}^{2(2n+1)} .$$

On $\mathbb{R}^{2n}$ we consider the quadratic form

$$y \oplus z \mapsto |y - z|^2 = \langle (y \oplus z)B, y \oplus z \rangle ,$$

$$(y \oplus z)B = (y - z) \oplus (z - y) , \qquad B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \in \mathrm{M}_{2(2n+1)}(\mathbb{R}) ,$$

then

$$\mathbb{E}\,|fY_t^{(m)} - fY_t^{(n)}|^2 = \mathbb{E}\,\langle (f \oplus f)(Y_t^{(m)} \oplus Y_t^{(n)})B, (f \oplus f)(Y_t^{(m)} \oplus Y_t^{(n)}) \rangle = \\ = \langle (f \oplus f)B \exp(tA_2^{(m,n)*}), f \oplus f \rangle ,$$

where $A_2^{(m,n)}$ describes the combined process $Y_t^{(m)} \oplus Y_t^{(n)}$.

6d5 Exercise. Let $(Y_t)_t$, $(Z_t)_t$ be as in 6d2, $m = n$, and $B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \in \mathrm{M}_{2n}(\mathbb{R})$. Then

$$BA_2^* = \frac{1}{2}\begin{pmatrix} 0 & (\sigma^Y - \sigma^Z)^2 \\ (\sigma^Y - \sigma^Z)^2 & 0 \end{pmatrix} + \frac{i}{2}\begin{pmatrix} 0 & i[\sigma^Y, \sigma^Z] + 2v^Z - 2v^Y \\ i[\sigma^Z, \sigma^Y] + 2v^Y - 2v^Z & 0 \end{pmatrix} .$$

Prove it.

Hint: recall (4g3), (4g5): $A_2 = A_1 \otimes \mathbf{1} + \mathbf{1} \otimes A_1 + D$; $B(A_1^* \otimes \mathbf{1} + \mathbf{1} \otimes A_1^*) = A_1B + BA_1^*$; $BD = -\begin{pmatrix} (\sigma^Y)^2 & -\sigma^Y\sigma^Z \\ -\sigma^Z\sigma^Y & (\sigma^Z)^2 \end{pmatrix}$; $A_1 + A_1^* + \sigma^2 = 0$.

For several driving Brownian motions the formula is quite similar; $(\sigma^Y - \sigma^Z)^2$ is replaced with $\sum_k (\sigma_k^Y - \sigma_k^Z)^2$, and $[\sigma^Y, \sigma^Z]$ is replaced with $\sum_k [\sigma_k^Y, \sigma_k^Z]$.

6d6 Lemma. Let $(Y_t)_t$ be a Brownian motion in $\mathrm{SO}(n)$ and $M, B \in \mathrm{M}_n(\mathbb{R})$, $\lambda \in (0, \infty)$ be such that $M^* = M$, $B^* = B$,

$$MA_2^* \le \lambda M ; \qquad BA_2^* \le M .$$

Then

$$Be^{tA_2^*} \le B + \frac{e^{\lambda t} - 1}{\lambda} M$$

for all $t \in [0, \infty)$.

(Of course, $A \le B$ means $B - A \ge 0$.)

Proof. Using (6d3), $\frac{d}{dt} M e^{tA_2^*} = M A_2^* e^{tA_2^*} \le \lambda M e^{tA_2^*}$, thus $e^{\lambda t} \frac{d}{dt}\big(e^{-\lambda t} M e^{tA_2^*} \big) \le 0$ and $e^{-\lambda t} M e^{tA_2^*} \le M$, that is,

$$M e^{tA_2^*} \le e^{\lambda t} M \,.$$

Further,

$$\frac{d}{dt} B e^{tA_2^*} = B A_2^* e^{tA_2^*} \le M e^{tA_2^*} \le e^{\lambda t} M \,;$$
$$B e^{tA_2^*} \le B + \int_0^t e^{\lambda s} M \, ds = B + \frac{e^{\lambda t} - 1}{\lambda} M \,.$$

□

In order to estimate $\mathbb{E} \, | f Y_t^{(m)} - f Y_t^{(n)} |^2$ we need M and λ such that $M A_2^{(m,n)*} \le \lambda M$ and

(6d7) $$\frac12 \begin{pmatrix} 0 & (\sigma_1^{(m)} - \sigma_1^{(n)})^2 + (\sigma_2^{(m)} - \sigma_2^{(n)})^2 \\ (\sigma_1^{(m)} - \sigma_1^{(n)})^2 + (\sigma_2^{(m)} - \sigma_2^{(n)})^2 & 0 \end{pmatrix} + \\ + \frac{i}{2} \begin{pmatrix} 0 & i[\sigma_1^{(m)}, \sigma_1^{(n)}] + i[\sigma_2^{(m)}, \sigma_2^{(n)}] \\ -i[\sigma_1^{(m)}, \sigma_1^{(n)}] - i[\sigma_2^{(m)}, \sigma_2^{(n)}] & 0 \end{pmatrix} \le M \,.$$

The task looks frightening. However, our setup is invariant under rotations of $\mathbb{T}$, thus the operators on $H = L_2(\mathbb{T}, \mathbb{R})$, written above, should commute with rotations of $\mathbb{T}$, and hopefully M will also be found among operators that commute with rotations of $\mathbb{T}$ and in addition, are of the form $M = M' \oplus M''$, which reduces the inequality $M A_2^{(m,n)*} \le \lambda M$ to two separate inequalities, $M' A_2^{(m)*} \le \lambda M'$ and $M'' A_2^{(n)*} \le \lambda M''$ (recall 6d4).

6e Birth and death on a commutative subalgebra

Elements f of the space $H = L_2(\mathbb{T}, \mathbb{R})$ are of the form

$$f(z) = \sum_{k \in \mathbb{Z}} f_k z^k \,, \quad f_k \in \mathbb{C} \,, \quad \sum_k |f_k|^2 < \infty \,, \quad f_{-k} = \overline{f_k} \,.$$

Operators diagonal in the basis $(z^k)_k$ commute with rotations. More exactly, the group of rotations of $\mathbb{T}$ splits H into two-dimensional subspaces $H_k = \{ a z^k + \overline{a} z^{-k} : a \in \mathbb{C} \}$, and operators C commuting with rotations leave H_k invariant,

$$(fC)_k = c_k f_k + b_k \overline{f_k} \,, \quad c_{-k} = \overline{c_k} \,, \qquad b_{-k} = \overline{b_k} \,.$$

We are mostly interested in the case $b_k = 0$,

$$(fC)_k = c_k f_k \,; \quad C = \operatorname{diag}(c_k)_k \,; \quad c_{-k} = \overline{c_k} \,.$$

Such operators are a commutative subalgebra. Especially (recall 6c4),

$$\sigma_1^2 + \sigma_2^2 = -d_\alpha^2 + \frac14 = \operatorname{diag}\big(k^2 + \tfrac14\big)_k \,,$$

since
$$d_\alpha = \operatorname{diag}(ik)_k \,.$$
However, $i\sigma_1, i\sigma_2$ are not diagonal (but 'three-diagonal'); namely, $i\sigma_1 = \frac{z+z^{-1}}{2} \circ \operatorname{diag}(ik)_k$; $(fi\sigma_1)_k = \frac12 i(k-\frac12)f_{k-1} + \frac12 i(k+\frac12)f_{k+1}$; similarly, $i\sigma_2 = \frac{z-z^{-1}}{2i} \circ \operatorname{diag}(ik)_k$; $(fi\sigma_2)_k = \frac12(k-\frac12)f_{k-1} - \frac12(k+\frac12)f_{k+1}$. We hope to construct the infinite-dimensional process
$$Y_t = \operatorname{Texp}\biggl(i\int_0^t \bigl(\sigma_1 \, dB_1(s) + \sigma_2 \, dB_2(s) \bigr) \biggr)$$
and get $A_2^* = A_2 = -\frac12 \bigl((\sigma_1\otimes\mathbf1 + \mathbf1\otimes\sigma_1)^2 + (\sigma_2\otimes\mathbf1+\mathbf1\otimes\sigma_2)^2 \bigr) = A_1\otimes\mathbf1 + \mathbf1\otimes A_1 + D$ (where $A_1 = -\frac12(\sigma_1^2+\sigma_2^2)$ and $D = i\sigma_1\otimes i\sigma_1 + i\sigma_2\otimes i\sigma_2$, recall (4g3), (4g5)), the generator of the dynamics on quadratic forms. The dynamics should preserve invariance under rotations of $\mathbb T$. And indeed, a formal calculation gives: if $C = \operatorname{diag}(c_k)_k$ then

$$\text{(6e1)} \quad CA_2^* = \operatorname{diag}\biggl(\frac12\Bigl(k-\frac12\Bigr)^2 (c_{k-1}-c_k) + \frac12\Bigl(k+\frac12\Bigr)^2 (c_{k+1}-c_k) \biggr)_k = \\ = \operatorname{diag}\biggl(\frac12\Bigl(k^2+\frac14\Bigr)(c_{k-1}-2c_k+c_{k+1}) + k\frac{c_{k+1}-c_{k-1}}{2} \biggr)_k \,.$$

You see, $A_1 = -\frac12 \operatorname{diag}(k^2+\frac14)_k$;
$C(A_1\otimes\mathbf1+\mathbf1\otimes A_1) = A_1^*C + CA_1 = 2A_1\circ C = -\operatorname{diag}(k^2+\frac14)_k \circ \operatorname{diag}(c_k)_k = -\operatorname{diag}\bigl((k^2+\frac14)c_k\bigr)_k$;
$CD = C(i\sigma_1\otimes i\sigma_1 + i\sigma_2\otimes i\sigma_2) = (i\sigma_1)^*Ci\sigma_1 + (i\sigma_2)^*Ci\sigma_2$;
$(fi\sigma_1Ci\sigma_1)_k = \frac12 i(k-\frac12)(fi\sigma_1C)_{k-1} + \frac12 i(k+\frac12)(fi\sigma_1C)_{k+1} =$
$= \frac12 i(k-\frac12)c_{k-1}(fi\sigma_1)_{k-1} + \frac12 i(k+\frac12)c_{k+1}(fi\sigma_1)_{k+1} =$
$= \frac12 i(k-\frac12)c_{k-1}\frac12 i(k-\frac32)f_{k-2} + \frac12 i(k-\frac12)c_{k-1}\frac12 i(k-\frac12)f_k +$
$+ \frac12 i(k+\frac12)c_{k+1}\frac12 i(k+\frac12)f_k + \frac12 i(k+\frac12)c_{k+1}\frac12 i(k+\frac32)f_{k+2}$;
similarly, $(fi\sigma_2Ci\sigma_2)_k =$
$= \frac12(k-\frac12)c_{k-1}\frac12(k-\frac32)f_{k-2} - \frac12(k-\frac12)c_{k-1}\frac12(k-\frac12)f_k -$
$- \frac12(k+\frac12)c_{k+1}\frac12(k+\frac12)f_k + \frac12(k+\frac12)c_{k+1}\frac12(k+\frac32)f_{k+2}$;
thus, $(fi\sigma_1Ci\sigma_1 + fi\sigma_2Ci\sigma_2)_k = -\frac12(k-\frac12)c_{k-1}(k-\frac12)f_k - \frac12(k+\frac12)c_{k+1}(k+\frac12)f_k$;
$(fCD)_k = \frac12\bigl((k-\frac12)^2c_{k-1} + (k+\frac12)^2c_{k+1}\bigr)f_k$;
$CD = \frac12\operatorname{diag}\bigl((k-\frac12)^2c_{k-1} + (k+\frac12)^2c_{k+1}\bigr)_k = \operatorname{diag}\bigl((k^2+\frac14)\frac{c_{k-1}+c_{k+1}}{2} + k\frac{c_{k+1}-c_{k-1}}{2}\bigr)_k$;
(6e1) follows.

The difference operator (6e1) is nothing but the infinitesimal generator of a birth-and-death process; namely, the birth $k\mapsto k+1$ has the rate $\frac12(k+\frac12)^2$, and the death $k+1\mapsto k$ has the same rate. The rate being unbounded (in k), we should bother: does the process explode? According to the general theory of such processes, explosion depends on solutions of two difference equations on $(c_k)_k$, $c_k\in\mathbb R$, $c_{-k}=c_k$; namely, $CA_2^*=0$ and $CA_2^*=\mathbf1$. If both solutions are bounded (in k), the process explodes, otherwise it does not. However, in our case it is easier to use eigenfunctions, $CA_2^* = \lambda C$.

The difference operator (6e1) is a discrete counterpart of the differential operator
$$\frac12 x^2\frac{d^2}{dx^2} + x\frac{d}{dx} = \frac12\frac{d}{dx}x^2\frac{d}{dx}$$

corresponding to the so-called Euler differential equation. Its eigenfunctions are just powers,[4]

$$\frac{1}{2}\frac{d}{dx}x^2\frac{d}{dx}x^p = \frac{1}{2}p(p+1)x^p\,,$$

and we may hope for similar eigenfunctions of the difference operator.

We try first $p=1$, $c_k=|k|$, $C=\operatorname{diag}(|k|)_k$; for $k>0$ (6e1) gives $(CA_2^*)_k = \frac12(k^2+\frac14)\cdot 0 + k\cdot 1 = k$. However, $(CA_2^*)_0 = \frac12(0+\frac14)\cdot 2 + 0 = \frac14$. We did not find an eigenfunction, but we get M and λ such that $MA_2^* \le \lambda M$; namely,

$$M_1 = \operatorname{diag}\big(|k|+\tfrac14\big)_k\,; \quad M_1A_2^* \le M_1\,,$$

since $M_1A_2^* = \operatorname{diag}\big(|k|\vee\frac14\big)_k$. For the birth and death process it means non-explosion and moreover, a finite first moment, bounded by $O(e^t)$ (assuming of course that it holds for the initial state).

For $p=2$ the situation is even simpler, — we get an eigenfunction:

$$M_2 = \operatorname{diag}\Big(k^2+\frac{1}{12}\Big)_k\,; \quad M_2A_2^* = 3M_2\,.$$

The birth and death process has a finite second moment, bounded by $O(e^{3t})$.

Higher eigenfunctions may be found by using

$$\begin{gathered} C_p = \operatorname{diag}\Big(\big(k-\tfrac{p-1}{2}\big)\big(k-\tfrac{p-1}{2}+1\big)\dots\big(k+\tfrac{p-1}{2}\big)\Big)_k\,; \\ C_pA_2^* = \frac12 p(p+1)C_p + \frac18 p^3(p-1)C_{p-2} \end{gathered}$$

for $p=2,4,6,\dots$ The p-th moment of the birth and death process is finite and bounded by $O\big(\exp(\frac12 p(p+1)t)\big)$. Especially,

$$M_4 = \operatorname{diag}\Big(k^4+\frac{13}{14}k^2+\frac{27}{560}\Big)\,; \qquad M_4A_2^* = 10M_4\,.$$

6f Convergence of finite-dimensional approximations

Properties of the birth and death process give us a hope that the desired Brownian rotation $(Y_t)_t$ can be constructed. To this end, however, we need estimations of $Y_t^{(n)}$. The cut-off (6d1) being rotation-invariant (since subspaces P_nH are), we may hope that the quadratic-evolution generator $A_2^{(n)}$, corresponding to $\big(Y_t^{(n)}\big)_t$, also preserves the commutative subalgebra. Here is a finite-dimensional counterpart of (6e1): if $C=\operatorname{diag}(c_k)_{k=-n,\dots,n}$ then

$$\begin{gathered} CA_2^{(n)*} = \operatorname{diag}(b_k)_{k=-n,\dots,n}\,, \\ b_k = \tfrac12\big(k-\tfrac12\big)^2(c_{k-1}-c_k) + \tfrac12\big(k+\tfrac12\big)^2(c_{k+1}-c_k) \quad \text{for } |k|<n\,, \\ b_{-n} = \tfrac12\big(n-\tfrac12\big)^2(c_{-n+1}-c_{-n})\,, \\ b_n = \tfrac12\big(n-\tfrac12\big)^2(c_{n-1}-c_n)\,. \end{gathered} \tag{6f1}$$

[4] In general $p\in\mathbb{C}$, but we need $p\in\mathbb{R}$ only.

For $|k| < n$ the calculation (and the result) is the same as before (for (6e1)), but this time it is rigorous. The case $k = -n$ is similar to the case $k = n$. The latter is obtained as follows.

First,

$$
\begin{gathered}
\big(\sigma_1^{(n)}\big)^2 + \big(\sigma_2^{(n)}\big)^2 = \operatorname{diag}(a_k)_{k=-n,\dots,n} \,, \\
a_k = \tfrac12 (k-\tfrac12)^2 + \tfrac12 (k+\tfrac12)^2 = k^2 + \tfrac14 \quad \text{for } |k|<n \,, \\
a_{-n} = a_n = \tfrac12 (n-\tfrac12)^2 \,.
\end{gathered} \tag{6f2}
$$

Indeed, $(fi\sigma_1^{(n)} i\sigma_1^{(n)})_n = \frac12 i(n-\frac12)(fi\sigma_1^{(n)})_{n-1} = \frac12 i(n-\frac12)\big(\frac12 i(n-\frac32)f_{n-2} + \frac12 i(n-\frac12)f_n\big)$ and $(fi\sigma_2^{(n)} i\sigma_2^{(n)})_n = \frac12 (n-\frac12)(fi\sigma_2^{(n)})_{n-1} = \frac12 (n-\frac12)\big(\frac12 (n-\frac32)f_{n-2} - \frac12 (n-\frac12)f_n\big)$, thus $\big(fi\sigma_1^{(n)} i\sigma_1^{(n)} + fi\sigma_2^{(n)} i\sigma_2^{(n)}\big)_n = -\frac12 (n-\frac12)^2 f_n$.

Further, $(fA_1^{(n)})_n = -\frac14 (n-\frac12)^2 f_n$;
$\big(fC(A_1^{(n)} \otimes \mathbf{1} + \mathbf{1} \otimes A_1^{(n)})\big)_n = (f(2A_1^{(n)} \circ C))_n = -\frac12 (n-\frac12)^2 c_n f_n$;
$(fi\sigma_1^{(n)} C i\sigma_1^{(n)})_n = \frac12 i(n-\frac12)(fi\sigma_1^{(n)} C)_{n-1} = \frac12 i(n-\frac12) c_{n-1} (fi\sigma_1^{(n)})_{n-1} =$
$= \frac12 i(n-\frac12) c_{n-1} \frac12 i(n-\frac32) f_{n-2} + \frac12 i(n-\frac12) c_{n-1} \frac12 i(n-\frac12) f_n$;
similarly, $(fi\sigma_2^{(n)} C i\sigma_2^{(n)})_n = \frac12 (n-\frac12) c_{n-1} \frac12 (n-\frac32) f_{n-2} - \frac12 (n-\frac12) c_{n-1} \frac12 (n-\frac12) f_n$;
thus, $\big(fi\sigma_1^{(n)} C i\sigma_1^{(n)} + fi\sigma_2^{(n)} C i\sigma_2^{(n)}\big)_n = -\frac12 (n-\frac12) c_{n-1} (n-\frac12) f_n$; $(fCD)_n = \frac12 (n-\frac12)^2 c_{n-1} f_n$;
(6f1) follows.

Comparing (6f1) with (6e1) we see that $(CP_n)A_2^{(n)*} \le (CA_2^*)P_n$ for $C = \operatorname{diag}(c_k)_k$, $c_k \in \mathbb{R}$, $c_{-k} = c_k$, provided that $c_n \le c_{n+1}$. The latter holds for M_1, M_2 and M_4; thus,

$$
M_1^{(n)} A_2^{(n)*} \le M_1^{(n)} \,, \quad M_2^{(n)} A_2^{(n)*} \le 3M_2^{(n)} \,, \quad M_4^{(n)} A_2^{(n)*} \le 10 M_4^{(n)} \,,
$$

where $M_p^{(n)} = M_p P_n$.

Now we are in position to return to (6d7). Another calculation similar to (6f2) gives (for $m < n$)

$$
\begin{gathered}
\big(\sigma_1^{(m)} - \sigma_1^{(n)}\big)^2 + \big(\sigma_2^{(m)} - \sigma_2^{(n)}\big)^2 = \operatorname{diag}(a_k)_{k=-n,\dots,n} \,, \\
a_k = \tfrac12 (k-\tfrac12)^2 + \tfrac12 (k+\tfrac12)^2 = k^2 + \tfrac14 \quad \text{for } m<|k|<n \,, \\
a_{-m} = a_m = \tfrac12 (m+\tfrac12)^2 \,, \\
a_{-n} = a_n = \tfrac12 (n-\tfrac12)^2 \,.
\end{gathered}
$$

We see that

$$
\begin{aligned}
\big(\sigma_1^{(m)} - \sigma_1^{(n)}\big)^2 + \big(\sigma_2^{(m)} - \sigma_2^{(n)}\big)^2 &\le \operatorname{diag}\big((k^2+\tfrac14)\mathbf{1}_{[m,n]}(|k|\big) = \\
&= (\sigma_1^2 + \sigma_2^2)(P_n - P_{m-1}) = (P_n - P_{m-1})(\sigma_1^2 + \sigma_2^2) \,.
\end{aligned}
$$

Also,

$$
[\sigma_1^{(m)}, \sigma_1^{(n)}] + [\sigma_2^{(m)}, \sigma_2^{(n)}] = 0 \,,
$$

since both $\sigma_1^{(m)}\sigma_1^{(n)} + \sigma_2^{(m)}\sigma_2^{(n)}$ and $\sigma_1^{(n)}\sigma_1^{(m)} + \sigma_2^{(n)}\sigma_2^{(m)}$ appear to be equal to $\big(\sigma_1^{(m)}\big)^2 + \big(\sigma_2^{(m)}\big)^2$.

For (6d7), it is sufficient to ensure

$$
\frac12 \begin{pmatrix} 0 & (\sigma_1^2+\sigma_2^2)(P_n - P_{m-1}) \\ (\sigma_1^2+\sigma_2^2)(P_n - P_{m-1}) & 0 \end{pmatrix} \le M \,.
$$

Taking into account that $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \le \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ (these are 2×2 matrices), we get

$$\begin{pmatrix} 0 & (\sigma_1^2+\sigma_2^2)(P_n - P_{m-1}) \\ (\sigma_1^2+\sigma_2^2)(P_n - P_{m-1}) & 0 \end{pmatrix} \le \\ \le \begin{pmatrix} (\sigma_1^2+\sigma_2^2)(P_n - P_{m-1}) & 0 \\ 0 & (\sigma_1^2+\sigma_2^2)(P_n - P_{m-1}) \end{pmatrix} .$$

We see that (6d7) holds for $M = 3M_2^{(n)} \oplus 3M_2^{(n)}$, since $k^2 + \frac14 \le 3(k^2 + \frac12)$. Also, $MA_2^{(m,n)*} \le 3M$ since $M_2^{(n)} A_2^{(m)*} \le 3M_2^{(n)}$ and $M_2^{(n)} A_2^{(n)*} \le 3M_2^{(n)}$. So, for every $f \in P_n H$, using Lemma 6d6,

$$\begin{aligned} \mathbb{E}\,|fY_t^{(m)} - fY_t^{(n)}|^2 &= \langle (f \oplus f) B \exp(tA_2^{(m,n)*}, f \oplus f \rangle \le \\ &\le \langle (f \oplus f)B, f \oplus f \rangle + \frac{e^{3t}-1}{3} \langle (f \oplus f)M, f \oplus f \rangle = \\ &= |f - f|^2 + \frac{e^{3t}-1}{3} \big(3\langle fM_2^{(n)}, f \rangle + 3\langle fM_2^{(n)}, f \rangle \big) = \frac{e^{3t}-1}{3} \cdot 6 \langle fM_2^{(n)}, f \rangle . \end{aligned}$$

So what?! It does not tend to 0 for $m \to \infty$ (and $n \to \infty$). How to use the fact that m is large? The projection $P_n - P_{m-1}$ should be small in some sence, much smaller than P_n. In which sense?

We should use M_4, not M_2. We have

$$(\sigma_1^2+\sigma_2^2)(P_n - P_{m-1}) \le \frac{1}{m^2} M_4^{(n)} ;$$

indeed, $k^2 + \frac14 = (k^4 + \frac14 k^2)/k^2 \le (k^4 + \frac{13}{14}k^2 + \frac{27}{560})/m^2$ for $|k| \in [m,n]$. Thus, we use $M = \frac{1}{m^2}\big(M_4^{(n)} \oplus M_4^{(n)}\big)$ and get

$$\mathbb{E}\,|fY_t^{(m)} - fY_t^{(n)}|^2 \le \frac{e^{10t}-1}{10} \frac{1}{m^2} \cdot 2\langle fM_4^{(n)}, f \rangle$$

for all $f \in P_n H$, whenever $m < n$. If $m \to \infty$ (and $n \to \infty$) while f and t are fixed, the right-hand side tends to 0, and we get a Cauchy sequence $\big(fY_t^{(n)}\big)_{n=1}^\infty$ in $L_2(\Omega, H)$. We define fY_t by

$$fY_t = \lim_{t \to \infty} fY_t^{(n)}$$

for $f \in (\cup_{n=1}^\infty P_n H) \subset H = L_2(\mathbb{T}, \mathbb{R})$. Choosing a subsequence that converges almost everywhere (on Ω, for a given t) we get (for almost every ω) a linear isometric map $Y_t(\omega) : (\cup_{n=1}^\infty P_n H) \to H$. We extend it by continuity to an isometric linear map $Y_t(\omega) : H \to H$. Note that

$$\mathbb{E}\,|fY_t - fY_t^{(n)}|^2 \xrightarrow[n\to\infty]{} 0$$

for all $f \in H$ (not just a dense subspace).

6g The infinite-dimensional process

Many questions about $(Y_t)_t$ have to wait more. Is $Y_t(\omega)$ an invertible operator? Is $fY_t(\omega)$ continuous in t? Does $Y_t(\omega)$ correspond to a diffeomorphism $\mathbb{T} \to \mathbb{T}$? Etc, etc. But some properties are easy to prove now.

Let G be the set of all isometric linear operators $U : H \to H$ (not just invertible); $H = L_2(\mathbb{T}, \mathbb{R})$, as before. Clearly, G is a semigroup. We equip it with the σ-field generated by the functions $U \mapsto \langle fU, g\rangle$ for $f, g \in H$. Now G is a measurable semigroup, and

$$Y_t \in L_0(\Omega, G) \quad \text{for } t \in [0, \infty) .$$

In the finite dimensional group $\mathrm{SO}(P_n H)$ we have not only the Brownian motion $(Y_t^{(n)})_t$ but also the abstract stochastic flow $(Y_{s,t}^{(n)})_{s\le t}$, and we may introduce $Y_{s,t}$ by

$$\mathbb{E}\,|fY_{s,t} - fY_{s,t}^{(n)}|^2 \xrightarrow[n\to\infty]{} 0 \quad \text{for all } f \in H ;$$

convergence follows from 6f and stationarity (indeed, $\mathbb{E}\,|fY_{s,t} - fY_{s,t}^{(n)}|^2$ depends on $t - s$ only).

6g1 Exercise. $(Y_{s,t})_{s\le t}$ is a G-valued abstract stochastic flow, and $\big((B_1(s,t), B_2(s,t)), Y_{s,t}\big)_{s\le t}$ is a morphism.

Prove it.

Hint: for each n, $\big((B_1(s,t), B_2(s,t)), Y_{s,t}^{(n)}\big)_{s\le t}$ is a morphism.

The group $\mathbb{T}$ acts on H by rotations U_z, $z \in \mathbb{T}$;

$$(fU_z)(z_1) = f(zz_1) \quad \text{for } z_1 \in \mathbb{T} .$$

The flow $(Y_{s,t})_{s\le t}$ is homogeneous not only in time but also in space, in the following sense.

6g2 Exercise. For every $z \in \mathbb{T}$, random variables $U_z Y_{s,t} U_z^{-1}$ (for $s \le t$) are an abstract stochastic flow, distributed identically to $(Y_{s,t})_{s\le t}$.

Prove it.

Hint: consider first a finite dimension; recall 4d (namely, Y_α there).