6 Time change

6a Time change

6a1 Theorem. [1] (Dambis, Dubins-Schwartz) Every Brownian local martingale $M(\cdot)$ is of the form

$$M(t) = M(0) + \tilde{B}\big(A(t)\big)$$

where $\tilde{B}(\cdot)$ is (distributed as) Brownian motion (on an extended probability space), and $A(\cdot)$ is a random increasing continuous function.

Assume for a while that $\sup_{[0,\infty)} |M(\cdot)| = \infty$ a.s.

Proof of Theorem 6a1 under the additional assumption. The stopping times $T_n^{(m)}$ introduced in Sect. 3a (when embedding simple random walks into the Brownian motion) are functions $C[0,\infty) \to [0,\infty]$; we may apply them to $M(\cdot)$ (instead of $B(\cdot)$). The process $M(t \wedge T_1^{(0)})$ is a martingale by 5b6, thus $\mathbb{P}\big(M(T_1^{(0)}) = -1 \big) = 0.5 = \mathbb{P}\big(M(T_1^{(0)}) = 1 \big)$. Continuing similarly to 3a we get a chain of embedded simple random walks $\big(M(T_n^{(m)})\big)_n$; here $T_n^{(m)}$ means $T_n^{(m)}\big(M(\cdot)\big)$, of course. The joint distribution of these random walks is the same as in Sect. 3a. We construct the corresponding Brownian motion $\tilde{B}(\cdot)$ as in 3a,

$$\lim_m M\big(T_{n_m}^{(m)}\big) = \tilde{B}(t) \quad \text{whenever} \quad \lim_m \frac{n_m}{2^{2m}} = t \,,$$

and get

$$M\big(T_n^{(m)}\big) = \tilde{B}\big(\tau_n^{(m)}\big) \,,$$

[1] See also [1] p. 75; [2] p. 174; [5] p. 173. The theorem holds for all continuous local martingales (Brownian or not).

where $\tau_n^{(m)}$ means $T_n^{(m)}\big(\tilde B(\cdot)\big)$.[1] Note that

$$\tau_{n_1}^{(m_1)} \le \tau_{n_2}^{(m_2)} \quad \text{if and only if} \quad T_{n_1}^{(m_1)} \le T_{n_2}^{(m_2)}$$

for all m_1, n_1, m_2, n_2. It follows that

$$\tau_n^{(m)} = A\big(T_n^{(m)}\big)$$

for some increasing function $A(\cdot)$; this function is unbounded since $\tau_n^{(m)}$ are unbounded, and continuous since the set $\{\tau_n^{(m)} : m, n = 1, 2, \dots\}$ is dense. The equality $M\big(T_n^{(m)}\big) = \tilde B\big(A(T_n^{(m)})\big)$ implies $M(t) = \tilde B(A(t))$ unless $M(\cdot)$ is constant in a neighborhood of t. By monotonicity it holds for all t. $\square$

Waiving the assumption $\sup_{[0,\infty)} |M(\cdot)| = \infty$ we face a difficulty: it may happen that the embedded simple random walk $\big(M(T_n^{(m)})\big)_n$ is defined only for finitely many n. Thus, it is no more a random walk. Rather, it is a (rescaled) unfinished random walk, as defined below.

6a2 Definition. An *unfinished random walk* is a sequence $(X_0, X_1, \dots)$ of measurable $X_n : \Omega \to \mathbb{Z} \cup \{\emptyset\}$ such that

$$\begin{gathered} X_0 = 0\,; \\ X_n = \emptyset \quad \text{implies} \quad X_{n+1} = \emptyset\,; \\ X_{n+1} \in \{X_n - 1, X_n + 1, \emptyset\} \quad \text{if } X_n \in \mathbb{Z}\,; \\ \mathbb{P}\big(X_{n+1} = x_{n+1} \,\big|\, X_0 = x_0, \dots, X_n = x_n\big) \le \frac12 \end{gathered}$$

whenever $x_0, \dots, x_{n+1} \in \mathbb{Z}$, $\mathbb{P}\big(X_0 = x_0, \dots, X_n = x_n\big) > 0$.

6a3 Exercise. The rescaled embedded walk $\big(2^m M(T_n^{(m)})\big)_n$ is an unfinished random walk (for every m).

Prove it.

6a4 Definition. A *continuation* of an unfinished random walk $(X_n)_n$ is a sequence of pairs (X'_n, Y_n) of measurable $X'_n : \Omega' \to \mathbb{Z} \cup \{\emptyset\}$, , $Y_n : \Omega' \to \mathbb{Z}$ such that

$$\begin{gathered} (X'_0, X'_1, \dots) \quad \text{is distributed like } (X_0, X_1, \dots)\,; \\ Y_n = X'_n \quad \text{whenever } X'_n \in \mathbb{Z}\,; \\ \mathbb{P}\big(Y_{n+1} - Y_n = -1 \,\big|\, X_0, Y_0, \dots, X_n, Y_n\big) = \frac12 = \\ = \mathbb{P}\big(Y_{n+1} - Y_n = -1 \,\big|\, X_0, Y_0, \dots, X_n, Y_n\big) \quad \text{a.s.} \end{gathered}$$

[1] Or do you prefer to write $M\big(T_n^{(m)}(M(\cdot))\big) = \tilde B\big(T_n^{(m)}(\tilde B(\cdot))\big)$?

Note that $(Y_n)_n$ is necessarily (distributed as) the simple random walk.

6a5 Exercise. Every unfinished random walk has a continuation.
Prove it.

6a6 Exercise. Prove Theorem 6a1 in full generality.

Note that $A(\cdot)$ need not be strictly increasing, since $M(\cdot)$ may be constant on some intervals (in contrast to $B(\cdot)$).

6a7 Remark. Theorem 6a1 is generalized readily to a local martingale $M(\cdot)$ on $[0, T)$; accordingly, $A(\cdot)$ is defined on $[0, T)$.

6a8 Corollary. Almost surely, $M(\cdot)$ either has a finite limit (at infinity), or is unbounded from below *and* above. Especially, the event '$M(t) \to +\infty$ as $t \to \infty$' is of probability 0, as well as the event '$M(t) \to -\infty$ as $t \to \infty$'. The same holds in the case of finite lifetime; then, of course, '$t \to \infty$' should be replaced with '$t \to T-$'.

6a9 Corollary. If $M(\cdot)$ is of finite variation on some (maybe random) time interval, then it is constant on this interval.

6b Quadratic variation

Given a time interval $[0, t]$ and a path of $M(\cdot)$ on $[0, t]$, we can find the corresponding $A(t)$ by counting steps of embedded random walks,

$$T^{(m)}_{n_m} \to t \quad \text{implies} \quad \frac{n_m}{2^{2m}} \to A(t)\,. \tag{6b1}$$

Thus, a segment of a path of $M(\cdot)$ determines uniquely the corresponding segment of a path of $\tilde{B}(\cdot)$. The converse is generally wrong. Here is a counterexample:

$$M(t) = \begin{cases} 0 & \text{if } t \in [0, 1], \\ B(t) - B(1) & \text{if } t \in [1, \infty) \text{ and } B(1) > 0, \\ 2\big(B(t) - B(1)\big) & \text{if } t \in [1, \infty) \text{ and } B(1) < 0. \end{cases}$$

6b2 Exercise. Find $A(\cdot)$ for this case, and verify the non-uniqueness.

In fact, (6b1) gives just one out of several 'deterministic' approaches to $A(\cdot)$; these approaches coincide for almost all, but not all paths. Regretfully, (6b1) does not suggest many important properties of $A(\cdot)$. Another, 'nondeterministic' approach, given below, does suggest.

6b3 Proposition. Let $M(\cdot)$ and $A(\cdot)$ be as in Theorem 6a1, then the following process is a local martingale:

$$t \mapsto M^2(t) - A(t) \, .$$

On the first sight it follows immediately from the fact that $M^2(t) - A(t) = \tilde{B}^2(A(t)) - A(t) + 2M(0)\tilde{B}(A(t)) + M^2(0)$, since $\tilde{B}^2(s) - s$ is distributed like $B^2(s) - s$. Be careful, however. It is clear that $\tilde{B}(\cdot)$ has the strong Markov property w.r.t. its own filtration, but it does not mean the strong Markov property w.r.t. the filtration of $B(\cdot)$.

6b4 Exercise. Give a counterexample to the following wrong claim: for every stopping time T there exists a stopping time $\tilde{T}$ such that $T(B(\cdot)) = \tilde{T}(\tilde{B}(\cdot))$ a.s.

It is possible to prove the strong Markov property of $\tilde{B}(\cdot)$ w.r.t. the filtration of $B(\cdot)$, thus giving a 'continuous' proof to 6b3. Alternatively, one may use a 'discrete' proof sketched below.

6b5 Exercise. Let an unfinished random walk $(X_n)_n$ be bounded in the sense that there exists K such that for all n, $X_n \in [-K, K] \cup \{\emptyset\}$ a.s. Define $N = \sup\{n : X_n \in \mathbb{Z}\}$. Then $\mathbb{E}\, N < \infty$ and

$$|\mathbb{E}\, X_N^2 - \mathbb{E}\, N| \le 2\, \mathbb{E}\, |X_N| \, .$$

Prove it.

Proof of Proposition 6b3. Let T be a stopping time such that $\exists t \;\; \mathbb{P}\big(T \le t\big) = 1$ and $\exists C \;\; \mathbb{P}\big(\max_{[0,T]} |M(\cdot)| \le C\big) = 1$, we'll prove that $\mathbb{E}\,\big(M^2(T) - A(T)\big) = 0$. (The conditional version of this equality, given the past, is proved similarly.)

We restrict the embedded random walk $(M(T_n^{(m)}))_n$ to n such that $T_n^{(m)} \le T$ and get a (rescaled) unfinished random walk. We apply 6b5 to it:

$$\begin{gathered} |\mathbb{E}\,\big(2^m M(T_N^{(m)})\big)^2 - \mathbb{E}\, N| \le 2\, \mathbb{E}\, |2^m M(T_N^{(m)})| \, ; \\ \Big| \mathbb{E}\, M^2(T_N^{(m)}) - \mathbb{E}\, \frac{N}{2^{2m}} \Big| \le 2^{-m} \cdot 2\, \mathbb{E}\, |M(T_N^{(m)})| \le 2^{-m} \cdot 2C \, . \end{gathered}$$

On the other hand,

$$|M(T_N^{(m)}) - M(T)| \le 2^{-m} \quad \text{a.s.}$$

and

$$\frac{N}{2^{2m}} \to A(T) \quad \text{a.s. as } m \to \infty \, .$$

□

6b6 Exercise. $A(\cdot)$ is the only random increasing continuous function such that the process $t \mapsto M^2(t) - A(t)$ is a local martingale.

Prove it.

6b7 Corollary. (a) If $M(\cdot)$ is a local martingale such that $M(0) = 0$ and the process $t \mapsto M^2(t) - t$ also is a local martingale then $M(\cdot)$ is (distributed as) Brownian motion.

(b) If $M(\cdot)$ is a local martingale on $[0,T)$ such that $M(0) = 0$ and the process $t \mapsto M^2(t) - t$ also is a local martingale on $[0,T)$ then there exists (on an extended probability space) a Brownian motion $\tilde{B}(\cdot)$ such that $M(t) = B(t)$ for all $t \in [0,T)$.

We state it only for Brownian local martingales, but it holds (with the same proof) for all continuous local martingales[1] and is well-known as Lévy's characterization of Brownian motion.[2]

The process $A(\cdot)$ is called the quadratic variation of $M(\cdot)$[3] (see [3], pp. 292, 303), the increasing process of $M(\cdot)$ (see [5], p. 119), the bracket $\langle M, M \rangle$ (see [5], p. 119 again). Let us denote this important process by

$$A(t) = \int_0^t \big(\mathrm{d}M(\cdot)\big)^2 \,;$$

it is just a notation, but very suggestive, especially in the form $\mathrm{d}A(t) = (\mathrm{d}M(t))^2$.

6b8 Exercise. Let $M_1(\cdot), M_2(\cdot)$ be Brownian local martingales, then the following function of $a, b \in \mathbb{R}$ is a quadratic form (for each t):

$$\int_0^t \big(a\mathrm{d}M_1(\cdot) + b\mathrm{d}M_2(\cdot)\big)^2 \,.$$

(Of course, $a\mathrm{d}M_1(\cdot) + b\mathrm{d}M_2(\cdot)$ means $\mathrm{d}(aM_1(\cdot) + bM_2(\cdot))$.)

Prove it.

6b9 Exercise. Define $\int_0^t \mathrm{d}M_1(\cdot)\mathrm{d}M_2(\cdot)$, verify that it is bilinear, and that

$$\int_0^t \mathrm{d}M(\cdot)\mathrm{d}M(\cdot) = \int_0^t \big(\mathrm{d}M(\cdot)\big)^2 \,,$$

[1] Do not think that it holds for discontinuous local martingales; a counterexample is $M(\cdot)$ such that $t \mapsto M(t) + t$ is the Poisson process.

[2] See also [1] p. 75; [2] p. 157; [5] p. 143; [6] p. 2.

[3] Despite the fact that $A(t)$ is not equal to the supremum of $\sum_{k=1}^n (M(t_k) - M_(t_{k-1}))^2$ over all n and $0 \le t_0 \le \dots \le t_n \le t$.

and that the following process is a local martingale:

$$t \mapsto M_1(t)M_2(t) - \int_0^t \mathrm{d}M_1(\cdot)\mathrm{d}M_2(\cdot)\,.$$

All said is generalized readily to local martingales on $[0,T)$. Of course, dealing with M_1, M_2 we should consider them both on the same $[0,T)$.

6c Planar Brownian motion

The planar (that is, two-dimensional) Brownian motion $B^{(2)}(\cdot)$ is nothing but two *independent* one-dimensional Brownian motions; say,

$$B^{(2)}(t)(\omega_1,\omega_2) = \big(B^{(1)}(t)(\omega_1), B^{(1)}(t)(\omega_2)\big)\,,$$

where $B^{(1)}$ is the one-dimensional Brownian motion on a probability space $\Omega^{(1)}$, and $B^{(2)}$ is the two-dimensional Brownian motion on the probability space $\Omega^{(2)} = \Omega^{(1)} \times \Omega^{(1)}$.

The two-dimensional Brownian motion is a random continuous function $[0,\infty) \to \mathbb{R}^2$. It has stationary independent increments. The distribution of $B^{(2)}(t)$ has a two-dimensional density

$$p_t^{(2)}(x) = p_t^{(2)}(x_1,x_2) = p_t^{(1)}(x_1)p_t^{(1)}(x_2) = \frac{1}{2\pi t}\exp\Big(-\frac{|x|^2}{2t}\Big)\,;$$

note the rotation invariance. It is a strong Markov process.[1] Still, all Brownian martingales are continuous, and 4c2 holds. Of course, from now on by a 'Brownian (local) martingale' we mean: w.r.t. the filtration of a given Brownian motion, be it one-dimensional or two-dimensional. The relation to the heat equation holds as before, but the differential operator $\frac{\partial}{\partial t} + \frac12 \frac{\partial^2}{\partial x^2}$ is now replaced with

$$\frac{\partial}{\partial t} + \frac12 \underbrace{\Big(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\Big)}_{\Delta}\,;$$

Δ is the Laplacian. Local martingales are defined as before.

The technique of embedded random walks (Sect. 3, Sect. 6a) is essentially one-dimensional, but still useful, as we'll see soon.

A smooth function $h : \mathbb{R}^2 \to \mathbb{R}$ is called *harmonic,* if $\Delta h = 0$. If h is harmonic then the process

$$M(t) = h\big(B^{(2)}(t)\big)$$

[1] Of course, the sub-σ-field $\mathcal{F}_t$ is now generated by both coordinates of $B^{(2)}(s)$, $s \in [0,t]$.

is a local martingale (by the two-dimensional counterpart of 5b3). It follows immediately that for every bounded domain $G \subset \mathbb{R}^2$ containing the origin, the exit point distribution[1] is the so-called harmonic measure,[2] — a probability measure μ on the boundary of G such that $\int h \, \mathrm{d}\mu = h(0)$ for all harmonic functions h on $\mathbb{R}^2$ (or on a neighborhood of the closure of G).

In order to find the quadratic variation of $M(\cdot)$ we apply the two-dimensional counterpart of Prop. 5b2 to $M^2(t) = h^2\big(B^{(2)}(t)\big)$ and get

$$\int_0^t \big(\mathrm{d}M(\cdot)\big)^2 = \int_0^t \Delta(h^2)\big(B^{(2)}(s)\big)\,\mathrm{d}s = \int_0^t |\nabla h\big(B^{(2)}(s)\big)|^2\,\mathrm{d}s\,,$$

since $\Delta h = |\nabla h|^2$ for harmonic h (think, why). In this sense,

$$\big(\mathrm{d}h(B^{(2)}(t))\big)^2 = |\nabla h\big(B^{(2)}(t)\big)|^2\,\mathrm{d}t \quad \text{for harmonic } h\,.$$

6c1 Exercise. For any two harmonic functions h_1, h_2,

$$\mathrm{d}h_1(B^{(2)}(t))\mathrm{d}h_2(B^{(2)}(t)) = \langle \nabla h_1, \nabla h_2\rangle(B^{(2)}(t))\mathrm{d}t\,.$$

Prove it.

The Brownian motion on the complex plane $\mathbb{C}$ will be denoted $B_{\mathbb{C}}(\cdot)$.

The function

$$z \mapsto \ln|z|\,, \quad \text{that is,} \quad (x,y) \mapsto \ln\sqrt{x^2+y^2}$$

is not a harmonic function on $\mathbb{C}$ (or $\mathbb{R}^2$), however, $\Delta \ln|\cdot| = 0$ on $\mathbb{C} \setminus \{0\}$. Does it mean that $\ln|B_{\mathbb{C}}(t) - 1|$ is a local martingale? We introduce the stopping time $T = \inf\{t : B_{\mathbb{C}}(t) = 1\}$ and note that $\ln|B_{\mathbb{C}}(t) - 1|$ is a local martingale on $[0, T)$ by the two-dimensional counterpart of 5d3. It cannot tend to $-\infty$ (recall 6a8), which shows that $\mathbb{P}\big(\exists t \;\; B_{\mathbb{C}}(t) = 1\big) = 0$. Similarly,

$$\mathbb{P}\big(\exists t \;\; B_{\mathbb{C}}(t) = z\big) = 0 \quad \text{for every } z \in \mathbb{C} \setminus \{0\}\,;$$

it follows easily that

$$\mathbb{P}\big(\exists t > 0 \;\; B_{\mathbb{C}}(t) = 0\big) = 0\,.$$

6c2 Exercise. Give an example of a Brownian local martingale $M(\cdot)$ satisfying $\sup_{t\in[0,\infty)} \mathbb{E}\,\mathrm{e}^{|M(t)|} < \infty$ but still not a martingale.

[1]That is, the distribution of $B^{(2)}(T_G)$ where $T_G = \min\{t : B^{(2)}(t) \notin G\}$.

[2]See also [4] p. 48.

6c3 Theorem. [1] (a) If $M_1(\cdot), M_2(\cdot)$ are local martingales such that $M_1(0) = 0$, $M_2(0) = 0$ and

$$\int_0^t \big(\mathrm{d}M_1(\cdot)\big)^2 = t\,, \quad \int_0^t \big(\mathrm{d}M_2(\cdot)\big)^2 = t\,, \quad \int_0^t \mathrm{d}M_1(\cdot)\mathrm{d}M_2(\cdot) = 0$$

for all t, then the two-dimensional process $\big(M_1(\cdot), M_2(\cdot)\big)$ is (distributed as) the planar Brownian motion.

(b) If $M_1(\cdot), M_2(\cdot)$ are local martingales on $[0,T)$ such that

$$\int_0^t \big(\mathrm{d}M_1(\cdot)\big)^2 = t\,, \quad \int_0^t \big(\mathrm{d}M_2(\cdot)\big)^2 = t\,, \quad \int_0^t \mathrm{d}M_1(\cdot)\mathrm{d}M_2(\cdot) = 0$$

for all $t \in [0,T)$, then there exist (on an extended probability space) two independent Brownian motions $\tilde B_1(\cdot), \tilde B_2(\cdot)$ such that $M_1(t) = \tilde B_1(t)$ and $M_2(t) = \tilde B_2(t)$ for all $t \in [0,T)$.

First, consider 6c3(a).

6c4 Exercise. Let $a_1, a_2 \in \mathbb{R}$, $a_1^2 + a_2^2 \ne 0$. Prove that the process

$$X(t) = a_1 M_1\Big(\frac{t}{a_1^2+a_2^2}\Big) + a_2 M_2\Big(\frac{t}{a_1^2+a_2^2}\Big)$$

is distributed like the Brownian motion.

6c5 Exercise. Let $a_1, a_2 \in \mathbb{R}$. Prove that

$$\mathbb{E} \exp \mathrm{i}\big(a_1 M_1(t) + a_2 M_2(t)\big) = \exp\Big(-\frac12 (a_1^2+a_2^2)t\Big)\,.$$

6c6 Exercise. Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$, $a_1^2 + a_2^2 \ne 0$, $b_1^2 + b_2^2 \ne 0$. Prove that the following process is distributed like the Brownian motion:

$$X(t) = a_1 M_1(A(t)) + a_2 M_2(A(t)) \quad \text{for } t \in [0,1]\,,$$
$$X(t) = X(1) + b_1\big(M_1(A(t)) - M_1(A(1))\big) + b_2\big(M_2(A(t)) - M_2(A(1))\big) \quad \text{for } t \in [1,\infty)\,,$$

where

$$A(t) = \begin{cases} \frac{t}{a_1^2+a_2^2} & \text{for } t \in [0,1], \\ A(1) + \frac{t-1}{b_1^2+b_2^2} & \text{for } t \in [1,\infty). \end{cases}$$

[1] See also [1] p. 78; [2] p. 157.

6c7 Exercise. Let $a_1, a_2, b_1, b_2 \in \mathbb{R}$. Prove that

$$\begin{aligned}\mathbb{E} \exp \mathrm{i}\big(a_1 M_1(1) + b_1(M_1(t) - M_1(1)) + a_2 M_2(1) + b_2(M_2(t) - M_2(1))\big) = \\ = \exp\Big(-\frac{1}{2}(a_1^2 + a_2^2) - \frac{1}{2}(b_1^2 + b_2^2)(t-1) \Big) \,.\end{aligned}$$

6c8 Exercise. Prove Theorem 6c3(a).

Item (b) of Theorem 6c3 is reduced to Item (a) by glueing two-dimensional processes,

$$\tilde{M}(\cdot)(\omega_1, \omega_2) = M(\cdot)(\omega_1) \overset{T(\omega_1)}{\sqcup} B^{(2)}(\cdot)(\omega_2) \,.$$

True, $M(T)$ is not defined; however, the left limit $M(T-)$ exists a.s. on the event $\{T < \infty\}$ by 6b7(b) applied to M_1 and M_2 separately.

6d Conformal local martingales

6d1 Definition. [1] A planar local martingale $M(\cdot) = \big(M_1(\cdot), M_2(\cdot)\big)$ on $[0, T)$ is called *conformal,* if

$$\int_0^t \big(\mathrm{d}M_1(\cdot)\big)^2 = \int_0^t \big(\mathrm{d}M_2(\cdot)\big)^2 \quad \text{and} \quad \int_0^t \mathrm{d}M_1(\cdot)\mathrm{d}M_2(\cdot) = 0$$

for all $t \in [0, T)$.

An important example: $M(t) = f\big(B_{\mathbb{C}}(t)\big)$ where $f : \mathbb{C} \to \mathbb{C}$ is an entire function. Indeed, $h_1 = \operatorname{Re} f$ and $h_2 = \operatorname{Im} f$ are harmonic functions on $\mathbb{C}$, and

$$|\nabla h_1(z)| = |f'(z)| = |\nabla h_2(z)| \,, \quad \langle \nabla h_1(z), \nabla h_2(z) \rangle = 0 \,.$$

A more general example: $M(t) = f\big(B_{\mathbb{C}}(t)\big)$ for $t \in [0, T)$, where $f : G \to \mathbb{C}$ is an analytic function, $G \subset \mathbb{C}$ a domain,[2] $0 \in G$, and $T = \inf\{t : B_{\mathbb{C}}(t) \notin G\}$.

6d2 Exercise. If $M(\cdot)$ is of the form

$$M(t) = M(0) + \tilde{B}(A(t))$$

where $\tilde{B}(\cdot)$ is distributed as $B_{\mathbb{C}}(\cdot)$ and $A(\cdot)$ is a random increasing continuous function, then $M(\cdot)$ is conformal.

Prove it.

[1] See also [5] p. 181.

[2] That is, a connected open set.

6d3 Theorem. [1] Every conformal local martingale $M(\cdot)$ on $[0,T)$ is of the form

$$M(t) = M(0) + \tilde{B}\big(A(t)\big)$$

where $\tilde{B}(\cdot)$ is distributed as the complex-valued Brownian motion $B_{\mathbb{C}}(\cdot)$, and

$$A(t) = \int_0^t \big(\mathrm{d}M_1(\cdot)\big)^2 = \int_0^t \big(\mathrm{d}M_2(\cdot)\big)^2 \quad \text{for } t \in [0,T)\,.$$

Proof. We define $\tilde{B}(\cdot)$ on $[0, A(\infty))$ by $M(t) = M(0) + \tilde{B}\big(A(t)\big)$ (which is correct...) and observe that $t \mapsto \tilde{B}_1^2(A(t)) - A(t)$ is a local martingale, therefore $s \mapsto \tilde{B}_1^2(s) - s$ is a local martingale,[2] that is, $\int_0^t \big(\mathrm{d}\tilde{B}_1(\cdot)\big)^2 = t$ for $t < A(\infty)$. Similarly, $\int_0^t \big(\mathrm{d}\tilde{B}_2(\cdot)\big)^2 = t$ and $\int_0^t \mathrm{d}\tilde{B}_1(\cdot)\mathrm{d}\tilde{B}_2(\cdot) = 0$. It remains to apply Theorem 6c3. $\square$

6d4 Exercise. Given $\varphi \in (0,\pi)$ and $r \in (1,\infty)$, we consider two stopping times

$$\begin{aligned} T'_\varphi &= \min\{t : |\arg(1 + B_{\mathbb{C}}(t))| = \varphi\}\,, \\ T''_r &= \min\{t : |1 + B_{\mathbb{C}}(t)| = r\} \end{aligned}$$

and the probability $\mathbb{P}\big(T'_\varphi < T''_r\big)$. Prove that this probability is a function of $\frac{1}{\varphi} \ln r$ only.

The cases $\varphi = \pi/4$, $\varphi = \pi/2$, $\varphi \to \pi-$ and $\varphi \to 0+$ are especially interesting.[3]

6e Hints to exercises

6a6: For each m use 6a3 and 6a5. Show that these constructions for m and $m+1$ are consistent, and get the needed Brownian motion.

6b2: recall the Brownian scaling.

6b4: recall 6b2.

6b5: induction; $p_-, p_+ \in [0, 0.5]$; $p_+\big((x+1)^2 - x^2 - 1\big) + p_-\big((x-1)^2 - x^2 - 1\big) = 2x(p_+ - p_-)$, and $|p_+ - p_-| \le 1 - p_- - p_+$.

6b6: use 6a9.

[1] See also [4] p. 45; [5] p. 182.

[2] Do not bother about 6b4, now we go the other direction: for every stopping time $\tilde{T}$ there exists a stopping time T such that $T(M(\cdot)) = \tilde{T}(\tilde{B}^{(2)}(\cdot))$.

[3] See also [5] p. 187, Exercise (2.18).

6b8: $(aM_1 + bM_2)^2$ is a linear combination of M_1^2, M_2^2 and $(M_1 + M_2)^2$; use 6b6.

6c1: recall 6b9.

6c2: try $M(t) = \text{const} \cdot \ln |B_{\mathbb{C}}(t) - 1|$.

6c4: use 6b7(a).

6c5: use 6c4.

6c6: use 6b7(a) again.

6c7: use 6c6.

6c8: generalizing 6c7, calculate $\mathbb{E} \exp \mathrm{i}\bigl(\sum \alpha_k (M_1(t_k) - M_1(t_{k-1})) + \sum \beta_k (M_2(t_k) - M_2(t_{k-1}))\bigr)$.

6d4: use the analytic function $z \mapsto z^\alpha$ (or alternatively, $z \mapsto \ln z$).

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