3 Random walks and Brownian motion

3a Simple walks embedded into Brownian motion

The simple (symmetric, one-dimensional) random walk is (by definition) the sequence of random variables $S_n = X_1 + \dots + X_n$ where $X_1, X_2, \dots$ are independent random signs, $\mathbb{P}\big(X_k = -1\big) = 0.5 = \mathbb{P}\big(X_k = +1\big)$. This is a discrete-time random process with stationary independent increments.

Returning to the Brownian motion B, we introduce stopping times T_n recursively:[1]

(3a1)
$$\begin{aligned} T_0 &= 0\,, \\ T_{n+1} &= \min\{t \in (T_n, \infty) : |B(t) - B(T_n)| = 1\} \quad \text{for } n = 0, 1, \dots \end{aligned}$$

The strong Markov property gives

(3a2) the discrete-time random process $(B(T_n))_n$ is distributed like the simple random walk.

This is called *embedded* random walk.

Note also that

(3a3) $T_1,\, T_2 - T_1,\, T_3 - T_2\,, \dots$ are i.i.d. random variables.

We may do the same for the process $\big(2B(t/4)\big)_t$ distributed like B (by the Brownian scaling):

$$\begin{gathered} T_0^{(1)} = 0\,, \\ T_{n+1}^{(1)} = \min\{t \in (T_n^{(1)}, \infty) : |2B(t) - 2B(T_n^{(1)})| = 1\} \quad \text{for } n = 0, 1, \dots\,; \\ (4T_n^{(1)})_n \text{ is distributed like } (T_n)_n\,, \\ \big(2B(T_n^{(1)})\big)_n \text{ is distributed like the simple random walk } (S_n)_n\,. \end{gathered}$$

[1]Do not confuse these T_n with T_x of Sect. 2. By the way, $\mathbb{E}\, T_x = \infty$ while $\mathbb{E}\, T_n < \infty$, see 3a6 (and 3a7).

The walk $\big(B(T_n)\big)_n$ is embedded into the walk $\big(2B(T_n^{(1)})\big)_n$ in roughly the same sense as the latter is embedded into the Brownian motion.

Continuing this way we get a chain of embedded random walks $\big(2^m B(T_n^{(m)})\big)_n$.

An interesting question: whether the random variables $B(T_n^{(m)})$ for all m,n determine uniquely a Brownian sample function, or not?

You may think that the answer is evidently negative, since all these random variables are insensitive to time change. If ω_1,ω_2 are such that, say, $\forall t \;\; B(t)(\omega_1) = B(2t)(\omega_2)$, then clearly $\forall m,n \;\; B(T_n^{(m)})(\omega_1) = B(T_n^{(m)})(\omega_2)$. The same holds if $\forall t \;\; B(t)(\omega_1) = B(\varphi(t))(\omega_2)$ for some increasing homeomorphism $\varphi : [0,\infty) \to [0,\infty)$.

Nevertheless, the answer is affirmative!

It means that the relation $\exists \varphi \; \forall t \;\; B(t)(\omega_1) = B(\varphi(t))(\omega_2)$ never holds (well, almost never). The Brownian motion cannot move more quickly (nor more slowly), even though $\mathrm{d}B(t)/\mathrm{d}t$ is ill-defined.

3a4 Proposition.

$$\lim_{m\to\infty} T_{n_m}^{(m)} = \lim_{m\to\infty} \frac{n_m}{2^{2m}} \quad \text{a.s.}$$

for every sequence $(n_1,n_2,\dots)$ such that the limit in the right-hand side exists.

The proof is given below (after 3a7).

You see, the rescaled (discrete) time of the m-th embedded random walk converges (as $m\to\infty$) to the (continuous) time of the Brownian motion.

This fact leads to another construction of the Brownian motion.[1],[2] One may start with a chain of simple random walks $S^{(m)} = (S_n^{(m)})_n$ (on a single probability space) such that each $S^{(m)}$ is embedded into $S^{(m+1)}$. (Such a sequence can be constructed easily out of a countable collection of independent random signs or binary digits.) Then one defines random variables $B(t)$ (on the same probability space) by

$$\lim_m S_{n_m}^{(m)} = B(t) \quad \text{whenever} \quad \lim_m \frac{n_m}{2^{2m}} = t\,.$$

Here is another implication of 3a4. We have $T_{2^{2m}}^{(m)} \to 1$ a.s., which evidently implies

$$B\big(T_{2^{2m}}^{(m)}\big) \to B(1) \quad \text{a.s.}$$

[1]See also: F.B. Knight (1962) 'On the random walk and Brownian motion', Trans. Amer. Math. Soc. **103**:2, 218–228.

[2]For the Brownian motion on the Sierpinski gasket, the construction via embedded walks is most natural. See: M.T. Barlow, E.A. Perkins (1988) 'Brownian motion on the Sierpiński gasket', Probab. Theory Related Fields **79**:4, 543–623 (MR966175).

A wonder: $B\big(T^{(m)}_{2^{2m}}\big)$ is distributed like $2^{-m}S_{2^{2m}}$, while $B(1)$ has the normal distribution $N(0,1)$; it follows that

$$\mathbb{P}\big(2^{-m}S_{2^{2m}} \le x\big) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-u^2/2}\,\mathrm{d}u \quad \text{as } m \to \infty$$

for all $x \in \mathbb{R}$.[1] This is asymptotic normality of the binomial distribution $B(n, 0.5)$, but only for $n \in \{1, 4, 16, 64, \dots\}$. We leave this matter till 3b, since now we want to prove 3a4.

3a5 Exercise. $\mathbb{E}\, B^2(T) = \mathbb{E}\, T$ for every stopping time T such that $\exists t\ \mathbb{P}\big(T \le t\big) = 1$.

Prove it.

3a6 Exercise. $\mathbb{E}\, T_1 = 1$.

Prove it.

3a7 Exercise. $\mathbb{E}\, T_1^n < \infty$ for all n.

Prove it.

Proof of Prop. 3a4. Let $2^{-2m}n_m \to t$. Using scaling, the strong Markov property and 3a6,

$$\mathbb{E}\, T^{(m)}_{n_m} = 2^{-2m}\mathbb{E}\, T_{n_m} = 2^{-2m} n_m \mathbb{E}\, T_1 = 2^{-2m} n_m \to t\,.$$

Similarly,

$$\operatorname{Var} T^{(m)}_{n_m} = 2^{-4m} \operatorname{Var} T_{n_m} = 2^{-4m} n_m \operatorname{Var} T_1\,;$$

note that $\operatorname{Var} T_1 < \infty$ by 3a7.[2] Chebyshev's inequality

$$\mathbb{P}\big(|T^{(m)}_{n_m} - \mathbb{E}\, T^{(m)}_{n_m}| \ge \varepsilon\big) \le \frac{\operatorname{Var} T^{(m)}_{n_m}}{\varepsilon^2}$$

gives

$$\sum_m \mathbb{P}\big(|T^{(m)}_{n_m} - 2^{-2m}n_m| \ge \varepsilon\big) < \infty$$

for all $\varepsilon > 0$. By the first Borel-Cantelli lemma,

$$T^{(m)}_{n_m} - \frac{n_m}{2^{2m}} \to 0 \text{ a.s.} \quad \text{as } m \to \infty\,.$$

$\square$

[1] You see, $\mathbf{1}_{(-\infty,x]}\big(B(T^{(m)}_{2^{2m}})\big) \to \mathbf{1}_{(-\infty,x]}\big(B(1)\big)$ a.s.

[2] In fact, $\operatorname{Var} T_1 = 2/3$.

3b Functional limit theorem

In order to get the asymptotic normality of the binomial distribution $B(n, 0.5)$ for all n (rather than $n = 2^{2m}$) we turn to another collection of stopping times $T_n^{(m)}$, namely,[1]

$$T_0^{(m)} = 0\,,$$
$$T_{n+1}^{(m)} = \min\{t \in (T_n^{(m)}, \infty) : |B(t) - B(T_n^{(m)})| = 1/\sqrt{m}\} \quad \text{for } n = 0, 1, \dots\,;$$
$$(mT_n^{(m)})_n \text{ is distributed like } (T_n)_n\,,$$
$$\bigl(\sqrt{m}B(T_n^{(m)})\bigr)_n \text{ is distributed like the simple random walk } (S_n)_n\,.$$

The m-th walk is embedded into the Brownian motion, and also into the $4m$-th walk (but not $(m+1)$-th walk).

3b1 Proposition.
$$\lim_{m\to\infty} T_{n_m}^{(m)} = \lim_{m\to\infty} \frac{n_m}{m} \quad \text{a.s.}$$
for every sequence $(n_1, n_2, \dots)$ such that the limit in the right-hand side exists.

The proof is a bit harder than the proof of 3a4, since $\operatorname{Var} T_{n_m}^{(m)} = \frac{1}{m^2} n_m \operatorname{Var} T_1 = O\bigl(\frac{1}{m}\bigr)$ does not lead to a convergent series. We turn to fourth moment.

3b2 Exercise. $\mathbb{E}\,(\xi_1 + \dots + \xi_n)^4 = n\mathbb{E}\,\xi_1^4 + (n^2 - n)\bigl(\mathbb{E}\,\xi_1^2\bigr)^2$ for any i.i.d. random variables $\xi_1, \dots, \xi_n$ such that $\mathbb{E}\,\xi_1 = 0$.

Prove it.

Combining it with (3a3), 3a6 and 3a7 we get

$$\mathbb{E}\,(T_n - n)^4 = O(n^2)\,;$$
$$\mathbb{E}\left(T_n^{(m)} - \frac{n}{m}\right)^4 = O\Bigl(\frac{n^2}{m^4}\Bigr)\,.$$

3b3 Exercise. Prove 3b1.

We have $T_m^{(m)} \to 1$ a.s., therefore $B(T_m^{(m)}) \to B(1)$ a.s., therefore $\mathbb{E}\,\varphi\bigl(B(T_m^{(m)})\bigr) \to \mathbb{E}\,\varphi\bigl(B(1)\bigr)$ for every bounded continuous $\varphi : \mathbb{R} \to \mathbb{R}$; thus,

$$\mathbb{E}\,\varphi\Bigl(\frac{1}{\sqrt{m}} S_m\Bigr) \to \mathbb{E}\,\varphi\bigl(B(1)\bigr)\,. \tag{3b4}$$

[1] Do not confuse $T_n^{(m)}$ here and $T_n^{(m)}$ of 3a.

The same holds if φ is required to be continuous almost everywhere (rather than everywhere). Taking $\varphi = \mathbf{1}_{(-\infty,x]}$ we get

$$\mathbb{P}\big(S_n \le x\sqrt{n}\big) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \mathrm{e}^{-u^2/2}\,\mathrm{d}u \quad \text{as } n \to \infty \tag{3b5}$$

for all $x \in \mathbb{R}$. Note that we get it without such analytic tools as Stirling's formula, Fourier transform etc. However, we can go much further.

3b6 Exercise. Let $X_1(\cdot), X_2(\cdot), \dots$ be random *increasing* functions $[0,1] \to \mathbb{R}$ (on the same probability space). If

$$\forall t \in [0,1] \;\; \mathbb{P}\big(|X_n(t) - t| \to 0\big) = 1$$

then

$$\mathbb{P}\big(\forall t \in [0,1] \;\; |X_n(t) - t| \to 0\big) = 1$$

and moreover,

$$\mathbb{P}\Big(\sup_{t\in[0,1]} |X_n(t) - t| \to 0\Big) = 1\,.$$

Prove it.

3b7 Exercise.

$$\max_{n=0,\dots,m} \Big| T_n^{(m)} - \frac{n}{m} \Big| \to 0 \quad \text{a.s. as } m \to \infty\,.$$

Prove it.

It follows that

$$\max_{n=0,\dots,m} \Big| B\big(T_n^{(m)}\big) - B\Big(\frac{n}{m}\Big) \Big| \to 0 \quad \text{a.s. as } m \to \infty\,, \tag{3b8}$$

since $B(\cdot)$ is uniformly continuous on (say) $[0,2]$ a.s. We see that a collection of random walks $\big(S_n^{(m)}\big)_n$ and the Brownian motion B can be constructed on a single probability space such that

$$\max_{n=0,\dots,m} \Big| \frac{1}{\sqrt{m}} S_n^{(m)} - B\Big(\frac{n}{m}\Big) \Big| \to 0 \quad \text{a.s. as } m \to \infty\,. \tag{3b9}$$

One may interpolate the function $\frac{n}{m} \mapsto \frac{1}{\sqrt{m}} S_n^{(m)}$ linearly and get a.s. uniform convergence of these random piecewise linear functions $X^{(m)}$ to the Brownian motion B. The same holds on every bounded time interval $[0,t]$, not just $[0,1]$. Therefore

$$\mathbb{E}\,\varphi(X^{(m)}) \to \mathbb{E}\,\varphi(B) \quad \text{as } m \to \infty \tag{3b10}$$

for every bounded $\varphi : C[0,\infty) \to \mathbb{R}$ continuous in the sense that $\varphi(f_n) \to \varphi(f)$ whenever $f_n \to f$ uniformly on bounded intervals $[0,t]$. The same holds if φ is required to be continuous for almost all f according to the Wiener measure (the distribution of the Brownian motion). In other words, $X^{(n)}$ converges to B in distribution. This is the *functional limit theorem* for the simple random walk.[1]

Also $\varphi(X^{(n)})$ converges to $\varphi(B)$ in distribution (since the composition of two almost everywhere continuous functions is almost everywhere continuous), be φ bounded or not.

Consider for example the function $\varphi : C[0,\infty) \to [0,1]$,

$$\varphi(f) = \sup\{t \in [0,1] : f(t) = 0\}$$

(recall (2a6)); here $\sup \emptyset = 0$.

3b11 Exercise. Let $f_n \to f$ uniformly on $[0,1]$.

(a) Then $\limsup_{n\to\infty} \varphi(f_n) \le \varphi(f)$.

(b) It may happen that $\varphi(f_n) \xrightarrow[n\to\infty]{} a < \varphi(f)$.

(c) If (b) happens then there exists a rational $t \in (0,1)$ such that at least one of the two relations

$$\min_{[t,1]} f(\cdot) = 0\,, \quad \max_{[t,1]} f(\cdot) = 0$$

holds.

Prove it.

For a given rational $t \in (0,1)$ the random variable $\max_{[t,1]} B(\cdot)$ is nonatomic (since it is the sum of two independent random variables, and one of them, $B(t)$, is nonatomic). Thus, Case (c) is negligible, and therefore φ is continuous almost everywhere.

It follows that L_n converges in distribution to L, where L is defined by (2a6), and

$$L_n = \frac{1}{n} \max\{k = 0,\dots,n : S_k = 0\}\,. \tag{3b12}$$

Using (2b7) we get

$$\mathbb{P}\big(L_n \le t\big) \to \frac{2}{\pi} \arcsin\sqrt{t} \quad \text{as } n \to \infty \tag{3b13}$$

for all $t \in [0,1]$.[2]

[1]This is rather the tip of the iceberg. For more see the following paper and references therein: S. Chatterjee, 'An alternative construction of the strong embedding for the simple random walk', arXiv:0711.0501.

[2]See also [1, Sect. 7.6, Example 6.3].

3c Invariance principle

Limit theorems of 3b can be generalized to non-simple random walks.

We introduce the hitting time

$$T_{-z,y} = \min \bigl\{ t : B(t) \in \{-z, y\} \bigr\} \tag{3c1}$$

for $y, z \in (0, \infty)$. Similarly to 3a5–3a6 we prove that

$$\mathbb{E}\, B(T_{-z,y}) = 0 \,, \tag{3c2}$$

therefore

$$\mathbb{P}\bigl(B(T_{-z,y}) = -z \bigr) = \frac{y}{y+z} \,, \quad \mathbb{P}\bigl(B(T_{-z,y}) = y \bigr) = \frac{z}{y+z} \,. \tag{3c3}$$

This is the general form of a centered (that is, zero mean) two-point probability measure. Similarly to 3a5–3a7,

$$\begin{gathered} \mathbb{E}\, T_{-z,y} = \mathbb{E}\, B^2(T_{-z,y}) = yz \,, \\ \mathbb{E}\, T^n_{-z,y} < \infty \quad \text{for all } n \,. \end{gathered}$$

3c4 Lemma. Every centered finitely supported[1] probability measure is a linear combination with positive coefficients of centered two-point (or one-point) probability measures.

This fact follows from general theorems about extremal points of convex sets, but anyway, here is an elementary proof.

Proof. Let $\mu = \sum_{k=1}^n p_k \delta_{x_k}$ be a centered finitely supported probability measure. Without loss of generality we assume that $\mu(\{0\}) = 0$ and $\int |x| \, \mu(\mathrm{d}x) = 1$. It is sufficient to find a pair of random variables Y, Z such that[2]

$$\mu = \mathbb{E} \Bigl(\frac{1}{2Y} \delta_Y + \frac{1}{2Z} \delta_{-Z} \Bigr) \,,$$

that is,

$$\mathbb{E}\, \frac{1}{2Y} \delta_Y = \sum_{k:x_k>0} p_k \delta_{x_k} \,, \quad \mathbb{E}\, \frac{1}{2Z} \delta_{-Z} = \sum_{k:x_k<0} p_k \delta_{x_k} \,.$$

Marginal distributions of Y and Z must be equal to $\sum_{k:x_k>0} 2x_k \delta_{x_k}$ and $\sum_{k:x_k<0} 2(-x_k) \delta_{x_k}$ respectively, which is evidently possible.[3] □

[1] That is, consisting of a finite set of atoms.

[2] This is the expectation of a random vector in the n-dimensional space of all $\sum c_k \delta_{x_k}$.

[3] You may choose Y, Z to be independent or, if you like, dependent in any way.

Given a centered finitely supported probability measure μ, we represent it in the form

$$\mu = p_0\delta_0 + \sum_{k=1}^n p_k\Big(\frac{y_k}{y_k+z_k}\delta_{-z_k} + \frac{z_k}{y_k+z_k}\delta_{y_k}\Big) \tag{3c5}$$

for some $n \in \{1,2,\dots\}$, $y_1,\dots,y_n > 0$, $z_1,\dots,z_n > 0$, and $p_0,\dots,p_n \ge 0$ (such that $p_0+\dots+p_n = 1$). We multiply the given probability space Ω (that carries the Brownian motion B) by the finite probability space $\{0,1,\dots,n\}$ (endowed with the probabilities $p_0,\dots,p_n$) and introduce the random variable

$$\begin{aligned} &T_1(\omega,k) = T_{-z_k,y_k}(\omega) \quad \text{for } (\omega,k) \in \Omega\times\{0,1,\dots,n\}\,, \\ &S_1(\omega,k) = B\big(T_1(\omega,k),\omega\big)\,. \end{aligned} \tag{3c6}$$

The conditional distribution of S_1, given k, is $\frac{y_k}{y_k+z_k}\delta_{-z_k} + \frac{z_k}{y_k+z_k}\delta_{y_k}$; thus, S_1 is distributed μ. Similarly to 3a, we iterate this construction:

$$\begin{aligned} T_0 &= 0\,, \\ T_{n+1} &= \min\big\{t\in(T_n,\infty) : B(t)-B(T_n) \in \{-z_{k_{n+1}}, y_{k_{n+1}}\}\big\}\,, \\ S_n &= B(T_n)\,; \end{aligned} \tag{3c7}$$

these are random variables on the probability space $\Omega\times\{0,1,\dots,n\}^\infty$, where $\{0,1,\dots,n\}^\infty$ is the space of (infinite) sequences $\{k_1,k_2,\dots\}$ with the product measure. As before, random variables $T_1, T_2-T_1, T_3-T_2,\dots$ are i.i.d.; random variables $S_1, S_2-S_1, S_3-S_2,\dots$ are also i.i.d., distributed μ. Thus, $(S_n)_n$ is a (non-simple) random walk. Also,

$$\begin{gathered} \mathbb{E}\,T_1 = \mathbb{E}\,S_1^2\,, \\ \mathbb{E}\,T_1^m < \infty \quad \text{for all } m\,. \end{gathered} \tag{3c8}$$

Similarly to 3b, we replace y,z with $y/\sqrt{m}$, $z/\sqrt{m}$ getting $T_n^{(m)}$ (for any m) and observe that $\big(mT_n^{(m)}\big)_n$ is distributed like $(T_n)_n$; also, $\big(\sqrt{m}B(T_n^{(m)})\big)_n$ is distributed like the random walk $(S_n^{(m)})_n$. Similarly to 3b1,

$$\lim_{m\to\infty} T_{n_m}^{(m)} = \sigma^2 \lim_{m\to\infty}\frac{n_m}{m} \quad \text{a.s.,} \tag{3c9}$$

where $\sigma^2 = \mathbb{E}\,S_1^2$. In particular, $T_m^{(m)} \to \sigma^2$ a.s., $B(T_m^{(m)}) \to B(\sigma^2)$ a.s., thus, assuming $\sigma = 1$,

$$\mathbb{P}\big(S_n \le x\sqrt{n}\big) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^x \mathrm{e}^{-u^2/2}\,\mathrm{d}u \quad \text{as } n\to\infty \tag{3c10}$$

for all $x \in \mathbb{R}$. (Still, without Stirling's formula, Fourier transform etc.)

It was assumed that S_1 is discrete, moreover, takes on a finite set of values. However, this assumption can be eliminated. Here are two ways to do so.

The first way: you can generalize 3c4 to all μ such that $\int x^2\,\mu(\mathrm{d}x) < \infty$ and $\int x\,\mu(\mathrm{d}x) = 0$; this is easy if you do not fear of measures on $\mathbb{R}$ in general. Subsequent arguments generalize easily as far as μ is compactly supported. Otherwise the fourth moment need not be finite, and we get convergence in probability (rather than a.s. convergence) in (3c9), which makes the transition to (3c10) a bit more complicated. For the *functional* limit theorem you need also redesign 3b6, 3b7, (3b8), (3b9) for convergence in probability (and still get (3b10)). This way is implemented in [1, Sect. 7.6].

The second way: for every such μ and every $\varepsilon > 0$ you can find a step function $f : \mathbb{R} \to \mathbb{R}$ (with a finite number of steps) such that $\int f^2(x)\,\mu(\mathrm{d}x) = \int x^2\,\mu(\mathrm{d}x)$, $\int f(x)\,\mu(\mathrm{d}x) = 0$ and $\int \big(x - f(x)\big)^2\,\mu(\mathrm{d}x) \le \varepsilon$. Then the CLT for $f(S_1) + f(S_2 - S_1) + \dots + f(S_n - S_{n-1})$ implies an approximate CLT for S_n (try it); finally take $\varepsilon \to 0\dots$ This approach (by approximation) will be applied soon to the functional limit theorem.

Before that we return for a while to the random walk $(S_n)_n$ such that S_n takes on a finite set of values. Similarly to 3b7 (assuming $\sigma = 1$, as before),

$$\max_{n=0,\dots,m} \Big| T_n^{(m)} - \frac{n}{m} \Big| \to 0 \quad \text{a.s. as } m \to \infty\,,$$

therefore

$$\max_{n=0,\dots,m} \Big| \underbrace{B(T_n^{(m)})}_{S_n^{(m)}/\sqrt{m}} - B\Big(\frac{n}{m}\Big) \Big| \to 0 \quad \text{a.s. as } m \to \infty\,;$$

the functional limit theorem follows similarly to (3b10):

$$\mathbb{E}\,\varphi(X^{(m)}) \to \mathbb{E}\,\varphi(B) \quad \text{as } m \to \infty$$

for every bounded $\varphi : C[0,\infty) \to \mathbb{R}$ continuous almost everywhere; here $X^{(m)}$ is the random piecewise linear function interpolating $\frac{n}{m} \mapsto \frac{1}{\sqrt{m}} S_n^{(m)}$.

An example: similarly to (3b13),

$$\mathbb{P}\big(L_n \le t \big) \to \frac{2}{\pi} \arcsin \sqrt{t} \quad \text{as } n \to \infty$$

for all $t \in [0,1]$; however, (3b12) is now replaced with, say,

$$L_n = \frac{1}{n} \max\{ k = 1, \dots, n : S_{k-1} S_k \le 0 \}$$

(think, why).

Now, as promised, we use approximation. Given a random walk $(S_n)_n$ such that $\mathbb{E}\,S_1 = 0$, $\mathbb{E}\,S_1^2 = 1$ (otherwise arbitrary), we construct step functions f_k satisfying

$$\mathbb{E}\,f_k(S_1) = 0\,, \quad \mathbb{E}\,f_k^2(S_1) = 1\,, \quad \mathbb{E}\,|S_1 - f_k(S_1)|^2 \le \frac{1}{k}\,.$$

The random walk $(S_n^{(k)})_n$, $S_n^{(k)} = f_k(S_1) + f_k(S_2 - S_1) + \dots + f_k(S_n - S_{n-1})$, is close to $(S_n)_n$ (if k is large); indeed, the differences $R_n = S_n^{(k)} - S_n$ (for a given k) are another random walk (centered, not just symmetric);

$$\mathbb{E}\,R_n = 0\,, \quad \mathbb{E}\,R_n^2 = n\,\mathbb{E}\,R_1^2 \le \frac{n}{k}\,.$$

3c11 Exercise. For every $c \in (0,\infty)$ and every m,

$$\mathbb{E}\,R_m^2 \ge c^2 \mathbb{P}\Big(\max_{n=0,\dots,m} |R_n| \ge c \Big)\,.$$

Prove it.

We have

$$\mathbb{P}\Big(\max_{n=0,\dots,m} |R_n| \ge c \Big) \le \frac{1}{c^2}\mathbb{E}\,R_m^2 \le \frac{m}{c^2 k}\,,$$

therefore

$$\mathbb{P}\Big(\frac{1}{\sqrt{m}} \max_{n=0,\dots,m} |S_n^{(k)} - S_n| \ge \varepsilon \Big) \le \frac{1}{\varepsilon^2 k}\,. \tag{3c12}$$

Given k and m, we consider the function

$$\frac{n}{m} \mapsto \frac{1}{\sqrt{m}} S_n^{(k)}$$

and interpolate it linearly, getting a piecewise linear random function $X_{k,m}(\cdot)$.

For each k the functional limit theorem holds;

$$X_{k,m} \to B \quad \text{in distribution as } m \to \infty\,.$$

On the other hand, for each m,

$$X_{k,m} \to X_m \quad \text{in distribution (moreover, in probability) as } k \to \infty\,,$$

where X_m is the linearly interpolated function $\frac{n}{m} \mapsto \frac{1}{\sqrt{m}} S_n$. Indeed, by (3c12),

$$\mathbb{P}\Big(\max_{t\in[0,1]} |X_{k,m}(t) - X_m(t)| \ge \varepsilon \Big) \le \frac{1}{\varepsilon^2 k} \to 0 \quad \text{as } k \to \infty\,. \tag{3c13}$$

Does it follow that $X_m \to B$ in distribution, as $m \to \infty$? In other words: can we interchange the limits (in distribution),

$$\lim_m \underbrace{\lim_k X_{k,m}}_{X_m} = \lim_k \underbrace{\lim_m X_{k,m}}_{B} \quad ?$$

The answer should be affirmative, since the convergence in k is uniform in m. Alas, distributions are not numbers...

There are two ways (again). One way is, to develop some theory of weak convergence: a metric on the space of probability distributions on $C[0,\infty)$, and all that. The other, shorter way is formalized by Lemma 3c14 below (it is in fact the relevant fragment of the weak convergence theory, adapted to our situation). You may restrict yourselves to the metric space $\mathcal{X} = C[0,1]$ (with its usual metric ρ), but the lemma holds for every metric space.

3c14 Lemma. Let $(\mathcal{X}, \rho)$ be a metric space; Ω and Ω' probability spaces; $Y_m, Y_{k,m} : \Omega' \to \mathcal{X}$ and $Z, Z_{k,m} : \Omega \to \mathcal{X}$ measurable maps such that[1]

(a) $\forall k, m \;\; Y_{k,m} \sim Z_{k,m}$ (that is, identically distributed);

(b) $\forall k \;\; \big(Z_{k,m} \to Z$ a.s. as $m \to \infty \big)$;

(c) $Y_{k,m} \to Y_m$ in probability as $k \to \infty$, uniformly in m (which means existence of $\varepsilon_k \to 0$ such that $\forall m \;\; \mathbb{P}\big(\rho(Y_{k,m}, Y_m) > \varepsilon_k\big) \le \varepsilon_k$).

Then

$$\mathbb{E}\,\varphi(Y_m) \to \mathbb{E}\,\varphi(Z) \quad \text{as } m \to \infty$$

for every bounded $\varphi : \mathcal{X} \to \mathbb{R}$ continuous almost everywhere w.r.t. the distribution of Z.

Here is how we use this lemma. The metric space $(\mathcal{X}, \rho)$ is $C[0,1]$. The probability space Ω' carries the given random walk $(S_n)_n$. The probability space Ω carries the Brownian motion $B(\cdot)$ treated as an $\mathcal{X}$-valued random variable Z. The $\mathcal{X}$-valued random variable Y_m is the piecewise linear random function interpolating $\frac{n}{m} \mapsto \frac{1}{\sqrt{m}} S_n$. Applying the function f_k we get the discrete approximation $(S_n^{(k)})_n$ to $(S_n)_n$, and (by scaling and interpolation) another piecewise linear random function, $Y_{k,m}$. We embed the random walk $\big(\frac{1}{\sqrt{m}} S_n^{(k)}\big)_n$ into the Brownian motion B on Ω and get (by interpolation) another copy $Z_{k,m}$ of this piecewise linear random function.

Condition (a) is evidently satisfied.

Condition (b) is satisfied by the functional limit theorem (applied to the k-th discrete approximation of the needed random walk).

Condition (c) is satisfied by 3c13 (think, why).

[1] We could also stipulate Ω'_m and $Y_m, Y_{k,m} : \Omega'_m \to \mathcal{X}$.

The conclusion of the lemma gives the functional limit theorem for the given random walk $(S_n)_n$.

3c15 Theorem. Let $\varphi : C[0,1] \to \mathbb{R}$ be a bounded continuous function. Let S_n be a random walk (that is, $S_0 = 0$ and random variables $S_{n+1} - S_n$ are i.i.d.), $\mathbb{E}\, S_1^2 = 1$, $\mathbb{E}\, S_1 = 0$. For each m let $X_m(\cdot)$ be the piecewise linear random function interpolating $\frac{n}{m} \mapsto \frac{1}{\sqrt{m}} S_n$ ($n = 0, \dots, m$). Then the limit

$$\lim_{m\to\infty} \mathbb{E}\, \varphi\big(X_m(\cdot)\big)$$

exists and does not depend on the distribution of S_1.

This is called the invariance principle:

$$\boxed{\text{All random walks are the same in the scaling limit.}}$$

And moreover,

$$\mathbb{E}\, \varphi\big(X_m(\cdot)\big) \to \mathbb{E}\, \varphi\big(B(\cdot)\big) \quad \text{as } m \to \infty$$

for every bounded $\varphi : C[0,1] \to \mathbb{R}$ continuous almost everywhere w.r.t. the distribution of $B(\cdot)$. Also, $C[0,1]$ may be replaced with $C[0,t]$ for any t, and even with $C[0,\infty)$ (with the locally uniform convergence).

Proof of Lemma 3c14. Without loss of generality we assume that $\varphi : \mathcal{X} \to [0,1]$.

For every $\delta > 0$ we define $\varphi_\delta : \mathcal{X} \to [0,1]$ by

$$\varphi_\delta(x) = \sup_{y:\rho(y,x)\le\delta} \varphi(y)$$

and observe that

$$\mathbb{E}\, \varphi_\delta(Z) \to \mathbb{E}\, \varphi(Z) \quad \text{as } \delta \to 0+$$

(think, why). By (c) and (a),

$$\mathbb{E}\, \varphi(Y_m) \le \varepsilon_k + \mathbb{E}\, \varphi_{\varepsilon_k}(Y_{k,m}) = \varepsilon_k + \mathbb{E}\, \varphi_{\varepsilon_k}(Z_{k,m})\,.$$

By (b),

$$\limsup_{m\to\infty} \big(\varphi_{\varepsilon_k}(Z_{k,m}) - \varphi_{2\varepsilon_k}(Z) \big) \le 0 \quad \text{a.s.;}$$

in combination with boundedness it implies

$$\limsup_{m\to\infty} \mathbb{E}\, \varphi_{\varepsilon_k}(Z_{k,m}) \le \mathbb{E}\, \varphi_{2\varepsilon_k}(Z)\,.$$

We have $\limsup_{m\to\infty} \mathbb{E}\, \varphi(Y_m) \le \varepsilon_k + \mathbb{E}\, \varphi_{2\varepsilon_k}(Z)$ for all k. Therefore $\limsup_{m\to\infty} \mathbb{E}\, \varphi(Y_m) \le \mathbb{E}\, \varphi(Z)$. Similarly, $\liminf_{m\to\infty} \mathbb{E}\, \varphi(Y_m) \ge \mathbb{E}\, \varphi(Z)$. $\square$

3d Hints to exercises

3a5: calculate $\mathbb{E}\, Y^2(t)$ for Y of 2f8.

3a6: $B^2(T_1 \wedge n) \le 1$ and $B^2(T_1 \wedge n) \to 1$ a.s.; use 3a5. (Here $a \wedge b = \min(a, b)$.)

3a7: recall 1d7, 1d8.

3b2: open the brackets; most of the expectations vanish.

3b6: consider $t \in \{0, \frac{1}{m}, \frac{2}{m}, \dots, 1\}$ first.

3b7: 3b1 and 3b6.

3b11: (a) $|f(\cdot)| \ge \varepsilon$ on $[\varphi(f) + \varepsilon, 1]$; (b) $f_n(t) = 1 - t + \frac{1}{n}$; (c) let $a < t < \varphi(f)$, then f_n cannot change the sign on $[t, 1]$.

3c11: recall the proof of 1e2 and note that $\mathbb{E}\,(a + R_i)^2 \ge a^2$.

References

[1] R. Durrett, *Probability: theory and examples,* 1996.

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