# ON THE PROBABILITY THAT A STATIONARY GAUSSIAN PROCESS WITH SPECTRAL GAP REMAINS NON-NEGATIVE ON A LONG INTERVAL

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ABSTRACT. Let $f$ be a continuous stationary Gaussian process on $\mathbb{R}$ whose spectral measure vanishes in a $\delta$-neighborhood of the origin. Then the probability that $f$ stays non-negative on an interval of length $L$ is at most $e^{-c\delta^2 L^2}$ with some absolute $c > 0$ and the result is sharp without additional assumptions.

## 1. INTRODUCTION

Let $f$ be a continuous stationary Gaussian process on $\mathbb{R}$ and let $L > 0$. We are interested in good bounds for the probability that $f$ stays non-negative on some fixed interval of length $L$ (since $f$ is stationary, this probability does not depend on the location of the interval, so we can always assume that our interval is just $[0, L]$). It has been recently observed in [FF], [KK] and [FFN] that in many interesting cases, one can get reasonably sharp bounds for this probability from both above and below in terms of the behavior of the spectral measure $\mu$ of the Gaussian process $f$ near the origin, usually under the assumption that $\mu$ has a non-trivial absolutely continuous component. In the present paper, we will prove the following

**Theorem 1.** *Let $f$ be a continuous stationary Gaussian process on $\mathbb{R}$ whose spectral measure $\mu$ has a gap, i.e.,*

$$\mu([-\delta, \delta]) = 0 \quad \text{for some } \delta > 0.$$

*Then*

(1) $$\mathcal{P}\{f \geqslant 0 \text{ on } [0, L]\} \leqslant e^{-c\delta^2 L^2}$$

*with some absolute constant $c > 0$.*

We will show in Section 5 that this bound cannot be improved in general. In particular, one can prove a matching bound from below for any stationary Gaussian process whose spectral measure is compactly supported and has a non-trivial absolutely continuous component, extending the results of [KK] where it was done for a class of processes with discrete time.

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An analogue of Theorem 1 (with the same proof) holds for stationary Gaussian processes on $\mathbb{Z}$.

The inequality (1) itself is not new: it appeared in [FFN] already. The novelty here is that the gap condition is the only one we impose on the spectral measure $\mu$; no other a priori assumptions of any kind are made about it. This seems to put our setup beyond the scope of the techniques used in [FFN] despite the fact that our present argument follows the same approach.

## 2. Basic facts about stationary Gaussian processes

A Gaussian process on $\mathbb{R}$ is a mapping $f : \mathbb{R} \times \Omega \to \mathbb{R}$, where $(\Omega, \mathcal{F}, \mathcal{P})$ is a probability space, such that it is jointly measurable and satisfies the condition that for every $x_1, \ldots, x_m \in \mathbb{R}$, the vector $(f(x_1, \cdot), \ldots, f(x_m, \cdot))$ has a (possibly degenerate) Gaussian distribution in $\mathbb{R}^m$. We say that $f$ is continuous if $f(\cdot, \omega)$ is a continuous function on $\mathbb{R}$ almost surely (i.e., for $\mathcal{P}$-almost every $\omega \in \Omega$). We say that $f$ is stationary if for every $x_1, \ldots, x_m \in \mathbb{R}$, the distribution of the vector $(f(x_1 + x, \cdot), \ldots, f(x_m + x, \cdot))$ does not depend on $x \in \mathbb{R}$. In what follows, we will often suppress the probability variable $\omega$ in the notation $f(x, \omega)$ and just write $f(x)$ instead. Also, we will always assume that our process $f$ is not identically 0.

If a Gaussian process $f$ is continuous and stationary, then its covariance kernel $K(x, y) = \mathcal{E}[f(x)f(y)]$ can be written as $K(x, y) = k(x - y)$ with some positive definite (in the sense that $\sum_{i,j=1}^{m} k(x_i - x_j)c_j c_j \geqslant 0$ for all $x_i \in \mathbb{R}$ and all real numbers $c_i$) continuous function $k : \mathbb{R} \to \mathbb{R}$. By the Bochner theorem, there exists a non-negative symmetric with respect to the origin finite measure $\mu$ on $\mathbb{R}$ such that

$$k(x) = \widehat{\mu}(x) = \int_{\mathbb{R}} e^{2\pi i x y} \, d\mu(y) \,.$$

This measure $\mu$ is called the spectral measure of $f$. Conversely, given any finite symmetric measure $\mu$ on $\mathbb{R}$ that decays not too slowly near infinity, we can construct a unique continuous stationary Gaussian process $f$ on $\mathbb{R}$ whose spectral measure is $\mu$.

If $f_j$ are independent continuous stationary Gaussian processes with spectral measures $\mu_j$ and the series $\sum_j f_j$ converges uniformly on compact subsets of $\mathbb{R}$ almost surely, then its sum is a continuous stationary Gaussian process whose spectral measure is $\mu = \sum_j \mu_j$. Conversely, if $f$ is a continuous stationary Gaussian process with spectral measure $\mu$ and $\mu$ is represented as a countable sum $\mu = \sum_j \mu_j$, then, under some mild extra assumptions, $f$ can be decomposed into a uniformly converging on compact subsets of $\mathbb{R}$ sum of independent continuous stationary Gaussian processes $f_j$ with spectral measures $\mu_j$.

We refer the reader who wants to learn more about continuous and smooth Gaussian processes to the Appendix in [NS] and books and articles mentioned therein.

The words "decaying not too slowly" and "under some mild extra assumptions" in the above two paragraphs can be given precise meaning by stating the corresponding (nearly) optimal assumptions explicitly. However, the following remark shows that

for the purpose of proving Theorem 1, we do not need the full strength of the corresponding delicate theory, but can get away with very crude sufficient conditions instead.

**Remark 1.** Let $f$ be any continuous stationary Gaussian process with some spectral measure $\mu$ (possibly, decaying very slowly). Then, for every non-negative compactly supported smooth mollifier $\eta : \mathbb{R} \to \mathbb{R}$, the convolution $f * \eta$ is a continuous (and even smooth) stationary Gaussian process whose spectral measure is $|\widehat{\eta}|^2 \mu$.

If $\eta$ is supported on $[0, \varepsilon]$, then the condition $f \geqslant 0$ on $[0, L]$ implies that $f * \eta \geqslant 0$ on $[0, L - \varepsilon]$, so

$$\mathcal{P}\{f \geqslant 0 \text{ on } [0, L]\} \leqslant \mathcal{P}\{f * \eta \geqslant 0 \text{ on } [0, L - \varepsilon]\}\,.$$

Since

$$(|\widehat{\eta}|^2 \mu)(\mathbb{R} \setminus [-R, R]) = \int_{\mathbb{R} \setminus [-R,R]} |\widehat{\eta}|^2 \, d\mu \leqslant \mu(\mathbb{R}) \max_{\mathbb{R} \setminus [-R,R]} |\widehat{\eta}|^2\,,$$

we can thus reduce the general case to the case when the spectral measure $\mu$ of the process has the property that $\mu(\mathbb{R} \setminus [-R, R])$ decays faster than any power of $R$ as $R \to \infty$, in which case all our claims about convergence, decompositions, existence, etc. in the course of the proof of Theorem 1 become totally routine.

## 3. THE MAIN LEMMA

The following lemma is the basis for all our further considerations and may be of independent interest as well.

**Lemma 1.** *There exist $n_0 \in \mathbb{N}$ and $c, c' > 0$ such that if $f$ is a continuous stationary Gaussian process on $\mathbb{R}$ whose spectral measure is supported on $[-\frac{1}{2}, \frac{1}{2}] \setminus [-\frac{1}{4}, \frac{1}{4}]$, then for every $n \geqslant n_0$, there exist a number $\sigma \in [0, \sqrt{\mu(\mathbb{R})}]$ and a non-negative measure $\nu = \sum_{k=0}^{n} \beta_k \delta_k$ depending on $\mu$ and $n$ such that $\nu(\mathbb{R}) = 1$,*

$$\mathcal{E}\left[\left(\int_{\mathbb{R}} f \, d\nu\right)^2\right] \leqslant e^{-6cn}\sigma^2\,,$$

*and for every deterministic function $\varphi : \mathbb{R} \to \mathbb{R}$, we have*

$$\mathcal{P}\left\{f + \varphi \geqslant 0 \text{ on } \operatorname{supp} \nu, \int_{\mathbb{R}} (f + \varphi) \, d\nu \leqslant e^{-cn}\sigma\right\} \leqslant e^{-c'n^2}\,.$$

*Here, as usual, $\delta_k$ stands for the Dirac unit point mass at $k$.*

*Proof.* Let $n \in \mathbb{N}$ be sufficiently large. Fix $N \in [1, n]$ to be chosen later (the reader should think of $N$ as of a small constant multiple of $n$) and consider the minimization problem

$$\int_{\mathbb{R}} |P(e^{2\pi i y})|^2 \, d\mu(y) \to \min, \qquad P(z) = \sum_{k=0}^{N} a_k z^k, \quad a_k \in \mathbb{C}, \quad \sum_{k=0}^{N} |a_k|^2 = 1\,.$$

Let $P$ be a minimizing polynomial and let $\sigma^2$ be the value of the minimum (since $P(z) = 1$ is an admissible polynomial, we certainly have $\sigma^2 \leqslant \mu(\mathbb{R})$). Write

$$P(z) = a \prod_{k=1}^{N} L_k(z)$$

where $a \in \mathbb{C}$ and each $L_k(z)$ is a linear polynomial either of the form $z - z_k$, or of the form $1 - z_k z$ with $|z_k| \leqslant 1$ (these two types correspond to the roots of $P$ inside and outside the unit disk respectively). Notice that $|L_k(z)| \leqslant 2$ for $|z| = 1$, so

$$1 = \int_{-\frac{1}{2}}^{\frac{1}{2}} |P(e^{2\pi i y})|^2 \, dy \leqslant 2^{2N} |a|^2 \,,$$

whence $|a| \geqslant 2^{-N}$.

We will now replace each factor $L_k(z)$ by a polynomial $\widetilde{L}_k(z)$ of low degree with positive coefficients summing to 1 (or, equivalently, satisfying $\widetilde{L}_k(1) = 1$) so that $|\widetilde{L}_k(z)| \leqslant 3|L_k(z)|$ for all $z$ on the left unit semicircle $\mathbb{T}_- = \{z : |z| = 1, \operatorname{Re} z \leqslant 0\}$. Consider two cases

**Case 1:** $\operatorname{dist}(z_k, \mathbb{T}_-) \geqslant \frac{1}{2}$.

In this case we just put $\widetilde{L}_k(z) = 1$. Then on $\mathbb{T}_-$, we have $|\widetilde{L}_k(z)| \leqslant 2|L_k(z)|$.

**Case 2:** $\operatorname{dist}(z_k, \mathbb{T}_-) < \frac{1}{2}$.

In this case the absolute value of the argument of $z_k$ is at least $\frac{\pi}{3}$ and, therefore, 0 is in the convex hull of $1, z_k, z_k^2, z_k^3$, so there exist $\alpha_0, \dots, \alpha_3 \geqslant 0$ with $\alpha_0 + \cdots + \alpha_3 = 1$ such that $\alpha_0 + \alpha_1 z_k + \alpha_2 z_k^2 + \alpha_3 z_k^3 = 0$. Let $U(z) = \alpha_0 + \alpha_1 z + \alpha_2 z^2 + \alpha_3 z^3$. Notice that $U(z_k) = 0$ and $|U'(z)| \leqslant 3$ in the unit disk, so $|U(\zeta)| \leqslant 3|\zeta - z_k|$ if $|\zeta| = 1$.

If $L_k(z) = z - z_k$, put $\widetilde{L}_k(z) = U(z)$. Then $|\widetilde{L}_k(z)| = |U(z)| \leqslant 3|z - z_k| = 3|L_k(z)|$ on $\mathbb{T}_-$.

If $L_k(z) = 1 - z_k z$, put $\widetilde{L}_k(z) = z^3 U(1/z)$. Then $|\widetilde{L}_k(z)| = |U(1/z)| \leqslant 3\left|\frac{1}{z} - z_k\right| = 3|L_k(z)|$ on $\mathbb{T}_-$.

Now put $\widetilde{P}(z) = \prod_{k=1}^{N} \widetilde{L}_k(z)$. Note that $\widetilde{P}$ is a polynomial of degree at most $3N$ with positive coefficients summing to 1. Moreover, on $\mathbb{T}_-$, one has

$$|\widetilde{P}| = \prod_{k=1}^{N} |\widetilde{L}_k| \leqslant 3^N \prod_{k=1}^{N} |L_k| \leqslant \frac{3^N}{|a|} |P| \leqslant 6^N |P| \,.$$

Let $m \in \mathbb{N}$. Consider the polynomial

$$Q(z) = \frac{1}{m + 4N + 1}(1 + z + \cdots + z^{m+4N}) \left(\frac{1+z}{2}\right)^m \widetilde{P}(z) = \sum_{k \geqslant 0} \beta_k z^k \,.$$

This polynomial still has non-negative coefficients summing up to 1 but, since $|1 + z| \leqslant \sqrt{2}$ on $\mathbb{T}_-$, it satisfies the bound

$$|Q(z)| \leqslant 6^N 2^{-m/2} |P(z)|, \quad z \in \mathbb{T}_- \,.$$

Next, the degree of $Q$ is at most $2m + 7N$. At last, notice that the coefficients of $Q$ can be obtained by convolving the coefficients of the polynomial $\left(\frac{1+z}{2}\right)^m \widetilde{P}(z)$ of

degree at most $m + 3N$ with the coefficients of $\frac{1}{m+4N+1}(1 + z + \cdots + z^{m+4N})$, which form a flat sequence of length $m + 4N$. Thus the coefficients of $Q$ with indices from $m + 3N$ to $m + 4N$ are equal to $\frac{1}{m+4N+1}$ (the common value of the coefficients of $\frac{1}{m+4N+1}(1 + z + \cdots + z^{m+4N})$, which is $\frac{1}{m+4N+1}$, times the *full* sum of the coefficients of $\left(\frac{1+z}{2}\right)^m \widetilde{P}(z)$, which is 1).

Choosing $m = 8N$ and $N = \left\lfloor \frac{n}{23} \right\rfloor$, say, and putting $\nu = \sum_{k \geqslant 0} \beta_k \delta_k$, (here, as above, $\beta_k$ are the coefficients of the polynomial $Q$) we see that $\nu$ is supported on $\{0, 1, \ldots, n\}$ and

$$\mathcal{E}\left[\left(\int_{\mathbb{R}} f \, d\nu\right)^2\right] = \int_{\mathbb{R}} |Q(e^{2\pi i y})|^2 \, d\mu(y) \leqslant 2^{-2N} \int_{\mathbb{R}} |P(e^{2\pi i y})|^2 \, d\mu(y)$$

$$= 2^{-2N}\sigma^2 \leqslant e^{-6cn}\sigma^2$$

with $c = \frac{\log 2}{100}$, say, provided that $n$ is not too small. Let us now fix this value of $c$ and estimate the probability $\mathcal{P}\left\{f + \varphi \geqslant 0 \text{ on supp } \nu, \int_{\mathbb{R}} (f + \varphi) \, d\nu \leqslant e^{-cn}\sigma\right\}$.

Notice that on the event in question, for $k = m + 3N, \ldots, m + 4N$, we must have

$$0 \leqslant f(k) + \varphi(k) \leqslant \frac{1}{\beta_k} \int_{\mathbb{R}} (f + \varphi) \, d\nu \leqslant (m + 4N + 1)e^{-cn}\sigma \leqslant ne^{-cn}\sigma.$$

If $\sigma = 0$, then this implies that the probability we are interested in is just 0. Otherwise, let $A$ be the covariance matrix of the vector $F = (f(m + 3N), \ldots, f(m + 4N))$. By the stationarity of $f$, it is the same as the covariance matrix of the vector $(f(0), \ldots, f(N))$. It follows immediately from the definition of the spectral measure and the construction of $\sigma$ that the least eigenvalue of $A$ is $\sigma^2$. Thus, the density of the distribution of the Gaussian vector $F$ in $\mathbb{R}^{N+1}$ is bounded by

$$(2\pi)^{-\frac{N+1}{2}} (\det A)^{-\frac{1}{2}} \leqslant 1 \cdot \sigma^{-N-1} = \sigma^{-N-1}.$$

On the other hand, on the event under consideration, $F$ belongs to a cube with sidelength $ne^{-cn}\sigma$ whose Euclidean volume is $(ne^{-cn}\sigma)^{N+1}$. Hence, the probability in question is at most $(ne^{-cn})^{N+1} \leqslant e^{-c'n^2}$, provided that $n$ is not too small. The lemma is completely proved. $\qquad\square$

Let us make two remarks:

**Remark 2.** Considering the process $f(x/a)$ instead of $f(x)$ with some $a > 0$, we can immediately generalize this result to the case when the spectral measure $\mu$ is supported on $\left[-\frac{a}{2}, \frac{a}{2}\right] \setminus \left[-\frac{a}{4}, \frac{a}{4}\right]$. In this case the measure $\nu$ will be supported on the set $\left\{0, \frac{1}{a}, \ldots, \frac{n}{a}\right\}$ but the rest of the formulation of Lemma 1 will remain exactly the same; in particular, the constants $n_0$, $c$ and $c'$ will not depend on $a$ in any way.

**Remark 3.** The argument in the proof of Lemma 1 applies with some minor changes to the case when the spectral gap is $[-\delta, \delta]$ with some small $\delta > 0$ instead of $\left[-\frac{1}{4}, \frac{1}{4}\right]$. This allows one to almost immediately get the bound $e^{-c(\delta)n^2}$ with some $c(\delta) > 0$ for the positivity probability in the discrete case. However, the dependence of $c(\delta)$ on $\delta$ one could get on this way would be suboptimal, so we will not use this most direct

approach but, instead, will resort to a more elaborate scheme that would allow us to treat the continuous case and spectral measures with unbounded supports as well.

## 4. THE DYADIC DECOMPOSITION OF THE SPECTRAL MEASURE AND THE PROOF OF THEOREM 1

Let $f$ be any continuous stationary Gaussian process with any spectral measure $\mu$ that decays not too slowly at infinity and satisfies the gap condition $\mu([-\delta, \delta]) = 0$. Note that the positivity probability is never greater than $\frac{1}{2}$, which is the probability that $f$ is non-negative at a single point, so it will suffice to prove inequality (1) under the assumption that $\delta L$ is greater than some fixed absolute constant.

We start with the decomposition $\mu = \sum_a \mu_a$ where $a$ runs over the numbers of the kind $2^k \delta$, $k \geqslant 2$, and $\mu_a$ is just the part of the measure $\mu$ supported on $[-\frac{a}{2}, \frac{a}{2}] \setminus [-\frac{a}{4}, \frac{a}{4}]$. This decomposition of the spectral measure corresponds to a decomposition of the continuous stationary Gaussian process $f$ into the sum of independent processes $f_a$. For each $a$, fix an integer $n_a \geqslant n_0$ to be chosen later.

By Remark 2, for each $a$, we can find a non-negative measure $\nu_a$ of total mass 1 supported on $\{0, \frac{1}{a}, \dots, \frac{n_a}{a}\}$ and a number $\sigma_a \in [0, \sqrt{\mu_a(\mathbb{R})}]$ such that

$$\mathcal{E}\left[\left(\int_{\mathbb{R}} f_a \, d\nu_a\right)^2\right] \leqslant e^{-6cn_a} \sigma_a^2,$$

and for every deterministic function $\varphi_a : \mathbb{R} \to \mathbb{R}$, we have

$$\mathcal{P}\left\{f_a + \varphi_a \geqslant 0 \text{ on } \operatorname{supp} \nu_a, \int_{\mathbb{R}} (f_a + \varphi_a) \, d\nu_a \leqslant e^{-cn_a} \sigma_a\right\} \leqslant e^{-c'n_a^2}.$$

Since $f_a$ is stationary, the same inequalities hold for any shift of the measure $\nu_a$. Consider now the (countably infinite in general) convolution $\nu = *_a \nu_a = \nu_{4\delta} * \nu_{8\delta} * \dots$. Since each measure $\nu_a$ is non-negative and satisfies $\nu_a(\mathbb{R}) = 1$, this convolution is well-defined and supported on $[0, L]$, provided that

$$(2) \qquad\qquad\qquad \sum_a \frac{n_a}{a} \leqslant L.$$

Using the Minkowski inequality, we get the bound

$$\left[\mathcal{E}\left(\int_{\mathbb{R}} f \, d\nu\right)^2\right]^{\frac{1}{2}} \leqslant \sum_a \left[\mathcal{E}\left(\int_{\mathbb{R}} f_a \, d\nu\right)^2\right]^{\frac{1}{2}}.$$

Note now that we can write $\nu$ as $\nu_a * \nu^{(a)}$ where $\nu^{(a)} = *_{a' \neq a} \nu_{a'}$ is also a measure of total mass 1. Then, denoting by $\nu_{a,t}$ the shift of the measure $\nu_a$ by $t \in \mathbb{R}$ (so $\nu_{a,t}(E) = \nu_a(E - t)$), and using the integral version of the Minkowski inequality, we get

$$\left[\mathcal{E}\left(\int_{\mathbb{R}} f_a \, d\nu\right)^2\right]^{\frac{1}{2}} = \left[\mathcal{E}\left(\int_{\mathbb{R}} \left[\int_{\mathbb{R}} f_a \, d\nu_{a,t}\right] d\nu^{(a)}(t)\right)^2\right]^{\frac{1}{2}}$$

$$\leqslant \int_{\mathbb{R}} \left[\mathcal{E}\left(\int_{\mathbb{R}} f_a \, d\nu_{a,t}\right)^2\right]^{\frac{1}{2}} d\nu^{(a)}(t) = \left[\mathcal{E}\left(\int_{\mathbb{R}} f_a \, d\nu_a\right)^2\right]^{\frac{1}{2}} \leqslant e^{-3cn_a} \sigma_a.$$

Here we used the fact that the Gaussian process $f_a$ is stationary, so the distribution of $\int_{\mathbb{R}} f_a \, d\nu_{a,t}$ does not depend on $t$.

Hence,

$$\left[ \mathcal{E}\left( \int_{\mathbb{R}} f \, d\nu \right)^2 \right]^{\frac{1}{2}} \leqslant \sum_a e^{-3cn_a} \sigma_a \leqslant \max_a (e^{-2cn_a} \sigma_a) \,,$$

provided that

$$(3) \qquad\qquad \sum_a e^{-cn_a} \leqslant 1 \,.$$

The maximum always exists because $\sigma_a$ tend to 0 as $a \to \infty$ (recall that $\sigma_a^2 \leqslant \mu_a(\mathbb{R})$ and $\sum_a \mu_a(\mathbb{R}) = \mu(\mathbb{R}) < +\infty$). We can also assume that the maximum is strictly positive since, otherwise, the only chance for $f$ to be non-negative on $\operatorname{supp} \nu$ is to be identically 0 there and that event has zero probability.

Let $\alpha$ be the value of $a$ for which the maximum is attained. By the standard Gaussian tail estimate, we have

$$\mathcal{P}\left\{ \int_{\mathbb{R}} f \, d\nu \geqslant \frac{1}{2} e^{-cn_\alpha} \sigma_\alpha \right\} \leqslant \exp\left( -\frac{e^{2cn_\alpha}}{8} \right) \leqslant e^{-\delta^2 L^2} \,,$$

provided that

$$(4) \qquad\qquad \min_a n_a \geqslant c'' \delta L \text{ with some } c'' > 0$$

and $\delta L$ is not too small.

Thus, it will suffice to bound the probability of the event $S$ that $f \geqslant 0$ on $\operatorname{supp} \nu$ and $\int_{\mathbb{R}} f \, d\nu \leqslant \frac{1}{2} e^{-cn_\alpha} \sigma_\alpha$. For $t \in \mathbb{R}$, let $S_t$ be the event that $f \geqslant 0$ on $\operatorname{supp} \nu$ and $\int_{\mathbb{R}} f \, d\nu_{\alpha,t} \leqslant e^{-cn_\alpha} \sigma_\alpha$. Writing $f = f_\alpha + \sum_{a \neq \alpha} f_a = f_\alpha + \varphi_\alpha$ and conditioning upon $f_a$ with $a \neq \alpha$, we see that for every fixed $t \in \operatorname{supp} \nu^{(\alpha)}$, the event $S_t$ is (conditionally) contained in the event that $f_\alpha + \varphi_\alpha \geqslant 0$ on $\operatorname{supp} \nu_{\alpha,t}$ and $\int_{\mathbb{R}} (f_\alpha + \varphi_\alpha) \, d\nu_{\alpha,t} \leqslant e^{-cn_\alpha} \sigma_\alpha$. Thus, the probability of $S_t$ does not exceed $e^{-c' n_\alpha^2}$.

Now define $g(\omega) = \int_{\mathbb{R}} \chi_{S_t}(\omega) \, d\nu^{(\alpha)}(t)$. Then, on the one hand,

$$\mathcal{E}[g] \leqslant \sup_{t \in \operatorname{supp} \nu^{(\alpha)}} \mathcal{P}\{S_t\} \leqslant e^{-c' n_\alpha^2}$$

while on the other hand, on $S$ we must clearly have $g \geqslant \frac{1}{2}$. This yields the bound

$$\mathcal{P}\{S\} \leqslant 2 e^{-c' n_\alpha^2} \leqslant e^{-c''' \delta^2 L^2}$$

provided that (4) holds and $\delta L$ is not too small.

It remains to show that we can, indeed, choose $n_a$ satisfying (2–4). We will just take a small $c'' > 0$ and, for $a = 2^k \delta$, put $n_a = \lfloor c'' 2^{k/2} \delta L \rfloor$. Then (2) rewrites as

$$\sum_{k \geqslant 2} c'' 2^{-k/2} \leqslant 1 \,,$$

which can be ensured by an appropriate choice of $c'' > 0$, while, (3), (4) and the inequality $\min_a n_a \geqslant n_0$ are satisfied as long as $\delta L$ is not too small.

This finishes the proof of the desired bound in the continuous case. To handle the discrete case (stationary Gaussian processes on $\mathbb{Z}$ with spectral gap), it suffices to

note that every such discrete process can be viewed as the restriction of a continuous process with the spectral measure supported on $[-\frac{1}{2}, \frac{1}{2}]$ with the same spectral gap. If we assume that $\delta > 0$ is a negative power of 2 (which we can always do without loss of generality) and restrict the dyadic decomposition in the argument above to $a \leqslant 1$, then all measures $\nu_a$ and their convolutions will be supported on $\mathbb{Z}$.

## 5. The sharpness of the bound

Now we will present a simple theorem that provides a wide class of spectral measures for which the result of Theorem 1 cannot be substantially improved.

**Theorem 2.** *Let $f$ be any continuous stationary Gaussian process whose spectral measure $\mu$ is supported on $[-R, R]$ and satisfies*

$$(5) \qquad \rho_n^2 = \inf\left\{ \int_\mathbb{R} |P|^2 \, d\mu \, : \, P(y) = 1 + \sum_{k=1}^n a_k y^k, a_k \in \mathbb{C} \right\} \geqslant e^{-2Cn}\mu(\mathbb{R})$$

*for all $n \geqslant 1$ with some $C > 0$. Then, for some $C' > 0$, $L_0 > 0$ depending on $C$ only, we have*

$$\mathcal{P}\{f \geqslant 0 \text{ on } [0, L]\} \geqslant e^{-C'R^2L^2}$$

*for all $L \geqslant L_0$.*

Note that this requirement is compatible with the spectral gap condition. For instance, the classical Remez theorem (see [CW], Lemma 4) immediately yields the following

**Corollary 1.** *Let $f$ be any continuous stationary Gaussian process whose spectral measure $\mu$ is compactly supported and has a non-trivial absolutely continuous component. Then, for some $C' > 0$, $L_0 > 0$, we have*

$$\mathcal{P}\{f \geqslant 0 \text{ on } [0, L]\} \geqslant e^{-C'L^2}$$

*for all $L \geqslant L_0$.*

In general, the compact support condition in the Corollary cannot be removed if one wants to preserve the conclusion in the current form (see [FFN], Corollary 5, for an example of a continuous stationary Gaussian process whose spectral measure is absolutely continuous and for which $\mathcal{P}\{f \geqslant 0 \text{ on } [0, L]\} \leqslant e^{-e^{cL}}$ for large $L$). However, having a non-trivial absolutely continuous part is by no means necessary for the conditions of Theorem 2 to be satisfied. At the end of this section we will present a purely discrete measure $\mu$ that has a spectral gap around the origin and still satisfies (5).

*Proof.* Since $\mu$ is compactly supported, the Gaussian process $f$ represents a random entire function of exponential type and we have the Taylor series decomposition

$$f(x) = \sum_{k \geqslant 0} \frac{1}{k!} f^{(k)}(0) x^k$$

converging almost surely on the entire real line.

Fix some $L > 0$. We will certainly have $f$ preserving sign on $[0, L]$ if we can find some positive numbers $a, a_1, a_2, \ldots$ such that $a \geqslant \sum_{k \geqslant 1} a_k \frac{L^k}{k!}$, $|f(0)| > a$, and $|f^{(k)}(0)| \leqslant a_k$ for all $k \geqslant 1$. Since $f$ and $-f$ are equidistributed, we have $f \geqslant 0$ on one half (with respect to the probability measure) of that event. In what follows, we assume without loss of generality that $R = \frac{1}{2\pi}$, $\mu(\mathbb{R}) = 1$.

When choosing $a$ and $a_k$, it will be convenient to take care of "small" and "large" $k$ separately. So, fix some big $K \geqslant 1$ (it will end up being a constant multiple of $L$ for large $L$) and consider first the derivatives $f^{(k)}(0)$ with $k \leqslant K$.

Condition (5) shows, in particular, that the Gaussian random variable $f(0)$ cannot be completely determined by the vector $F = (f'(0), \ldots, f^{(K)}(0))$. More precisely, we can represent $f(0)$ as $\rho_K g + AF$ where $g$ is the standard real Gaussian random variable independent of $F$ and $A$ is some linear mapping from $\mathbb{R}^K$ to $\mathbb{R}$.

Indeed, let $A = (a_1, \ldots, a_K) \in \mathbb{C}^K$ be a minimizer of

$$\mathcal{E}[(f(0) - AF)^2] = \int_{\mathbb{R}} \left| 1 + \sum_{k=1}^{K} (2\pi i)^k a_k y^k \right|^2 d\mu(y).$$

Note that the expression on the right hand side of this identity shows that the minimization problem we are talking about is just the problem of finding the closest point to the constant function 1 in the finite-dimensional subspace of the complex Hilbert space $L^2(\mu)$ spanned by $y^k$ ($k = 1, \ldots, K$), so it always has a solution, which is just the orthogonal projection of 1 to that subspace.

On the other hand, the expression on left hand side shows that the minimizing vector $A$ can always be taken real (since $f(0), f^{(k)}(0)$ are all real Gaussian random variables, removing the imaginary part does not increase the functional), and, for this choice of a minimizing vector, $f(0) - AF$ is a real Gaussian random variable satisfying $\mathcal{E}[(f(0) - AF)f^{(k)}(0)] = 0$ for all $k = 1, \ldots, K$. By the special properties of jointly Gaussian random variables, it means that $f(0) - AF$ is independent of $F$. At last, by the definition of $\rho_K$ (see (5)), we get

$$E[(f(0) - AF)^2] = \rho_K^2$$

so we can write $f(0) - AF$ as $\rho_K$ times a standard real Gaussian $g$.

Since we want to impose the restriction that $F$ is small, our only chance to get $f(0)$ reasonably large is to use the $\rho_K g$ component of this decomposition, which dictates the choice $a = \rho_K \geqslant e^{-CK}$. We will also put $a_k = \alpha > 0$ for $k = 1, \ldots, K$. Since we need to ensure that $a \geqslant \sum_{k \geqslant 1} a_k \frac{L^k}{k!}$ in the end, we choose $\alpha = \frac{a}{2} e^{-L}$, so that

$$\sum_{k=1}^{K} \frac{L^k}{k!} a_k < e^L \alpha = \frac{a}{2}.$$

Our next goal will be to estimate from below the probability that $|f(0)| > a$ while $|f^{(k)}(0)| < \alpha$ for all $k = 1, \ldots, K$. Since

$$E[f^{(k)}(0)^2] = (2\pi)^{2k} \int_{\mathbb{R}} y^{2k} d\mu(y) \leqslant 1,$$

for all $k$, the diagonal elements of the covariance matrix of $F$ are bounded by 1. Therefore its norm is at most $K$, whence $F$ can be written as $BG$ where $B : \mathbb{R}^K \to \mathbb{R}^K$ is a linear transformation of norm at most $\sqrt{K}$ and $G = (g_1, \dots, g_K)$ is the standard Gaussian vector in $\mathbb{R}^K$.

Thus, denoting the Euclidean norm of the vector $F$ by $|F|$, as usual, and observing that $\alpha < 1$, we have

$$\mathcal{P}\{|f^{(k)}(0)| \leqslant \alpha \text{ for all } k = 1, \dots, K\}$$
$$\geqslant \mathcal{P}\{|F| \leqslant \alpha\} \geqslant \mathcal{P}\{|G| < K^{-\frac{1}{2}}\alpha\} \geqslant \mathcal{P}\{|g_k| < K^{-1}\alpha \text{ for all } k = 1, \dots, K\}$$
$$\geqslant \left( \frac{2\alpha e^{-\frac{1}{2}K^{-2}\alpha^2}}{\sqrt{2\pi}\,K} \right)^K \geqslant \left( \frac{\alpha}{3K} \right)^K,$$

say.

Conditioning upon $F$, we see that for every $\bar{F} \in \mathbb{R}^K$,

$$\mathcal{P}\{|f(0)| > \rho_K \mid F = \bar{F}\} \geqslant \inf_{t \in \mathbb{R}} \mathcal{P}\{|\rho_K g + t| > \rho_K\} = \mathcal{P}\{|g| > 1\} = p_0 > 0\,,$$

whence

$$\mathcal{P}\{|f(0)| > a;\ |f^{(k)}(0)| \leqslant a_k \text{ for all } k = 1, \dots, K\}$$
$$= \mathcal{P}\{|f(0)| > \rho_K;\ |f^{(k)}(0)| \leqslant \alpha \text{ for all } k = 1, \dots, K\}$$
$$\geqslant p_0 \left( \frac{\alpha}{3K} \right)^K.$$

Plugging in the value $\alpha = \frac{a}{2}e^{-L} = \frac{1}{2}\rho_K e^{-L} \geqslant \frac{1}{2}e^{-CK-L}$ (recall that by (5) we have $\rho_K \geqslant e^{-CK}$), we get the lower bound

$$p_0 \left( \frac{\alpha}{3K} \right)^K \geqslant p_0 \left( \frac{e^{-CK-L}}{6K} \right)^K.$$

For $K \geqslant L$, the right hand side of the last inequality is at least $e^{-\widetilde{C}K^2}$ with $\widetilde{C} = C + 7 + p_0^{-1}$, say.

Now let us take care of $k > K$. We have already seen that

$$E[f^{(k)}(0)^2] \leqslant 1\,,$$

so we have the standard Gaussian tail bound

$$\mathcal{P}\{|f^{(k)}(0)| > a_k\} \leqslant e^{-\frac{1}{2}a_k^2}\,.$$

Thus, for any choice of $a_k > 0$, the probability that the inequality $|f^{(k)}(0)| \leqslant a_k$ is violated for some $k > K$ is at most $\sum_{k>K} e^{-\frac{1}{2}a_k^2}$ and, therefore,

$$\mathcal{P}\{|f(0)| > a;\ |f^{(k)}(0)| \leqslant a_k \text{ for all } k \geqslant 1\}$$
$$= \mathcal{P}\{|f(0)| > a;\ |f^{(k)}(0)| \leqslant a_k \text{ for all } k = 1, \dots, K;\text{ and } |f^{(k)}(0)| \leqslant a_k \text{ for } k > K\}$$
$$\geqslant e^{-\widetilde{C}K^2} - \sum_{k>K} e^{-\frac{1}{2}a_k^2}\,.$$

We want to make sure that $\sum_{k>K} e^{-\frac{1}{2}a_k^2}$ stays well below $e^{-\widetilde{C}K^2}$ so that the subtraction in the probability estimate is harmless. It can be achieved, say, by putting $a_k = \sqrt{2(\widetilde{C}K^2 + k)}$ for $k > K$, in which case the sum in question is bounded by $e^{-\widetilde{C}K^2} \sum_{k>K} e^{-k} \leqslant \frac{1}{2}e^{-\widetilde{C}K^2}$, so we still have the lower bound $\frac{1}{2}e^{-\widetilde{C}K^2}$ for the probability of the event we are interested in.

Since we have already ensured that $\sum_{1\leqslant k\leqslant K} \frac{L^k}{k!}a_k < \frac{a}{2}$, it remains to choose $K$ so that $\sum_{k>K} \frac{L^k}{k!}a_k \leqslant \frac{a}{2}$ as well. Recalling that $a = \rho_K \geqslant e^{-CK}$, we see that it will suffice to ensure that

$$\sum_{k>K} \frac{L^k}{k!}\sqrt{2(\widetilde{C}K^2 + k)} \leqslant \frac{1}{2}e^{-CK}.$$

The classical bound $k! \geqslant \left(\frac{k}{e}\right)^k$ and the inequality

$$\sqrt{2(\widetilde{C}K^2 + k)} \leqslant \sqrt{2(\widetilde{C}k^2 + k)} \leqslant 2\widetilde{C}k \leqslant e^{2\widetilde{C}k}$$

imply that the left hand side is at most

$$\sum_{k>K} \left(\frac{Le^{2\widetilde{C}+1}}{K}\right)^k.$$

If $K \geqslant 2e^{2\widetilde{C}+1}L$ (i.e., the common ratio of this geometric progression is at most $\frac{1}{2}$), then the sum converges and does not exceed the term of the progression corresponding to $k = K$, which is $\left(\frac{Le^{2\widetilde{C}+1}}{K}\right)^K$. To ensure that it is less than $\frac{1}{2}e^{-CK} > \left(e^{-C-1}\right)^K$, it is enough to choose $K$ such that

$$\frac{Le^{2\widetilde{C}+1}}{K} \leqslant e^{-C-1},$$

i.e., $K \geqslant e^{C+2\widetilde{C}+2}L$. Compared to all the previous conditions imposed on $K$ ($K \geqslant 1$, $K \geqslant L$, $K \geqslant 2e^{2\widetilde{C}+1}L$), this one is the most restrictive for $L \geqslant 1$ but it still allows one to choose $K$ below or at some fixed constant multiple of $L \geqslant 1$, so the desired estimate follows. $\qquad\square$

Now, it remains to present a symmetric discrete measure $\mu$ with a spectral gap that satisfies the assumptions of Theorem 2. We will merely take for $\mu$ the measure whose restriction to $[0, +\infty)$ is $\sum_{n\geqslant 2} \frac{1}{n2^n} \sum_{k=1}^n \delta_{\frac{n+k}{4\pi n}}$. It is supported on $[-\frac{1}{2\pi}, \frac{1}{2\pi}] \setminus [-\frac{1}{4\pi}, \frac{1}{4\pi}]$

and satisfies (5) just by the Lagrange interpolation formula

$$
1 = P(0) = \sum_{k=1}^{n+1} \Big[ \prod_{1 \leqslant j \leqslant n+1, j \neq k} \frac{0 - \frac{n+1+j}{4\pi(n+1)}}{\frac{k-j}{4\pi(n+1)}} \Big] P\left( \frac{n+1+k}{4\pi(n+1)} \right)
$$

$$
\leqslant \sum_{k=1}^{n+1} \frac{(2n+2)!}{(k-1)!(n+1-k)!(n+1)!} \left| P\left( \frac{n+1+k}{4\pi(n+1)} \right) \right|
$$

$$
\leqslant 3^{2n+1}(2n+2) \sum_{k=1}^{n+1} \left| P\left( \frac{n+1+k}{4\pi(n+1)} \right) \right|
$$

$$
\leqslant 3^{2n+1} 2^{n+2} (n+1)^2 \int_{\mathbb{R}} |P| \, d\mu \leqslant 10^{3n} \sqrt{ \int_{\mathbb{R}} |P|^2 \, d\mu } \, .
$$

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