Persistence of Gaussian stationary processes: a spectral perspective

Naomi Feldheim [*]   Ohad Feldheim [†]   Shahaf Nitzan [‡]

Abstract

We study the persistence probability of a centered stationary Gaussian process on $\mathbb{Z}$ or $\mathbb{R}$, that is, its probability to remain positive for a long time. We describe the delicate interplay between this probability and the behavior of the spectral measure of the process near zero and infinity.

1 Introduction

1.1 General Introduction

The persistence of a stochastic process f above a certain level ℓ, that is, the probability that $f(t) > \ell$ for all t in some large interval, is a classical topic of study (see the recent survey by Aurzada-Simon [3]). Here we investigate the persistence probability for the class of Gaussian stationary processes (GSP's) above the mean. This quantity has been extensively studied since the 1950's, by Slepian [24], Newell-Rosenblatt [20] and many others, with old and new applications in mathematical physics, engineering and other areas of probability [8, 9, 23]. Nonetheless, until recently, good estimates of the persistence decay were known only for particular cases (e.g. [2, 24]), and for families of processes with either non-negative or summable correlations. The state of the art of the latter was recently achieved by Dembo-Mukherjee [9], who were able to determine the log persistence of non-negatively correlated GSP's up to a constant factor.

A few years ago, by introducing a spectral point of view, the first two authors were able to provide general conditions under which the log persistence is bounded between two linear functions [11]. This extended a result by Antezana-Buckley-Marzo-Olsen for the *sinc-kernel* process [2], and provided the first general result on persistence of GSP's which does not require summability or non-negativity of correlations. However, these tools alone were insufficient to provide answers to two long-standing questions formulated by Slepian in his well known 1962 paper [24]:

What are possible asymptotic behaviors of the persistence probability of a GSP on large intervals? What features of the covariance function determine this behavior?

Spectral methods were recently used by Krishna-Krishnapur [15] in order to give a lower bound of e^{-cN^2} on the persistence of any GSP over $\mathbb{Z}$, provided that the spectral measure has a non-trivial absolutely continuous part. This gave rise to other interesting questions, stated in [15] and related to us also by M. Sodin [25]:

Is there a GSP that achieves persistence of the order of e^{-cN^2}? Is it possible for a GSP over $\mathbb{R}$ to have even lower persistence?

In this paper we combine the spectral methods of [11] with tools from real and harmonic analysis in order to provide nearly complete answers to all of these questions, in the case that the spectral measure has a non–trivial absolutely continuous component. While our methods do not employ [2] directly, they are nonetheless inspired by the behavior of the sinc-kernel process. Our results promote a point of view which regards persistence as a spectral property, governed by the interplay between the spectral behavior near zero and near infinity.

1.2 Mathematical overview and discussion

Let $T \in \{\mathbb{Z}, \mathbb{R}\}$. A *Gaussian process* on T is a random function $f : T \to \mathbb{R}$ whose finite marginals, i.e. $(f(t_1), \ldots, f(t_n))$ for any $t_1, \ldots, t_n \in T$, have multi-variate centered Gaussian distribution. We say that f is *stationary* if its distribution is invariant under translations by elements of T. For an introduction to Gaussian processes see [1].

The persistence probability of a Gaussian stationary process (GSP) f on $[0, N]$ is defined by

$$P_f(N) := \mathbb{P}\Big(f(t) > 0, \ \forall t \in (0, N] \cap T \Big).$$

Notice that we consider persistence above the mean (the zero level). Our methods may be applied to study other constant levels, though we expect some qualitative differences in the results.

A GSP is determined uniquely by its covariance kernel $r(t) = \mathbb{E}[f(0)f(t)]$, $t \in T$. Throughout the paper we implicitly assume Gaussian stationary processes to be almost-surely continuous and with a continuous covariance kernel. Since r is continuous and positive-definite, Bochner's theorem implies that there exists a finite, symmetric, non-negative measure ρ on T^* (where $\mathbb{R}^* \simeq \mathbb{R}$ and $\mathbb{Z}^* \simeq [-\pi, \pi]$) such that

$$r(t) = \widehat{\rho}(t) = \int_{T^*} e^{-i\lambda t} \, d\rho(\lambda). \tag{1}$$

The measure ρ is called *the spectral measure* of the process f. It is well known that any finite, symmetric, non-negative measure on T^* corresponds to a unique GSP on T (see Lemma 3.7 for a construction).

While our main results are presented in Section 2, we state here a simplified version which demonstrates our findings for particularly well-behaved spectral measures. We write $A(N) \lesssim B(N)$

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to denote that $A(N) \leq C\, B(N)$ for some $C > 0$ and all N, and $A(N) \asymp B(N)$ to denote that both $A(N) \lesssim B(N)$ and $B(N) \lesssim A(N)$.

Theorem 1. *Let f be a GSP over $\mathbb{R}$ or $\mathbb{Z}$. Suppose that its spectral measure is absolutely continuous with density $w(\lambda)$ satisfying $\int |\lambda|^{\delta} w(\lambda) d\lambda < \infty$ for some $\delta > 0$, and $c_1 \lambda^{\alpha} \leq w(\lambda) \leq c_2 \lambda^{\alpha}$ for all λ in a neighborhood of 0 (and some $\alpha > -1$, $c_1, c_2 > 0$). Then:*

$$\log P_f(N) \begin{cases} \asymp -N^{1+\alpha} \log N, & \alpha < 0 \\ \asymp -N, & \alpha = 0 \\ \lesssim -N \log N, & \alpha > 0. \end{cases}$$

Moreover, if $w(\lambda)$ vanishes on an interval containing 0, then $\log P_f(N) \lesssim -N^2$. In this case, if the process is over $\mathbb{R}$ and it satisfies in addition $w(\lambda) \geq \lambda^{-\eta}$ for some $\eta > 0$ and all $|\lambda| > 1$, then $\log P_f(N) \leq -e^{CN}$.

In Section 2 we provide results which are more general than Theorem 1. In particular, we do not require ρ to have density, but rather some *non-trivial absolutely continuous component*. The results are then given by *the spectral mass near the origin $\rho([0, \frac{1}{N}])$.*

For measures with a non–vanishing absolutely continuous component, our results prove the *"spectral gap conjecture"* [15, 25]. This conjecture states that any process whose spectral measure vanishes on an interval around 0 should have persistence smaller than e^{-CN^2}. Prior to this paper there has not been any rigorous example of a GSP whose persistence decays faster than the order $e^{-CN \log N}$, although it was believed that the lower bound of e^{-CN^2} given by Krishna-Krishnapur in [15] for processes over $\mathbb{Z}$ should be attainable.

Interestingly, this stochastic result corresponds with the following analytic theorem of Eremenko-Novikov [10]: The Fourier transform of a measure with a spectral gap has a positive asymototic density of zeroes on the real line. Roughly speaking, both results reflect the idea that functions with a spectral gap have a strong tendency to oscillate. Another result in this flavor was obtained recently by Borichev-Sodin-Weiss [6]. They showed that any finite-valued stationary process on $\mathbb{Z}$ which has a spectral gap must be periodic, thus giving a probabilistic counterpart to a theorem by Helson (see references within [6]).

One may notice that no matching lower bound is given in Theorem 1 for the case $\alpha > 0$. Over $\mathbb{Z}$ we give such a matching lower bound (see Corollary 1 below), but over $\mathbb{R}$ this is impossible. This is due to an interesting phenomenon which happens only in continuous time: *When the spectral measure vanishes at 0, then the heavier is the spectral tail at infinity – the smaller is the persistence probability.* This phenomena is reflected in the estimate $\log P_f(N) \leq -e^{CN}$ appearing in Theorem 1, and a precise formulation of it appears in Corollary 5. A possible interpretation is that the heavy tail makes the process very rough, and this non-smoothness makes it even harder to stay positive (as oppose to a smooth process, for which positivity at a certain point makes it more likely for its

whole neighborhood to be positive). However, if the spectral measure is compactly supported, we believe matching lower bounds should hold (as is the case over $\mathbb{Z}$). This remains to be studied.

As noted earlier, one novelty in our work is the ability to capture persistence behavior without requiring absolute summability or non-negativity of correlations. For non-negatively correlated processes, that is, when $r(t) \geq 0$ for all $t \in T$, the asymptotic behavior of $P_f(N)$ could be obtained directly from $r(t)$ without using the spectral measure. This was done by Dembo-Mukherjee, first in [8] for the case $\alpha = 0$ and later in [9] for $\alpha < 0$ (using the notation of Theorem 1). Notice that, when $r(t) \geq 0$ the spectral measure at $\lambda = 0$ cannot vanish, so the case $\alpha > 0$ is impossible.

1.3 Outline of the paper

The rest of the paper is organized as follows.

Section 2 contains precise formulation of our results. The main results are general inequalities on persistence of GSPs – Theorem 2 (lower bound) and Theorem 3 (upper bound). Then we state several corollaries with explicit bounds. Theorem 1 is discussed in this section as well. We also present Theorem 4 which is an analytic tool in the flavor of persistence developed in this paper.

Section 3 contains useful tools, such as spectral properties and decompositions of GSPs, ball and tail estimates, and well-known Gaussian inequalities. Section 4 is dedicated to the proofs of lower bounds (Theorem 2, and Corollaries 1 and 2). Section 5 is dedicated to the proofs of upper bounds (Theorem 3 and Corollaries 3, 4 and 5). Finally, Section 6 contains the proof of the analytic (non-probabilistic) result, Theorem 4, which is used in Section 5.

2 Results

In this section we provide a more precise presentation of the results. Let f be a GSP over $\mathbb{R}$ or $\mathbb{Z}$ with spectral measure ρ. For $\delta \in \mathbb{R}$ denote the δ-moment of ρ by

$$m_\delta = m_\delta(\rho) := \int_{T^*} |\lambda|^\delta \, d\rho(\lambda).$$

Throughout, we assume that ρ is normalized and has some finite positive moment, that is,

$$\exists \delta > 0 : m_\delta < \infty, \quad \text{and} \quad m_0 = \rho(T^*) = 1. \tag{2}$$

To capture the spectral behavior near zero we will employ both negative moments and the total measure on small intervals which we denote by

$$\sigma_N^2 := \rho([0, \tfrac{1}{N}]), \quad \text{for } N > 0.$$

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These two ways to describe a measure near 0 are related, e.g. by the following (see Observation 3.1):

$$\begin{cases} \text{If } \sigma_N^2 \leq bN^{-(\gamma+\varepsilon)} \text{ for some } b, \varepsilon > 0 \text{ and all } N > 0, \text{ then } m_{-\gamma} < \infty. \\ \text{If } m_{-\gamma} < \infty, \text{ then } \sigma_N^2 \leq bN^{-\gamma} \text{ for some } b > 0 \text{ and all } N > 0. \end{cases} \tag{3}$$

The absolutely continuous component of ρ is denoted by ρ_{ac}, and the notation $\rho_{ac} \neq 0$ means that it is not trivial. The support of ρ_{ac} is denoted by $\mathrm{sprt}\,(\rho_{ac})$, and $|E|$ denotes the Lebesgue measure of a set E. We reserve the letter Z to denote a standard Gaussian random variable, i.e. $Z \sim \mathcal{N}(0,1)$.

2.1 General inequalities for persistence

We present here two general inequalities for persistence of GSPs.

Theorem 2 (general lower bound). *Let f be a GSP with spectral measure ρ obeying (2), and let $\delta > 0$ be such that $m_\delta = m_\delta(\rho) < \infty$. Then there exist*

$$\beta = \begin{cases} \beta(\delta, m_\delta), & T = \mathbb{R} \\ 2\sqrt{2}, & T = \mathbb{Z} \end{cases} \quad and \quad \ell_0 = \begin{cases} \ell_0(\delta, m_\delta), & T = \mathbb{R} \\ 0, & T = \mathbb{Z} \end{cases}$$

such that, for all $\ell > \ell_0$ and $N > 0$, we have:

$$P_f(N) \geq \mathbb{P}(\sigma_N Z > \ell) \cdot \mathbb{P}(\beta|Z| < \ell)^N.$$

Assume that f is a GSP whose spectral measure ρ satisfies $\rho_{ac} \neq 0$. This implies that there exist $\nu > 0$ and $E \subseteq \mathbb{R}$ such that E is a bounded set of positive measure on which $d\rho_{ac} \geq \nu\, dx$. We let q be such that $\frac{1}{q}E \subseteq [-\pi, \pi]$. Further, assume that $\gamma \geq 0$ is such that $m_{-\gamma}(\rho) < \infty$ and let $k \in \mathbb{N} \cup \{0\}$ and $0 \leq s < 2$ be such that $\gamma = 2k + s$. Put $r = \max\{k, s/2\}$. We have the following.

Theorem 3 (general upper bound). *There exist universal positive constants c_0, c_1, and a constant depending only on the variable s, $c(s)$, such that the following holds. Let f be a GSP with spectral measure ρ obeying (2), and let $E, \nu, q, \gamma, k, s, r$ be as described above. Set,*

$$\alpha = c_0|E|, \quad and \quad \beta = \begin{cases} (c_1 k)^{-k} \sqrt{\frac{\nu|E|}{m_{-2k}}}, & k > 0 \\ c(s)\sqrt{\frac{\nu|E|}{m_{-s}}}, & k = 0. \end{cases}$$

Then there exists $N_0(E) > 0$ such that for every $N > \max\{N_0, 4k\}$ and

$$\ell_0(N) = 2N^{-r}\max\{\sqrt{\tfrac{1}{2}\log(\tfrac{m_{-2k+2}}{2m_{-2k}}N^2)}, 1\}$$

the following holds for all $\ell > \ell_0(N)$:

$$P_f(N) \leq \mathbb{P}(N^{-r}Z > \ell) + 2qN\,\mathbb{P}(\beta|Z| < \ell)^{\alpha N}.$$

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Remark 1. *Over* $\mathbb{R}$, *if* E *contains an interval* J, *then one can take* $N_0 = \frac{c}{|J|}$ *with a universal* $c > 0$.

Theorems 2 and 3 give a recipe for estimating $P_f(N)$: given N and ρ, one should choose a level $\ell = \ell(N, \rho)$ so that the factors in the relevant estimate are of the same order. This recipe is used to derive the explicit bounds which follow.

2.2 Lower bounds

Here we provide explicit lower bounds for the persistence probability, which will be derived from Theorem 2 in the end of Section 4. Throughout, we assume f be a GSP with spectral measure ρ satisfying the conditions in (2).

Corollary 1 (explicit lower bounds). *Assume that* $m_\delta < \infty$ *for some* $\delta > 0$. *If* $\sigma_N^2 \geq bN^{-\gamma}$ *holds on a subsequence of* N, *where* $b, \gamma > 0$ *are some constants, then there exists* $C, N_0 > 0$ *such that along this subsequence, for* $N > N_0$,

$$
\log P_f(N) \geq \begin{cases} -CN^\gamma \log N, & \gamma < 1 \\ -CN, & \gamma = 1 \\ -C(\gamma - 1)N \log N, & \gamma > 1, \ T = \mathbb{Z}. \end{cases}
$$

Here N_0 *and* C *depend only on* $b, \gamma, \delta, m_\delta$.

Note that in the case $\gamma > 1$, which corresponds to vanishing spectrum at the origin, we give a lower bound only over $T = \mathbb{Z}$. As was discussed in Section 1.2, a similar lower bound is not true over $\mathbb{R}$ (see Corollary 5 below). However, over $\mathbb{Z}$ we obtain additional estimates in the case of deeply vanishing spectrum at the origin.

Corollary 2 (lower bounds for vanishing spectrum). *Over* $T = \mathbb{Z}$, *assume that for a certain* N *we have* $N\sigma_N^2 < 1$. *Then for some universal* $C > 0$ *we have*

$$
P_f(N) \geq \exp\left(CN \log(N\sigma_N^2)\right).
$$

In particular, if $\sigma_N^2 \geq e^{-AN^\alpha}$ for some $A > 0$ and $0 < \alpha \leq 1$, then $\log P_f(N) \geq -C'N^{1+\alpha}$. Note that Corollary 2 reproduces the cases $\gamma = 1$ and $\gamma > 1$ of Corollary 1 (over $\mathbb{Z}$). However, it does not capture the lower bound of e^{-cN^2} which holds for any spectral measure with density, as proven by Krishna-Krishnapur in [15].

2.3 Upper bounds

Here we state explicit upper bounds for the persistence probability, which will be derived from Theorem 3 in the end of Section 5. Throughout, we assume f be a GSP with spectral measure ρ satisfying the conditions in (2).

Corollary 3 (explicit upper bounds). *Assume that $\rho_{ac} \neq 0$ and $m_{-\gamma} < \infty$ for some $\gamma \geq 0$. Then there exist $N_0, C > 0$ such that for all $N > N_0$:*

$$\log P_f(N) \leq \begin{cases} -CN^\gamma \log N, & \gamma < 1 \\ -CN, & \gamma = 1 \\ -C(\gamma - 1)N \log N, & \gamma > 1. \end{cases}$$

Here N_0 and C *depend on $\gamma, m_{-\gamma},$* $m_{-2k}, m_{-s},$ *ν, E, where E is a set on which $d\rho_{ac} \geq \nu dx$ and $\gamma = 2k + s$ with $k \in \mathbb{N}_0$ and $s < 2$.* The dependence of C on E is linear in $|E|$.

If the spectral measure has an infinite order zero at the origin, then the persistence becomes much smaller. In particular, the estimate e^{-cN^2} for the persistence of a process with a spectral gap is implied by the following Corollary. In addition, this corollary provides an interpolation between this estimate and the results stated in Corollary 3.

Corollary 4 (upper bounds for vanishing spectrum). *Assume that $\rho_{ac} \neq 0$. Then for large enough N we have*

$$\log P_f(N) \leq -CNk(N) \log \left(\frac{cN}{k(N)} \right),$$

where C and c are positive constants depending on ρ, and

$$k(N) := \max \left\{ k \in \mathbb{N} \cap (0, N] : \ k \, m_{-2k}^{1/k} \leq N \right\}.$$

In particular, for any $A > 0$ we have:

- If $\rho \equiv 0$ on $[-\frac{1}{A}, \frac{1}{A}]$ then $m_{-k} < A^k$ for all $k > 0$ and $\log P_f(N) \leq -C_1 N^2$.

- If $m_{-2k} < k^{Ak}$ for all $k > 0$, then $\log P_f(N) \leq -N^{1+\frac{1}{1+A}} \log N$.
  (e.g. spectral density $e^{-\frac{1}{|\lambda|^A}} \mathbf{1}_{[-1,1]}(\lambda)$.)

- If $m_{-2k_0} < \infty$ for some k_0 but $m_{-2k} = \infty$ for $k > k_0$, then $\log P_f(N) \leq -C_3 N \log N$.
  (this is implied also by the case $\gamma > 1$ of Corollary 3.)

The next result shows that over $\mathbb{R}$, if the spectral measure vanishes at zero and has a heavy enough tail at infinity, then the persistence probability is tiny.

Corollary 5 (tiny persistence). *Let f be a GSP over $\mathbb{R}$, whose spectral measure ρ has an absolutely continuous component with density $w(\lambda)$. Let $\alpha > 0$. There exists a constant $C > 0$ such that the following hold for large enough N.*

1. *If $\rho = 0$ on $[-1, 1]$ and $w(\lambda) \geq \lambda^{-\alpha}$ for $|\lambda| > 1$, then $\log P_f(N) \leq -e^{CN}$.*

2. *If $m_{-2}(\rho) \leq \infty$ and $w(\lambda) \geq \lambda^{-\alpha}$ for $|\lambda| > 1$, then $\log P_f(N) \leq -CN^{1+\frac{1}{\alpha}} \log N$.*

Using our methods one can generate many more such examples, for instance, it is possible to get $P_f(N) \leq \exp(-e^{C\sqrt{N}})$ with only first-order vanishing at 0 (by using a tail of $\frac{1}{\lambda \log^2 \lambda}$ on $[1, \infty)$).

2.4 An analytic result

Finally, we present an analytic result (in the flavor of persistence) which was developed for this work and plays a role in proving our upper bounds. It quantifies the fact that, if a function on an interval takes values in $[-L, L]$ and has positive k-th derivative, then the average of this k-th derivative cannot be too large with respect to L. The statement holds true both on $T = \mathbb{Z}$ and on $T = \mathbb{R}$, where the notion of derivative over $\mathbb{Z}$ is the usual discrete one (see (31) below).

Theorem 4. *Let* $T \in \{\mathbb{Z}, \mathbb{R}\}$, $N > 0$. *Fix* $k \in \mathbb{N}$ *such that* $k \leq N$. *Suppose that* $f : T \to \mathbb{R}$ *is* k-*times differentiable and* $f^{(k)} > 0$ *on* $[-N, N] \subset T$. *Then*

$$\frac{1}{N} \int_{-\frac{9}{20}N}^{\frac{9}{20}N} f^{(k)} \leq \left(\frac{c_0 k}{N} \right)^k \sup_{[-N,N]} |f|,$$

where $c_0 > 0$ *is a universal constant.*

The proof appears in Section 6 and employs results from approximation theory, as we show that, in fact, the k-th degree Chebyshev polynomial is in some sense extremal for this inequality.

2.5 A remark regarding the proof of Theorem 1

Lastly, we remark on the proof of Theorem 1. The lower bounds in Theorem 1 follow immediately from Corollary 1 and, in the case that $\alpha > 0$, Corollary 3 implies the upper bound in Theorem 1. However, in the cases where $\alpha \leq 0$, this corollary gives the upper bounds only up to a small error. In fact, in the latter cases one can obtain a slightly better version of Corollary 3, which in particular will imply the upper bounds in Theorem 1. This version is obtained by replacing the negative moment condition with an appropriate condition on σ_N (recalling (3)). We omit the formulation of this version and its proof to avoid a cumbersome presentation (the proof is very similar to the proofs presented in this paper).

3 Preliminaries

This section contains tools which will be used throughout our proofs. These tools include properties implied by finite spectral moments, decompositions of a GSP, basic calculus for GSPs (that is, properties of its derivative and anti-derivative processes), ball and tail estimates, and some famous Gaussian inequalities. We begin with some notations and then sort our tools by topic.

3.1 Notations

Recall that we let T be either $\mathbb{R}$ or $\mathbb{Z}$ and correspondingly, we let T^* be $\mathbb{R}$ or $[-\pi, \pi]$. Let ρ be a positive, symmetric, and finite measure over T^* and denote by $\mathcal{L}^2_\rho(T^*)$ the Hilbert space of functions $\{\varphi : \int |\varphi|^2 d\rho < \infty\}$ with the inner product $\langle \varphi_1, \varphi_2 \rangle = \int_{T^*} \varphi_1 \overline{\varphi_2} d\rho$. In case ρ has density $\mathbb{I}_E$ with $E \subseteq T^*$ being a compact, symmetric measurable set, we abbreviate this space by $\mathcal{L}^2_E$.

Unless stated otherwise, we always assume $f : T \to \mathbb{R}$ to be a GSP with spectral measure ρ and covariance function r. As before, for $\delta \in \mathbb{R}$ we denote the δ-moment of ρ by $m_\delta = \int_{T^*} |\lambda|^\delta \, d\rho(\lambda)$. Recall that we assume $m_\delta < \infty$ for some $\delta > 0$.

We denote by $[N]$ the set of integers $\{1, 2, \ldots, N\} = (0, N] \cap \mathbb{Z}$. A set of integers $\Lambda \subset \mathbb{Z}$ is said to have positive (central) density if $D^-(\Lambda) := \liminf_{N \to \infty} \frac{|\Lambda \cap [-N, N]|}{2N} > 0$.

The symbol $\overset{d}{=}$ indicates equality in distribution (between two random variables or processes on the same space). The symbol $\oplus$ is used to sum two independent processes.

3.2   Spectral moments

Below are a few observations regarding spectral moments. The first relates negative moments with the spectral mass near 0, as was stated in (3).

Observation 3.1. *Let $\gamma, \varepsilon > 0$.*

  i. If $\rho([0, \lambda]) \leq b\lambda^{\gamma+\varepsilon}$ for some $b > 0$ and all $\lambda > 0$, then $m_{-\gamma} < \infty$.

  ii. If $m_{-\gamma} < \infty$ then $\rho([0, \lambda]) \leq b\lambda^\gamma$ with $b = \frac{1}{2}m_{-\gamma}$.

Proof. For the first item, assume that $\rho([0, \lambda]) \leq b\lambda^{\gamma+\varepsilon}$ and use integration by parts:

$$\int_0^\infty \frac{d\rho(\lambda)}{\lambda^\gamma} = -\lim_{\lambda \to 0} \frac{\rho([0, \lambda])}{\lambda^\gamma} + \gamma \int_0^\infty \frac{\rho([0, \lambda])}{\lambda^{\gamma+1}} d\lambda$$
$$\leq 0 + \gamma \left(b \int_0^1 \lambda^{-1+\varepsilon} d\lambda + \int_1^\infty \frac{d\lambda}{\lambda^{1+\gamma}} \right) < \infty.$$

For the second part, notice that $\int_0^\infty \frac{d\rho(\lambda)}{\lambda^\gamma} \geq \frac{1}{\tau^\gamma} \int_0^\tau d\rho(\lambda)$ for any $\tau > 0$. $\qquad\square$

The second observation relates positive moments with the behavior of the covariance function near 0.

Observation 3.2. *If $m_\delta < \infty$ for some $\delta \in (0, 2]$, then*

$$\forall t \in T : \quad r(t) \geq r(0) - C(\delta)m_\delta |t|^\delta,$$

where $C(\delta)$ is a positive constant depending only on δ. In particular, $C(2) = \frac{1}{2}$.

Proof. By definition $r(t)$ is symmetric and we have:

$$r(0) - r(t) = \int_{\mathbb{R}} \left(1 - \cos(\lambda t) \right) d\rho(\lambda) \leq C|t|^\delta \int_{\mathbb{R}} |\lambda|^\delta d\rho(\lambda),$$

where $C = C(\delta) = \sup_{x \in \mathbb{R}} \frac{1 - \cos(x)}{|x|^\delta} < \infty$ is finite when $\delta \in (0, 2]$. In particular, $C(2) = \frac{1}{2}$. $\qquad\square$

We conclude with a property of negative moments.

Claim 3.3. *For any spectral measure ρ, there exists $\tau \in (0, \infty)$ such that $m_{-2k+2} \leq \tau \cdot m_{-2k}$ for all $k \in \mathbb{N}$.*

Proof. Let $M > 1$ be such that $\int_0^M d\rho \geq \frac{1}{2} \int_0^\infty d\rho$. This implies that $\int_0^M d\rho \geq \int_M^\infty d\rho$ and so, we have for every $j \in N$,

$$\int_0^M \frac{d\rho(\lambda)}{\lambda^j} \geq \frac{1}{M^j} \int_0^M d\rho(\lambda) \geq \frac{1}{M^j} \int_M^\infty d\rho(\lambda) \geq \int_M^\infty \frac{d\rho(\lambda)}{\lambda^j}.$$

It follows that for every $j \in N$ we have $\int_0^M \frac{d\rho(\lambda)}{\lambda^j} > \frac{1}{2} \int_0^\infty \frac{d\rho(\lambda)}{\lambda^j}$. We can therefore conclude that

$$m_{-2k} \geq \int_0^M \frac{d\rho(\lambda)}{\lambda^{2k}} \geq \frac{1}{M^2} \int_0^M \frac{d\rho(\lambda)}{\lambda^{2k-2}} \geq \frac{1}{2M^2} \int_0^\infty \frac{d\rho(\lambda)}{\lambda^{2k-2}} = \frac{1}{2M^2} \, m_{-2k+2}.$$

$\square$

3.3 Decomposition of a GSP

3.3.1 Spectral decomposition

Assume that $\rho_{ac} \neq 0$. This condition may be written more explicitly as follows: There exists a number $\nu > 0$ and a compact measurable set $E \subset T^*$ such that

$$d\rho = \nu \mathbb{1}_E(\lambda) d\lambda + d\mu, \quad \text{where } \mu \text{ is a non-negative measure.} \tag{4}$$

By rescaling f we will assume that $E \subset [-\pi, \pi]$ (clf. Proposition 5.1). The next decomposition is an extension of [11, Obs. 2], where it was assumed that E is an interval.

Claim 3.4. *Suppose that condition (4) holds with $E \subseteq [-\pi, \pi]$. Then, there exist $\Lambda = \{\lambda_n\} \subset \mathbb{Z}$ of positive density A, and a constant $B > 0$, such that $f(\lambda_n) \overset{d}{=} BZ_n \oplus g_n$, where Z_n are i.i.d. standard Gaussian random variables and g_n is a Gaussian process on $\mathbb{Z}$.*
Moreover, given $\varepsilon > 0$ one can have $A = (1 - \varepsilon)\frac{|E|}{2\pi}$, $B = \sqrt{c(\varepsilon)\nu|E|}$, where $c(\varepsilon) > 0$ is a constant depending only on ε.

The proof of Claim 3.4 is based on the following result by Bourgain and Tzafriri, which is a consequence of their celebrated "Restricted Invertibility Theorem" [4]. The 'moreover' part is due to Vershynin [26, Thm. 1.5].

Theorem A (Bourgain, Tzafriri, Vershynin). *Let $E \subseteq [-\pi, \pi]$ be a set of positive Lebesgue measure. Then, there exist $\Lambda = \{\lambda_n\} \subset \mathbb{Z}$ and constants $A, D > 0$, such that:*

(i) $\forall \{a_n\} \in l^2(\mathbb{Z}): \quad D \sum |a_n|^2 \leq \left\| \sum a_n e^{-i\lambda_n x} \right\|_{L^2(E)}^2 \leq \sum |a_n|^2.$

(ii) $\liminf_{N \to \infty} \frac{|\Lambda \cap [-N, N]|}{2N} > A.$

Moreover, given $\varepsilon > 0$ one can have $A = (1 - \varepsilon)\frac{|E|}{2\pi}$ and $D = c(\varepsilon)|E|$, where $c(\varepsilon) > 0$ is a constant depending only on ε.

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Proof of Claim 3.4. Let $\varepsilon > 0$, and let $\Lambda = \{\lambda_n\} \subset \mathbb{Z}$ be the sequence whose existence is guaranteed by Theorem A. We have:

$$\left\| \sum_n a_n e^{-i\lambda_n x} \right\|_{L_\rho^2}^2 = \sum_{n,m} a_n a_m \int e^{-i(\lambda_n - \lambda_m)x} d\rho = \sum_{n,m} a_n a_m r(\lambda_n - \lambda_m) = a^T \Sigma a,$$

where $\Sigma = (r(\lambda_n - \lambda_m))_{n,m \in \mathbb{Z}}$ is the infinite covariance matrix of the Gaussian process $(f(\lambda_n))_{n \in \mathbb{Z}}$. We note that this matrix is symmetric, as r is symmetric. By item (i) of Theorem A, and condition (4), we have for all $\{a_n\} \in l^2(\mathbb{Z})$,

$$D\nu \sum |a_n|^2 \leq \nu \left\| \sum a_n e^{-i\lambda_n x} \right\|_{L^2(E)}^2 \leq \left\| \sum a_n e^{-i\lambda_n x} \right\|_{L_\rho^2}^2 = a^T \Sigma a.$$

It follows that $\Sigma - \nu DI$ defines a positive-definite operator on $\ell^2(\mathbb{Z})$ (here $I(n,m) = \mathbb{1}\{n = m\}$ is the identity). Therefore, it is the covariance of some Gaussian process $g : \mathbb{Z} \to \mathbb{R}$ (see e.g. [18, Sec. 4, Thm. 2]). We obtain that

$$f(\lambda_n) \overset{d}{=} \sqrt{\nu D} Z_n \oplus g_n, \quad Z_n \sim \mathcal{N}(0,1) \text{ i.i.d.}$$

This establishes the result with $\beta = \sqrt{\nu D} = \sqrt{c(\varepsilon)\nu|E|}$. $\qquad\square$

The decomposition of Claim 3.4 will be useful to us together with the two following claims, the first of which appeared in [11, Prop. 3.1].

Claim 3.5. *Let $(Z_j)_{j \in \mathbb{Z}}$ be i.i.d. Gaussian random variables. Let $q, b_1, \ldots, b_N \in \mathbb{R}$ be numbers such that $\frac{1}{N} \sum_{j=1}^N b_j \leq q$. Then:*

$$\mathbb{P}\left(Z_j + b_j \geq 0,\ 1 \leq j \leq N\right) \leq \mathbb{P}(Z_1 \leq q)^N.$$

Proof. Write $\Phi(b) := \mathbb{P}(Z_1 \leq b) = \mathbb{P}(Z_1 \geq -b)$. One may check that $x \mapsto \log \Phi(x)$ is monotone and concave (for $x > 0$ it is straightforward, for $x \leq 0$ one should use the tail estimate in Lemma 3.14(a)). Thus we have:

$$\log \mathbb{P}\left(Z_j + b_j \geq 0,\ 1 \leq j \leq N\right) = \sum_{j=1}^N \log \Phi(b_j) \leq N \cdot \log \Phi\left(\frac{1}{N} \sum_{j=1}^N b_j\right) \leq N \cdot \log \Phi(q).$$

$\qquad\square$

Claim 3.6. *Let $N \in \mathbb{N}$ and let $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{R}$ be a function which satisfies $\frac{1}{N} \sum_{n=0}^{N-1} f(n) \leq L$, for some $L \in \mathbb{R}$. Then, for every $S \subseteq \mathbb{Z}/N\mathbb{Z}$ there exists $\tau \in \mathbb{Z}/N\mathbb{Z}$, such that*

$$\frac{1}{|S|} \sum_{n \in S} f(n + \tau) \leq L.$$

Proof. Let $g : \mathbb{Z}/N\mathbb{Z} \to \mathbb{R}$ be the function defined by $g(\tau) = \frac{1}{|S|} \sum_{n \in S} f(n+\tau)$. Then $\frac{1}{N} \sum_{\tau=0}^{N-1} g(\tau) \leq L$, which implies that there exists $\tau \in \mathbb{Z}/N\mathbb{Z}$ such that $g(\tau) \leq L$. $\qquad \square$

3.3.2 Hilbert decomposition

We turn to a different type of decomposition. The next proposition gives a classical *basis representation* of GSP's.

Lemma 3.7. *Let ρ be a* symmetric, non-negative *measure on T^* with a finite positive moment, and let φ_n be an orthonormal basis in $\mathcal{L}_\rho^2(T^*)$ which satisfies, for every $n \in \mathbb{N}$, $\varphi_n(-\lambda) = \overline{\varphi_n(\lambda)}$. Denote $\psi_n(t) = \int_{\mathbb{R}} e^{-i\lambda t} \varphi_n(\lambda) d\rho(\lambda)$. Then*

$$f(t) = \sum_n \zeta_n \psi_n(t), \quad \zeta_n \sim \mathcal{N}(0,1) \ i.i.d.$$

is a continuous GSP over T with spectral measure ρ.

We note that every space $\mathcal{L}_\rho^2(T^*)$, with a measure ρ satisfying the requirements above, admits such an orthonormal basis. In this case, the condition on the elements of the basis, $\varphi_n(-\lambda) = \overline{\varphi_n(\lambda)}$, implies that the functions $\psi_n(t)$ are real.

Proof. Standard arguments (see [13, Chapter 3, Thm. 2] or [12, Lemma 2.2.3]) yield that the series defining f converges almost surely to a Gaussian function, with covariance

$$K(t,s) = \mathbb{E}[f(t)f(s)] = \sum_n \psi_n(t)\psi_n(s).$$

Denote $e_t(\lambda) = e^{i\lambda t}$ then, since $\{\varphi_n\}$ is an orthonormal basis in $\mathcal{L}_\rho^2(T^*)$, we have

$$\sum_n \psi_n(t)\psi_n(s) = \sum_n \langle \varphi_n, e_t \rangle_{\mathcal{L}_\rho^2(T^*)} \langle \varphi_n, e_s \rangle_{\mathcal{L}_\rho^2(T^*)} = \langle e_t, e_s \rangle_{\mathcal{L}_\rho^2(T^*)} = \widehat{\rho}(t-s).$$

Thus f is stationary with spectral measure ρ. Almost sure continuity of f follows from the moment condition on ρ (in fact, it follows from the weaker condition $\int \log^{1+\varepsilon}(1+|\lambda|)d\rho(\lambda) < \infty$ for some $\varepsilon > 0$, see [1, Ch. 1.4.1]). $\qquad \square$

From Lemma 3.7 we deduce the following claim.

Claim 3.8. *Let $\varphi \in \mathcal{L}_\rho^2$ be a real symmetric function such that $\|\varphi\|_{\mathcal{L}_\rho^2} = 1$. Write $\psi(t) = \int_{\mathbb{R}} e^{-i\lambda t} \varphi(\lambda) d\rho(\lambda)$, then we have the decomposition*

$$f(t) \stackrel{d}{=} \zeta \cdot \psi(t) \oplus g,$$

where $\zeta \sim \mathcal{N}(0,1)$ and g is a Gaussian process which is independent of ζ.

Proof. Such a function φ can be completed into a basis of L_ρ^2, which satisfies the conditions of Lemma 3.7. The result immediately follows. $\qquad \square$

3.4 Calculus of GSP's

Next we discuss the relationship between the spectral measure and differentiation or integration. The derivative of a function $f : T \to \mathbb{R}$, where $T \in \{\mathbb{Z}, \mathbb{R}\}$, is defined to be

$$f'(t) = \begin{cases} \lim_{\delta \to 0} \frac{f(t+\delta) - f(x)}{\delta}, & T = \mathbb{R} \\ f(t+1) - f(t), & T = \mathbb{Z}. \end{cases}$$

Observation 3.9 (derivative). *Suppose that $m_\delta < \infty$ for some $\delta > 2$. Then f is a.s. continuously differentiable, and f' is a GSP with spectral measure μ defined by:*

$$d\mu(\lambda) = \begin{cases} \lambda^2 d\rho(\lambda), & T = \mathbb{R} \\ 2(1 - \cos\lambda) d\rho(\lambda) & T = \mathbb{Z}. \end{cases} \tag{5}$$

Proof. In the case $T = \mathbb{R}$, the fact that f' exists a.s. and is a stationary continuous Gaussian process follows from our moment condition (clf. [1, Ch. 1.4.1]). Differentiating the relation $\mathbb{E}[f(t)f(s)] = \int_\mathbb{R} e^{-i\lambda(t-s)} d\rho(\lambda)$ once by t and once by s yields

$$\widehat{\mu}(t - s) = \mathbb{E}[f'(t)f'(s)] = \int_\mathbb{R} e^{-i\lambda(t-s)} \lambda^2 d\rho(\lambda).$$

In the case $T = \mathbb{Z}$, differentiability is immediate and we compute:

$$\begin{aligned} \widehat{\mu}(m - n) = \mathbb{E}[f'(m)f'(n)] &= \mathbb{E}[(f(m+1) - f(m))(f(n+1) - f(n))] \\ &= 2r(m-n) - r(m-n+1) - r(m-n-1) \\ &= \int (2 - e^{-i\lambda} - e^{i\lambda}) e^{-i\lambda(m-n)} d\rho(\lambda) = \int e^{-i\lambda(m-n)} 2(1 - \cos\lambda) d\rho(\lambda). \end{aligned}$$

$\square$

Observation 3.10 (stationary anti-derivative). *Suppose that $m_{-2} < \infty$ and $m_\delta < \infty$ for some $\delta > 0$. Then there exists a GSP $F : T \to \mathbb{R}$ such that $F' \stackrel{d}{=} f$.*

Proof. Let μ be the measure defined by

$$d\mu(\lambda) = \begin{cases} \frac{d\rho(\lambda)}{\lambda^2}, & T = \mathbb{R} \\ \frac{d\rho(\lambda)}{2(1-\cos\lambda)} & T = \mathbb{Z}. \end{cases} \tag{6}$$

By the premise μ is a finite, non-negative, symmetric measure, and therefore defines a GSP which we denote by F. Moreover, in the case $T = \mathbb{R}$ we have $m_{2+\delta}(\mu) = m_\delta(\rho) < \infty$. Thus by Observation 3.9 it follows that F is a.s. continuously differentiable and $F' \stackrel{d}{=} f$. $\square$

Observation 3.10 asserts that if $m_{-2} < \infty$ then the anti–derivative process is stationary, and in particular its variance is uniformly bounded. The next lemma, which is a generalization of

Proposition 3.2 in [11], provides estimates for the variance of the anti-derivative of a GSP even when the latter is not stationary. We formulate and prove it in continuous time, noting that the discrete analogue follows by simple modifications.

Lemma 3.11 (general anti-derivative). *Let $b, \gamma \geq 0$ be such that $\gamma < 2$. Suppose that $\rho([0, \lambda]) \leq b\lambda^\gamma$ for all $\lambda > 0$. Then for all $N > 0$:*

$$\mathrm{var}\left(\int_0^N f(t)dt\right) \leq b\, C(\gamma)\, \mathrm{var}\,(f(0)) \cdot N^{2-\gamma},$$

where $C(\gamma) = \frac{8}{2-\gamma}$.

Proof. Without loss of generality assume $\mathrm{var}\,(f(0)) = 1$. We calculate the variance:

$$\mathrm{var}\left(\int_0^N f(t)dt\right) = \mathbb{E}\left[\left(\int_0^N f(t)dt\right)^2\right] = \iint_{[0,N]^2} \mathbb{E}(f(t)f(s))dt\,ds$$

$$= \int_0^N \int_0^N \widehat{\rho}(t-s)dt\,ds = N \int_{|t|<N} \left(1 - \frac{|t|}{N}\right)\widehat{\rho}(t)dt.$$

The change in order of integration and expectation follows from Fubini's theorem. The inverse Fourier transform of $N(1 - \frac{|t|}{N})\mathrm{I\!I}_{[-N,N]}(t)$ is given by $\frac{N^2}{2\pi} \mathrm{sinc}^2(\frac{N}{2}\lambda)$ where $\mathrm{sinc}(x) = \frac{\sin x}{x}$ (the definition of Fourier transform is given in (1)). We use the estimate

$$\mathrm{sinc}^2(N\lambda/2) \leq \begin{cases} 1, & |\lambda| \leq \frac{1}{N}, \\ 4(N\lambda)^{-2}, & |\lambda| > \frac{1}{N}, \end{cases}$$

combined with Parseval's formula [14, Sec 2.2], to get for $\gamma \neq 2$:

$$\mathrm{var}\left(\int_0^N f(t)dt\right) = N^2 \int_{\mathbb{R}} \mathrm{sinc}^2(N\lambda/2)d\rho(\lambda)$$

$$\leq N^2\rho([0, \tfrac{1}{N}]) + 4 \int_{1/N}^\infty \frac{d\rho(\lambda)}{\lambda^2}$$

$$\leq N^2\rho([0, \tfrac{1}{N}]) + 4 \left(2 \int_{1/N}^\infty \frac{\rho([0,\lambda])}{\lambda^3}d\lambda - N^2\rho([0, \tfrac{1}{N}])\right)$$

$$\leq 8b \int_{1/N}^\infty \lambda^{\gamma-3}d\lambda \leq \frac{8b}{2-\gamma}N^{2-\gamma}.$$

$\square$

Lastly we need estimates on the supremum of a GSP and its anti-derivative. We achieve this using Dudley's metric entropy bound [1, Thm. 1.3.3], which reads as follows. For a Gaussian process H on I, we define a canonical semi-metric by $d_H(a, b) := \sqrt{\mathbb{E}(H(a) - H(b))^2}$. For any $\varepsilon > 0$, let $N(\varepsilon)$ be the minimal number of d_H-balls of radius ε which cover I. Then Dudley's bound

states that there exists a universal constant $K > 0$ such that

$$\mathbb{E}\sup_I H \leq K \int_0^{\mathrm{diam(I)}} \sqrt{\log N(\varepsilon)} d\varepsilon, \tag{7}$$

where diam(I) is the diameter of I under d_H. The following lemmas are applications of this bound.

Lemma 3.12. *Let f be a GSP with given $m_0 = r(0)$ and $m_2 < \infty$. Denote $a = \sqrt{\frac{m_2}{m_0}}$. Then there is a universal constant $K > 0$ such that for all $N > 1$ we have*

$$\mathbb{E}\left(\sup_{[0,N]} f\right) \leq K\sqrt{m_0 \cdot \max\{\log(aN), 1\}},$$

Proof. By stationarity we have $d_f(x,y) = \sqrt{2(r(0) - r(x-y))}$, so that $\mathrm{diam}([0,N]) \leq \sqrt{2m_0}$ and by Observation 3.2, $d_f(x,y) \leq \sqrt{m_2}|x-y|$. Thus $N(\varepsilon) \leq \max(1, \sqrt{m_2}\frac{N}{\varepsilon})$, and therefore if $2\sqrt{2m_0} \leq \sqrt{m_2}N$, then by Dudley's bound

$$\mathbb{E}\sup_{[0,N]} f \leq K \int_0^{\sqrt{2m_0}} \sqrt{\log\left(\frac{\sqrt{m_2}N}{\varepsilon}\right)} d\varepsilon = K\sqrt{m_2}N \int_{\sqrt{\frac{m_2}{2m_0}}N}^{\infty} \frac{\sqrt{\log u}}{u^2} du.$$

Note that for $A > 1$ one has:

$$\int_A^{\infty} \frac{\sqrt{\log u}}{u^2} du = \frac{\sqrt{\log A}}{A} + \frac{1}{2}\int_A^{\infty} \frac{1}{u^2\sqrt{\log u}} du \leq \frac{\sqrt{\log A}}{A}\left(1 + \frac{1}{2\log A}\right),$$

which implies that under these conditions $\mathbb{E}\sup_{[0,N]} f \leq \widetilde{K}\sqrt{m_0 \log(aN)}$ where $a = \sqrt{\frac{m_2}{2m_0}}$ and $\widetilde{K}$ is some universal constant. On the other hand, if $2\sqrt{2m_0} \geq \sqrt{m_2}N$ Dudley's bound gives

$$\mathbb{E}\sup_{[0,N]} f \leq K \int_0^{\sqrt{m_2}N} \sqrt{\log\left(\frac{\sqrt{m_2}N}{\varepsilon}\right)} d\varepsilon = K\sqrt{m_2}N \int_1^{\infty} \frac{\sqrt{\log u}}{u^2} du \leq \widetilde{K}\sqrt{m_0},$$

so the desired bound holds. $\qquad\square$

Lemma 3.13. *Let $b > 0$ and $0 \leq \gamma < 2$. Let f be a GSP such that $\rho([0,\lambda]) \leq b\lambda^{\gamma}$ for all $\lambda > 0$, and $m_0 = \int d\rho$. Then there exists $c(\gamma) > 0$ such that for all $N > 1$:*

$$\mathbb{E}\sup_{x\in[0,N]}\left(\int_0^x f(t)dt\right) \leq c(\gamma)\sqrt{bm_0}\,N^{1-\frac{\gamma}{2}}.$$

Proof. Denote $H(x) = \int_0^x f(t)dt$. Then H is a Gaussian process, whose canonical semi-metric may be bounded by Lemma 3.11:

$$d_H(x,y) = \sqrt{\mathrm{var}\left(\int_x^y f(t)dt\right)} = \sqrt{\mathrm{var}\left(\int_0^{y-x} f(t)dt\right)} \leq C \cdot \max\{|y-x|^{1-\frac{\gamma}{2}}, 1\},$$

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where $C = c_0(\gamma)\sqrt{bm_0}$. From now on c_j, $j \in \mathbb{N}$, will denote a constant which depends only on γ. Denoting $\alpha := 1 - \frac{\gamma}{2}$, we see that $\mathrm{diam}([0, N]) \leq CN^\alpha$, and $N(\varepsilon) \leq \max\left(1, \frac{N}{(\varepsilon/C)^{1/\alpha}}\right)$ (by taking balls of Eucleadian radius $(\varepsilon/C)^{1/\alpha}$). By Dudley's bound (7), for $0 \leq \gamma < 2$ we have:

$$\mathbb{E}\sup_{x \in [0,N]} H(x) \leq K \int_0^{CN^\alpha} \sqrt{\log\left(\frac{C^{1/\alpha}N}{\varepsilon^{1/\alpha}}\right)} d\varepsilon = c_1(\gamma)CN^\alpha \int_1^\infty \frac{\sqrt{\log u}}{u^2} du = c_2(\gamma)\sqrt{bm_0}N^{1-\frac{\gamma}{2}}.$$

$\square$

3.5   Ball and tail estimates

The terms "ball" and "tail" events refer to a stochastic process remaining inside or outside a ball, respectively. These were immensely studied, see e.g. [17]. The following bounds, which will be repeatedly used, are ball and tail estimates for the one-dimensional Gaussian variable $Z \sim \mathcal{N}(0, 1)$.

Lemma 3.14. *For all $x > 0$:*

(a) $\frac{1}{\sqrt{2\pi}}\left(\frac{1}{x} - \frac{1}{x^3}\right)e^{-x^2/2} \leq \mathbb{P}(Z > x) \leq \frac{1}{\sqrt{2\pi}}\frac{1}{x}e^{-x^2/2}$.
In particular, for $x \geq 2$: $e^{-x^2} \leq \mathbb{P}(Z > x) \leq e^{-x^2/2}$.

(b) $\sqrt{\frac{2}{\pi}}xe^{-x^2/2} \leq \mathbb{P}(|Z| \leq x) \leq x$.
In particular, for $0 < x \leq 1$: $\frac{1}{4}x \leq \mathbb{P}(|Z| \leq x) \leq x$.

We omit the proof, as its first part is a standard bound on the Gaussian tail (see [1, eq. (1.2.2)]), while the second part follows from a straightforward integral estimate.

The estimates in Lemma 3.14 imply the following comparison of tail probabilities.

Claim 3.15. *For any $\delta > 0$ there exists $\theta > 0$ such that $\mathbb{P}(Z \leq x) \leq \mathbb{P}(|Z| \leq \theta x)$ for $x > \delta$.*

Proof. We first note that the inequality in the statement can be rewritten as $\mathbb{P}(Z > x) \geq \mathbb{P}(|Z| > \theta x)$. Let $\delta > 0$. There exists $\theta_1 = \theta_1(\delta)$ such that $\mathbb{P}(Z > 2) \geq \mathbb{P}(|Z| > \theta_1\delta)$. Set $\theta = \max\{2, \theta_1\}$. To show that the inequality above holds for all $x \geq \delta$ we consider two cases. First, assume that $x \geq 2$. Then, by part (a) of Lemma 3.14 we have

$$\mathbb{P}(Z > x) \geq e^{-x^2} \geq 2e^{-(2x)^2/2} \geq \mathbb{P}(|Z| > 2x) \geq \mathbb{P}(|Z| > \theta x).$$

Next, consider the case where $\delta \leq x \leq 2$. Then,

$$\mathbb{P}(Z > x) \geq \mathbb{P}(Z > 2) \geq \mathbb{P}(|Z| > \theta_1\delta) \geq \mathbb{P}(|Z| > \theta x).$$

$\square$

Now we turn to bound the ball probability of a Gaussian process. For discrete time this is given by the Khatri-Sidak inequality [19, Ch. 2.4], which is a particular case of the recently proved Gaussian correlation inequality [16, 21].

Lemma 3.16 (Khatri-Sidak). *for any $\ell > 0$ and any centered Gaussian vector Z one has:*

$$\mathbb{P}(|Z_j| \leq \ell, \quad \forall j \in [N]) \geq \prod_{j=1}^{N} \mathbb{P}(|Z_j| \leq \ell).$$

The next lemma extends this inequality to continuous time, provided that ℓ is large enough. We use the standard "chaining method", which is nicely presented in [17].

Lemma 3.17 (Large ball). *Let h be a GSP over $\mathbb{R}$ which satisfies $m_\delta < \infty$ with a given $\delta > 0$. Then there exist $\ell_0 > 0$ and $c \geq 1$ such that for all $\ell > \ell_0$ and all $N \in \mathbb{N}$ the following holds:*

$$\mathbb{P}\left(|h(t)| \leq \ell, \ \forall t \in [0, N]\right) \geq \mathbb{P}\left(c|h(0)| \leq \ell\right)^N.$$

The constants c and ℓ_0 depend only on δ, m_δ and m_0.

Proof. Assume, without loss of generality, that $m_0 = \operatorname{var} h(0) = 1$. Fix a number $c \geq 1$ (to be specified later) and let $\ell_0 \in \mathbb{N}$ be large enough such that, in particular,

$$e^{-\left(\frac{\ell_0}{c}\right)^2} \leq \tfrac{1}{12}. \tag{8}$$

By part (a) of Lemma 3.14, for any $\ell \geq \ell_0$ we have:

$$\mathbb{P}\left(|h(0)| \leq \frac{\ell}{c}\right)^N \leq (1 - 2e^{-\frac{\ell^2}{c^2}})^N \leq \exp(-2Ne^{-\frac{\ell^2}{c^2}}), \tag{9}$$

where in the right inequality we used the fact that $\log(1 - x) \leq -x$ for all $0 < x < 1$. Let $\alpha = 1 + \sum_{k=1}^{\infty} \frac{1}{k^2}$. For $k \in \mathbb{N}$, define the event

$$A_k = \left\{ \left|h(j2^{-k}) - h((j-1)2^{-k})\right| \leq \frac{\ell}{\alpha k^2}, \quad \forall j \in [N2^k] \right\},$$

while $A_0 = \left\{|h(j)| \leq \frac{\ell}{\alpha}, \quad \forall j \in [0, N]\right\}$. Since every real number equals $n + \sum_{k=1}^{\infty} \varepsilon_k 2^{-k}$ for some $n \in \mathbb{Z}$ and $\varepsilon_k \in \{0, 1\}$, the almost-sure continuity of h implies that,

$$\bigcap_{k \geq 0} A_k \subset \{|h(t)| \leq \ell, \ \forall t \in [0, N]\}.$$

Therefore,

$$\mathbb{P}\left(|h(t)| \leq \ell, \ \forall t \in [0, N]\right) \geq \mathbb{P}\left(\bigcap_{k \geq 0} A_k\right) = \lim_{K \to \infty} \mathbb{P}\left(\bigcap_{0 \leq k \leq K} A_k\right).$$

Now, by Observation 3.2 we have $\operatorname{var}\left(h(2^{-k}) - h(0)\right) = 2(1 - r(2^{-k})) \leq \beta 2^{-dk}$, where $d = \min(\delta, 2)$

and $\beta = \beta(d, m_d)$ is a given constant. By Lemma 3.16, stationarity and Lemma 3.14(a), we have:

$$\mathbb{P}(\cap_{0 \leq k \leq K} A_k) \geq \mathbb{P}\left(|h(0)| \leq \frac{\ell}{\alpha}\right)^N \prod_{1 \leq k \leq K} \mathbb{P}\left(|h(2^{-k}) - h(0)| \leq \frac{\ell}{\alpha k^2}\right)^{N2^k}$$

$$\geq \left(1 - 2e^{-\frac{\ell^2}{2\alpha^2}}\right)^N \prod_{1 \leq k \leq K} \left(1 - 2e^{-\frac{\ell^2 2^{dk}}{2\beta\alpha^2 k^4}}\right)^{N2^k}.$$

Since $-2x \leq \log(1 - x)$ for all $0 < x < \frac{1}{2}$, we find that if ℓ_0 is large enough then all $\ell \geq \ell_0$ satisfy

$$\mathbb{P}\left(|h(t)| \leq \ell, \ \forall t \in [0, N]\right) \geq \exp\left(-4N\left(e^{-\frac{\ell^2}{2\alpha^2}} + \sum_{k=1}^{K} 2^k e^{-\frac{\ell^2 2^{dk}}{2\beta\alpha^2 k^4}}\right)\right).$$

In light of (9), we need only check that there is a choice of ℓ_0 and c so that for any $\ell > \ell_0$ one has:

$$\exp\left(-4N\left(e^{-\frac{\ell^2}{2\alpha^2}} + \sum_{k=1}^{\infty} 2^k e^{-\frac{2^{dk}}{2\beta\alpha^2 k^4}\ell^2}\right)\right) \geq \exp(-2Ne^{-\frac{\ell^2}{c^2}}),$$

which is equivalent to

$$\sum_{k=1}^{\infty} 2^k e^{-\frac{2^{dk}}{2\beta\alpha^2 k^4}\ell^2} \leq \frac{1}{2}e^{-\left(\frac{\ell}{c}\right)^2} - e^{-\frac{1}{2}\left(\frac{\ell}{\alpha}\right)^2}.$$

If $c^2 \geq 4\alpha^2$, it is enough to show that

$$\sum_{k=1}^{\infty} 2^k e^{-\frac{2^{dk}}{2\beta\alpha^2 k^4}\ell^2} \leq \frac{1}{2}e^{-\left(\frac{\ell}{c}\right)^2} - e^{-2\left(\frac{\ell}{c}\right)^2},$$

which by applying (8) reduces to

$$\sum_{k=1}^{\infty} 2^k e^{-\frac{2^{dk}}{2\beta\alpha^2 k^4}\ell^2} \leq \frac{1}{3}e^{-\frac{\ell^2}{c^2}}.$$

Setting $\frac{2}{c^2} = \min\left(\min_{k \in \mathbb{N}} \frac{2^{dk}}{2\beta\alpha^2 k^5}, \frac{1}{2\alpha^2}\right)$ and using the identity $\sum_{k=1}^{\infty} q^k \leq 2q$ for $q < \frac{1}{2}$, we have:

$$\sum_{k=1}^{\infty} 2^k e^{-\frac{2^{dk}}{2\beta\alpha^2 k^4}\ell^2} \leq \sum_{k=1}^{\infty} (2e^{-2\ell^2/c^2})^k \leq 4e^{-2\ell^2/c^2} \leq \frac{1}{3}e^{-\frac{\ell^2}{c^2}},$$

where in each of the last two inequalities we used the estimate (8). This completes the proof. $\square$

3.6 Two famous Gaussian inequalities

We end with two famous Gaussian bounds. The first is a comparison between ball probabilities due to Anderson [19, Ch. 2.3].

Lemma 3.18 (Anderson). *Let X, Y be two independent, centered Gaussian processes on I. Then for any $\ell > 0$,*

$$\mathbb{P}\left(\sup_I |X \oplus Y| \leq \ell\right) \leq \mathbb{P}\left(\sup_I |X| \leq \ell\right).$$

The second lemma is due independently to Borell and Tsirelson-Ibragimov-Sudakov, see [1, Thm. 2.1.1].

Lemma 3.19 (Borell-TIS). *Let X be a centered Gaussian process on I which is almost surely bounded. Then for all $u > 0$ we have:*

$$\mathbb{P}\left(\sup_I X - \mathbb{E}\sup_I X > u\right) \leq \exp\left(-\frac{u^2}{2\sigma_I}\right),$$

where $\sigma_I = \sup_{t \in I} \operatorname{var} X(t)$.

4 Lower bounds

4.1 General lower bound: proof of Theorem 2

We use the Hilbert decomposition discussed in Subsection 3.3.2. Fix $N > 0$ and define

$$\varphi_N := \frac{1}{\sqrt{2}\sigma_N} \mathbb{1}_{[-\frac{1}{N},\frac{1}{N}]\cap\operatorname{sprt}\rho},$$

where, as in the introduction, $\sigma_N^2 = \rho([0, 1/N])$. Then, $\varphi_N \in \mathcal{L}_\rho^2$ and $\|\varphi\|_{\mathcal{L}_\rho^2} = 1$. Write $\psi_N(t) = \int_{T^*} e^{-i\lambda t}\varphi(\lambda)d\rho(\lambda)$. Claim 3.8 implies that

$$f(t) \stackrel{d}{=} \zeta\psi_N(t) \oplus R(t)$$

where $\zeta \sim N(0, 1)$ and R is a centered Gaussian (not necessarily stationary) process. Thus, for any $\ell > 0$ we have:

$$P_f(N) \geq \mathbb{P}\left(\zeta\psi_N(t) \geq \ell, \ \forall t \in [0, N]\right) \cdot \mathbb{P}\left(|R(t)| \leq \ell, \ \forall t \in [0, N]\right). \tag{10}$$

To estimate the first term, we note that for $t \in [0, N]$ the function ψ_N satisfies

$$\psi_N(t) = \frac{1}{\sqrt{2}\sigma_N}\int_0^{\frac{1}{N}} \cos(t\lambda)d\rho(\lambda) \geq \frac{1}{\sqrt{2}\sigma_N}\cos\left(\frac{t}{N}\right)\sigma_N^2 \geq \frac{1}{2\sqrt{2}}\sigma_N.$$

Therefore,

$$\mathbb{P}(\zeta\psi_N(t) \geq \ell, \forall t \in [0, N]) \geq \mathbb{P}\left(\sigma_N\zeta \geq 2\sqrt{2}\ell\right).$$

For the second term, we use Lemma 3.18 to get:

$$\mathbb{P}(|R(t)| \leq \ell, \ t \in [0, N]) \geq \mathbb{P}(|f(t)| \leq \ell, \ t \in [0, N]).$$

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We now apply ball estimates – Lemma 3.16 over $\mathbb{Z}$, and Lemma 3.17 over $\mathbb{R}$ – to obtain:

$$\mathbb{P}\Big(|f(t)| \le \ell, \ t \in [0, N]\Big) \ge \mathbb{P}\Big(\beta|f(0)| \le \ell\Big)^N,$$

where over $\mathbb{Z}$ the above is valid for all $\ell > 0$ and with $\beta = 1$, while over $\mathbb{R}$, as var $f(0) = 1$, it is valid with $\ell > \ell_0(m_\delta, \delta)$ and a certain $\beta = \beta(\delta, m_\delta)$. By plugging our estimates back into (10) and substituting $\tilde{\ell} = 2\sqrt{2}\ell$, the result follows.

4.2 Explicit lower bounds: proof of Corollary 1

Let $\ell_0 = 0$ and $\beta = 1$ over $\mathbb{Z}$, while $\ell_0 = \ell_0(m_\delta, \delta)$ and $\beta = \beta(\delta, m_\delta)$ over $\mathbb{R}$ as are given in Theorem 2. Applying Lemma 3.14(a) to Theorem 2 gives the following estimate for all $\ell > \max\{\ell_0, \sigma_N\}$ and $N > 0$:

$$P_f(N) \ge \exp\left(-\ell^2/\sigma_N^2\right) \cdot \mathbb{P}(\beta|Z| \le \ell)^N. \tag{11}$$

By our premise, there are some $b, \gamma > 0$ such that $\sigma_N^2 \ge bN^{-\gamma}$ along a subsequence of N. Let N be a member of that subsequence. We shall choose the level $\ell = \ell(N)$ and estimate the terms in (11) in each of three cases.

Case 1: spectrum exploding at 0 ($\gamma < 1$). Put $\ell = \ell(N) = \beta\sqrt{2\log N}$, then there exists N_0 such that for $N > N_0$ we have $\ell(N) > \ell_0$. This yields for the first term

$$\exp\left(-\ell^2/\sigma_N^2\right) \ge e^{-CN^\gamma \log N},$$

while by Lemma 3.14(a) we have for the second term

$$\mathbb{P}\Big(\beta|Z| \le \ell\Big)^N \ge \Big(1 - 2e^{-\log N}\Big)^N = \Big(1 - \tfrac{2}{N}\Big)^N \ge e^{-2}.$$

Case 2: spectrum bounded near 0 ($\gamma = 1$). Fix an arbitrary $\ell > \max\{\ell_0, \beta\}$. Then

$$\exp\left(-\ell^2/\sigma_N^2\right) \ge e^{-\ell^2 N/b},$$

while

$$\mathbb{P}(\beta|Z| \le \ell)^N \ge \mathbb{P}(|Z| \le 1)^N = e^{-cN}.$$

Case 3: spectrum vanishing at 0 ($\gamma > 1$) over $\mathbb{Z}$. Put $\ell^2 = N\sigma_N^2 \ge bN^{1-\gamma}$. The first term is

$$\exp\left(-\ell^2/\sigma_N^2\right) = e^{-N}.$$

Using Lemma 3.14(b) we bound the second term:

$$\mathbb{P}\Big(|Z| \le \ell\Big)^N \ge \mathbb{P}\Big(|Z| \le \sqrt{b}N^{(1-\gamma)/2}\Big)^N \ge \Big(\tfrac{\sqrt{b}}{4}N^{(1-\gamma)/2}\Big)^N \ge e^{-C(\gamma-1)N \log N}.$$

In all cases, the estimate stated in Corollary 1 follows.

4.3 Vanishing spectrum over $\mathbb{Z}$: proof of Corollary 2

Over $T = \mathbb{Z}$, assume that $N\sigma_N^2 < 1$. We imitate the proof of the case $\gamma > 1$ of Corollary 1, and set $\ell^2 = N\sigma_N^2$. We estimate both parts of (11). As above, the first term is estimated by e^{-N}. By Lemma 3.14(b) the second term satisfies

$$\mathbb{P}\Big(|Z| \leq \ell\Big)^N = \mathbb{P}\Big(|Z| \leq \sigma_N\sqrt{N}\Big)^N \geq e^{CN\log(N\sigma_N^2)}.$$

The estimate follows.

5 Upper bounds

In this section we prove Theorem 3 and its corollaries. We will use the following reduction.

Claim 5.1. *It is enough to prove Theorem 3 assuming that condition* (4) *holds for* $E \subseteq [-\pi, \pi]$.

Proof. Over $\mathbb{Z}$ the claim is trivial. Let $T = \mathbb{R}$. Suppose that Theorem 3 holds when $E \subseteq [-\pi, \pi]$. In the general case, let $\rho = \nu \mathbb{1}_E + \mu$ for some $\nu > 0$ and measurable $E \subset \mathbb{R}$. Set the constants α and β, as well as the function $\ell_0(N)$, as defined in Theorem 3. Let $q > 1$ be such that $\frac{1}{q}E \subseteq [-\pi, \pi]$ and let $\tilde{f}$ be the GSP defined by $\tilde{f}(x) = f(\frac{x}{q})$. Then $\tilde{f}$ has spectral measure $\tilde{\rho} = q\nu \mathbb{1}_{\frac{1}{q}E} + \tilde{\mu}$ whose moments satisfy $m_\gamma(\tilde{\rho}) = q^{-\gamma}m_\gamma(\rho)$ for any $\gamma \in \mathbb{R}$. Thus the corresponding values of $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\ell}_0(N)$ satisfy: $\tilde{\alpha} = q^{-1}\alpha$, $\tilde{\beta} = q^{-r}\beta$ and $\tilde{\ell}_0(qN) = q^{-r}\ell_0(N)$, where as above, $r = \max\{k, s/2\}$. Set $N_0(\frac{1}{q}E)$ as defined in Theorem 3 and let $N_0(E) = N_0(\frac{1}{q}E)$. Note that since $q > 1$, if $N > \max\{N_0(\frac{1}{q}E), 4k\}$ then qN satisfies the same condition. So, given $N > \max\{N_0(\frac{1}{q}E), 4k\}$ and $\ell > \ell_0(N)$ we may apply Theorem 3 for $\tilde{f}$, with $\tilde{N} = qN$ and $\tilde{\ell} = q^{-r}\ell > \tilde{\ell}_0(\tilde{N})$, to get:

$$P_f(N) = \mathbb{P}\Big(f(t) > 0 \;\forall t \in [0, N]\Big) = \mathbb{P}\Big(\tilde{f}(t) > 0 \;\forall t \in [0, qN]\Big)$$

$$\leq \mathbb{P}\left((qN)^{-r}Z > q^{-r}\ell\right) + 2qN\mathbb{P}\left(q^{-r}\beta|Z| \leq q^{-r}\ell\right)^{\frac{\alpha}{q}\cdot qN}$$

$$\leq \mathbb{P}\left(N^{-r}Z > \ell\right) + 2qN\mathbb{P}\left(\beta|Z| \leq \ell\right)^{\alpha N}.$$

$\square$

5.1 General upper bound: proof of Theorem 3

We turn to prove Theorem 3. We will give full details for the case $T = \mathbb{R}$, as the proof for $T = \mathbb{Z}$ is almost identical. Throughout the proof we denote by C universal constants which may change from line to line, and by $C(s)$ constants depending only on s, which again, may change from line to line. We divide the proof into several steps.

Step I: Let $\gamma > 0$ be such that $m_{-\gamma} < \infty$ and write $\gamma = 2k + s$, where $k \in \mathbb{N} \cup \{0\}$ and $0 \leq s < 2$. If $k \neq 0$ then, by applying Observation 3.10 k times, we find that there exists a GSP F_k which satisfies $F_k^{(k)} \stackrel{d}{=} f$. We denote the spectral measure of F_k by μ_k and note that $d\mu_k = \frac{d\rho(\lambda)}{\lambda^{2k}}$.

We therefore have, due to stationarity,

$$\operatorname{var}\left(F_k(t)\right) = m_{-2k}(\rho) \qquad \forall t \in \mathbb{R},$$

and, due to Lemma 3.12,

$$\mathbb{E}\left(\sup_{[0,N]} F_k\right) \le C\sqrt{m_{-2k}(\rho)}\max\{\log\left(c_k N\right), 1\}.$$

where $c_k := \sqrt{\frac{m_{-2k+2}(\rho)}{m_{-2k}(\rho)}}$.

If $k = 0$, we integrate one time to get the process $F_{\frac{s}{2}}(t) := \int_0^t f(\tau)d\tau$, so that $F'_{\frac{s}{2}} \stackrel{d}{=} f$. Notice that in this case, $F_{\frac{s}{2}}$ is a Gaussian process, but not necessarily stationary. From Observation 3.1 we deduce that $\rho([0,\lambda]) \le \frac{1}{2}m_{-s}(\rho)\lambda^s$ for all $\lambda > 0$. Since we assume that $m_0 = 1$, Lemma 3.11 implies that, for every $N > 0$,

$$\sup_{t \in [0,N]} \operatorname{var}\left(F_{\frac{s}{2}}(t)\right) \le C(s)m_{-s}(\rho)N^{2-s}.$$

while Lemma 3.13 implies that, for every $N > 1$, we have

$$\mathbb{E}\left(\sup_{[0,N]} F_{\frac{s}{2}}(t)\right) \le C(s)\sqrt{m_{-s}(\rho)}N^{1-\frac{s}{2}}.$$

Recall the notation $r = \max\{k, s/2\}$. For $\ell > 2N^{-r}\max\{\sqrt{\log(c_k N)}, 1\}$ set

$$M(N) = \mathbb{E}\left(\sup_{[0,N]} F_r\right) + \sqrt{2\sup_{t \in [0,N]} \operatorname{var}\left(F_r(t)\right)}N^r\ell.$$

Our estimates yield

$$M(N) \le \begin{cases} C\sqrt{m_{-2k}}(\max\{\sqrt{\log(c_k N)}, 1\} + N^k\ell), & k > 0 \\ C(s)\sqrt{m_{-s}}N^{1-\frac{s}{2}}(1 + N^{s/2}\ell), & k = 0 \end{cases},$$

or rather, due to the condition on ℓ,

$$M(N) \le \begin{cases} C\sqrt{m_{-2k}}N^k\ell, & k > 0 \\ C(s)\sqrt{m_{-s}}N\ell, & k = 0. \end{cases} \tag{12}$$

Step II: Consider the event

$$G = \left\{\sup_{[0,N]} F_r \le M(N)\right\},$$

We will estimate the persistence probability through:

$$P_f(N) \le \mathbb{P}(G^c) + \mathbb{P}(\{f > 0 \text{ on } [0,N]\} \cap G) =: P_1 + P_2. \tag{13}$$

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To estimate P_1, we apply the Borell-TIS inequality (Lemma 3.19). We get

$$P_1 = \mathbb{P}\left(\sup_{[0,N]} F_r > \mathbb{E}\left(\sup_{[0,N]} F_r\right) + \sqrt{2 \sup_{[0,N]} \text{var}\,(F_r)} N^r \ell\right) \le e^{-N^{2r}\ell^2} \le \mathbb{P}(N^{-r}Z > \ell), \quad (14)$$

where in the last step we use Lemma 3.14(a) and the fact that $\ell N^r > 2$.

Step III: We turn to the estimate of P_2. By stationarity,

$$P_2 = \mathbb{P}\left(\left\{f > 0 \text{ on } [-\tfrac{N}{2}, \tfrac{N}{2}]\right\} \cap \left\{\sup_{[-N/2,N/2]} F_r \le M(N)\right\}\right).$$

Since $k \le N$ we may apply Theorem 4 to get

$$P_2 \le \mathbb{P}\left(\left\{f > 0 \text{ on } [-\tfrac{N}{2}, \tfrac{N}{2}]\right\} \cap \left\{\frac{1}{N}\int_{-\frac{9}{40}N}^{\frac{9}{40}N} f \le L\right\}\right), \quad (15)$$

where, by (12),

$$L = \begin{cases} (Ck)^k \sqrt{m_{-2k}}\,\ell & k > 0 \\ C(s)\sqrt{m_{-s}}\,\ell, & k = 0. \end{cases} \quad (16)$$

Set $I_N = [-\tfrac{9}{40}N, \tfrac{9}{40}N - 1] \cap \mathbb{Z}$. Observe that if $f > 0$ and $\frac{1}{N}\int_{-0.45N}^{0.45N} f \le L$, then

$$\left|\left\{v \in [0,1] : \frac{1}{N}\sum_{n \in I_N} f(n+v) < 2L\right\}\right| \ge \frac{1}{2}.$$

Let $v \sim \text{Unif}([0,1])$ be a unifrom random variable which is independent of f. On the product of the probability spaces of f and v define the events

$$V = \left\{f(n+v) > 0, n \in I_N\right\} \cap \left\{\frac{1}{N}\sum_{n \in I_N} f(n+v) < 2L\right\},$$

and

$$U = \left\{f(t) > 0, t \in [0,N]\right\} \cap \left\{\frac{1}{N}\int_{-0.45N}^{0.45N} f \le L\right\}.$$

Then

$$\mathbb{P}_{f,v}(V) \ge \mathbb{P}_{f,v}(V \mid U)\mathbb{P}_f(U) \ge \frac{1}{2}\mathbb{P}_f(U).$$

The estimate in (15) and the stationarity of f now imply that

$$P_2 \le 2\mathbb{P}_{v,f}(V) = 2\mathbb{P}_f\left(\left\{f(n) > 0, n \in I_N\right\} \cap \left\{\frac{1}{N}\sum_{n \in I_N} f(n) < 2L\right\}\right). \quad (17)$$

Step IV: By Claim 5.1 we assume, without loss of generality, that $d\rho(\lambda) \ge \nu \mathbb{1}_E(\lambda)d\lambda$ for $\nu > 0$

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and $E \subseteq [-\pi, \pi]$. By Claim 3.4 used with $\varepsilon = \frac{1}{2}$, there exist $\Lambda = \{\lambda_n\} \subseteq \mathbb{Z}$ of density $A = \frac{1}{4\pi}|E|$ and a number $B = \sqrt{C\nu|E|}$ such that

$$f(\lambda_n) \overset{d}{=} BZ_n \oplus g_n, \quad \text{where } \{Z_n\} \text{ are i.i.d. } \mathcal{N}(0, 1). \tag{18}$$

By the definition of the density $D^-(\Lambda)$, there exists N_0, depending only on E, such that for all $N > N_0$ we have $|\Lambda \cap I_N| > A|I_N|/2$. From this point we assume N to satisfy this condition. Denote

$$d = \left\lfloor \frac{A|I_N|}{4} \right\rfloor, \tag{19}$$

where $\lfloor a \rfloor$ is the integer value of a. Let Λ_N be the set containing the smallest $2d - 1$ elements of $\Lambda \cap I_N$.

For $\tau \in [0, |I_N| - 1] \cap \mathbb{Z}$ consider the two disjoint sets $(\Lambda_N + \tau) \cap I_N$ and $(\Lambda_N + \tau - |I_N|) \cap I_N$. One of these sets has more elements then the other and, in particular, at least d elements. Let $\widetilde{S}(\tau)$ be the first d elements of that set. Further, let $S(\tau)$ be the corresponding translate of $\widetilde{S}(\tau)$ by either τ or $\tau - |I_N|$ so that $S(\tau) \subseteq \Lambda_N$.

If $\frac{1}{N}\sum_{j \in I_N} f(j) < 2L$ then, by Claim 3.6, there exists $\tau \in [0, |I_N| - 1]$ so that,

$$\frac{1}{|\Lambda_N|} \sum_{j \in (\Lambda_N + \tau) \cap I_N} f(j) + \frac{1}{|\Lambda_N|} \sum_{j \in (\Lambda_N + \tau - |I_N|) \cap I_N} f(j) < 2L.$$

Recalling that $|\Lambda_N| = 2d - 1 < 2d$, this implies, in particular, that

$$\frac{1}{d} \sum_{j \in \widetilde{S}(\tau)} f(j) < 4L.$$

This allows us to apply a simple union bound to the expression in (17) and find that

$$P_2 \leq 2 \sum_{\tau=0}^{|I_N|-1} \mathbb{P}\left(\left\{f(j) > 0, j \in \widetilde{S}(\tau)\right\} \cap \left\{\frac{1}{d} \sum_{j \in \widetilde{S}(\tau)} f(j) < 4L\right\}\right)$$

$$= 2 \sum_{\tau=0}^{|I_N|-1} \mathbb{P}\left(\left\{f(j) > 0, j \in S(\tau)\right\} \cap \left\{\frac{1}{d} \sum_{j \in S(\tau)} f(j) < 4L\right\}\right),$$

where the last step is due to the stationarity of the process f.

Let

$$\Sigma := \left\{\underline{x} := (x_1, \ldots, x_d) : x_1, \ldots, x_d \geq 0, \frac{1}{d}\sum_{n=1}^{d} x_n \leq 4L\right\} \subset \mathbb{R}^d,$$

and denote $\underline{Z} := (Z_1, ..., Z_d)$ where the Z_i are i.i.d. random variables with distribution $\mathcal{N}(0, 1)$. Since for every τ, $S(\tau) \subset \Lambda$ and $|S(\tau)| = d$, the decomposition in (18), combined with the estimate

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$|I_N| < N$, implies that

$$P_2 \le 2N \sup_{\underline{g} \in \mathbb{R}^d} \mathbb{P}_Z \left(B\underline{Z} \in \Sigma - \underline{g} \right), \tag{20}$$

Step V: We claim that,

$$\sup_{\underline{g} \in \mathbb{R}^d} \mathbb{P}_Z \left(B\underline{Z} \in \Sigma - \underline{g} \right) \le \mathbb{P} \left(B|Z_1| \le CL \right)^d \tag{21}$$

where $C > 0$ is a universal constant. To this end we consider two cases.

Case 1: $48LB^{-1} < 1$. For a fixed $\underline{g} \in \mathbb{R}^d$ we have

$$\mathbb{P}_Z \left(B\underline{Z} \in \Sigma - \underline{g} \right) \le B^{-d} \left| \Sigma - \underline{g} \right| = B^{-d}|\Sigma| = \frac{(4LdB^{-1})^d}{d!}$$
$$\le (12LB^{-1})^d \le \mathbb{P} \left(B|Z_1| \le 48L \right)^d,$$

where the last step holds by Lemma 3.14(b).

Case 2: $48LB^{-1} \ge 1$. For a fixed $\underline{g} \in \mathbb{R}^d$ we first note that if $\frac{1}{d}\sum_{n=1}^d g_n > 4L$, then the shifted simplex $\Sigma - \underline{g}$ does not contain $\underline{0}$. Therefore, in that case, there exists another shift $\underline{h}$, with $\frac{1}{d}\sum_{n=1}^d h_n \le 4L$, such that $\mathbb{P}_Z(B\underline{Z} \in \Sigma - \underline{g}) \le \mathbb{P}_Z(B\underline{Z} \in \Sigma - \underline{h})$ (one may take h to be the point where $\|x\|$ attains its minimum on $\Sigma - g$, noticing that the density of Z is monotone decreasing in $\|x\|$). We can therefore assume, without loss of generality, that $\frac{1}{d}\sum_{n=1}^d g_n \le 4L$. By Claim 3.5 we have

$$\mathbb{P} \left(B\underline{Z} \in \Sigma - \underline{g} \right) \le \mathbb{P} \left(BZ_n + g_n \ge 0, n = 1, ..., d \right) \le \mathbb{P} \left(BZ_1 \le 4L \right)^d.$$

Now, Applying Claim 3.15 with $\delta = 1/16$ we get, for some (universal) $\theta > 0$, that $\mathbb{P} \left(BZ_1 \le 4L \right) < \mathbb{P} \left(B|Z_1| \le 4\theta L \right)$. This establishes (21) with $C = \max(48, 4\theta)$.

Inserting (21) into (20) we find that

$$P_2 \le 2N\mathbb{P} \left(B|Z_1| \le CL \right)^d. \tag{22}$$

Step VI: We insert the estimates (14) and (22) into (13), to find that under the conditions of the theorem,

$$P_f(N) \le \mathbb{P}(N^{-r}Z > \ell) + 2N\mathbb{P} \left(B|Z_1| \le CL \right)^d$$
$$= \mathbb{P}(N^{-r}Z > \ell) + 2N\mathbb{P} \left(\beta|Z_1| \le \ell \right)^{\alpha N}, \tag{23}$$

where due to (16) and the fact that $B = \sqrt{C\nu|E|}$,

$$\beta = \begin{cases} (Ck)^{-k}\sqrt{\frac{\nu|E|}{m_{-2k}}}, & k > 0 \\ C(s)\sqrt{\frac{\nu|E|}{m_{-s}}}, & k = 0, \end{cases}$$

while due to (19), $\alpha = C|E|$, for some universal constant C. This completes the proof of the theorem.

We end this section with a proof for Remark 1. Recall that N_0 was such that for $N > N_0$ and any interval $I = [-N/2, N/2]$ we have $|\Lambda \cap I| > D^-(\Lambda)\,|I|/2$, where Λ is the set from Theorem A. Now, if E is contains an interval (or rather, due to symmetry, E contains a set of the form $J \cup (-J)$ where J is an interval), one can directly check that the lattice $\Lambda_E = \frac{\pi}{|J|}\mathbb{Z}$ obeys the properties in Theorem A. Therefore, indeed, we may take $N_0 = \frac{\pi}{|J|}$.

5.2 Explicit upper bounds: proof of Corollary 3

In this section we prove Corollary 3 by choosing an appropriate $\ell = \ell(N, \beta, \gamma)$ in Theorem 3. We leave it to the reader to verify that in all cases our choice satisfies

$$\ell \gg 2N^{-r}\max\{\sqrt{\log(\sqrt{\tfrac{m_{-2k+2}}{2m_{-2k}}}N)}, 1\}$$

for large enough N (depending on β, γ), thus fulfilling the requirements of the theorem.

Case 1: spectrum exploding at 0 ($0 < \gamma < 1$). Set $\ell > 0$ so that $e^{-\ell^2/\beta^2} = \frac{\log N}{N^{1-\gamma}}$. Applying Lemma 3.14(a) and the inequality $\log(1-x) \leq -x$ we obtain for some positive constant c_1,

$$
\begin{aligned}
P_f(N) &\leq \mathbb{P}(N^{-\gamma/2}Z > \ell) + 2qN\mathbb{P}\left(\beta|Z| \leq \ell\right)^{\alpha N} \\
&\leq e^{-N^\gamma \ell^2/2} + 2qN\left(1 - 2e^{-\ell^2/\beta^2}\right)^{\alpha N} \\
&\leq e^{-N^\gamma \ell^2/2} + 2qNe^{-\alpha N \cdot 2e^{-\ell^2/\beta^2}} \\
&\leq e^{-c_1(1-\gamma)\beta^2 N^\gamma \log N} + 2qNe^{-2\alpha N^\gamma \log N} \leq \exp\left(-CN^\gamma \log N\right).
\end{aligned}
$$

Case 2: spectrum bounded near 0 ($\gamma = 1$). In this case we choose $\ell = 1$ (or any other constant) to obtain the exponential bound:

$$P_f(N) \leq \mathbb{P}(Z > \sqrt{N}) + 2N\,\mathbb{P}\left(\beta|Z| \leq 1\right)^{\alpha N} \leq e^{-CN}.$$

Case 3: spectrum vanishing at 0 ($\gamma > 1$). First we note that, using Theorem 3 and both parts of Lemma 3.14 we get, as long as $\ell N^{\gamma/2} > 2$,

$$P_f(N) \leq \mathbb{P}(N^{-\gamma/2}Z > \ell) + 2qN\,\mathbb{P}\left(\beta|Z| \leq \ell\right)^{\alpha N} \leq e^{-N^\gamma \ell^2/2} + 2qN\left(\frac{\ell^2}{\beta^2}\right)^{\alpha N/2} \tag{24}$$

Setting $\ell^2 = \beta^2(\gamma - 1)N^{1-\gamma}\log N$ we get for some constant c_1,

$$
\begin{aligned}
P_f(N) &\leq e^{-\frac{1}{2}\beta^2(\gamma-1)N \log N} + 2qN\,e^{-c_1(\gamma-1)N \log N} \\
&\leq \exp(-C(\gamma - 1)N \log N).
\end{aligned}
$$

In all cases, the estimate holds for $N > N_0(E, \alpha, \beta, \gamma)$ and $C(\alpha, \beta, \gamma)$. Moreover, C is propor-

tional either to α or β^2, thus is linear in $|E|$.

5.3 Vanishing of infinite order: proof of Corollary 4

Tried to avoid asking for all negative moments: For a given $N > 0$ we may apply Theorem 3 with $\gamma = 2k$, so long as $m_{-2k} < \infty$, $k \leq N$ and our chosen level $\ell(N) > \ell_0(N)$. We will choose $k = k(N)$ later. Denote $\tau := \min\{t > 0 : m_{-2k+2} \leq \sqrt{t}\, m_{-2k}\}$, and note that $\tau < \infty$ by Claim 3.3. Let

$$\ell(N) = N^{-k} \cdot \sqrt{c_0|E|kN \log \frac{\tau N}{c_1 k}}, \tag{25}$$

where all the constants $(c_0, c_1, |E|)$ are as in Theorem 3. Notice that indeed, this choice obeys $\ell(N) > \ell_0(N)$, for every large enough N and $k \leq \frac{N}{4}$. By (24) and the explicit forms of α, β and ℓ we get:

$$
\begin{aligned}
P_f(N) &\leq \exp(-N^{2k}\ell^2/2) + 2qN \exp\left(\alpha N(\log \ell - \log \beta)\right) \\
&\leq \exp\left(-\frac{c_0}{2}|E|kN \log \frac{\tau N}{c_1 k}\right) \\
&\quad + 2qN \exp\left\{c_0|E|N\left(-k \log \frac{N}{c_1 k} + \tfrac{1}{2}\log \frac{c_0 m_{-2k}}{\nu} + \tfrac{1}{2}\log\left(Nk \log \frac{\tau N}{c_1 k}\right)\right)\right\}
\end{aligned}
\tag{26}
$$

Now, we use $k = k(N) \in \mathbb{N}_0$ which satisfies

$$c_1 k \left(\frac{c_0 m_{-2k}}{\nu}\right)^{1/k} \leq N, \quad \text{and} \quad k \leq N. \tag{27}$$

For this k we have $\dfrac{1}{2}\log \dfrac{c_0 m_{-2k}}{\nu} \leq \dfrac{k}{2}\log \dfrac{N}{c_1 k}$, so (26) becomes

$$
\begin{aligned}
P_f(N) &\leq \exp\left(-\frac{c_0}{2}|E|kN \log \frac{\tau N}{c_1 k}\right) + 2qN \exp\left\{c_0|E|N\left(-\frac{k}{2}\log \frac{N}{c_1 k} + \tfrac{1}{2}\log\left(Nk \log \frac{\tau N}{c_1 k}\right)\right)\right\} \\
&\leq \exp\left(-CNk \log \frac{cN}{k}\right),
\end{aligned}
$$

For some constants C and c depending on ρ. Finally, we note that, if ν and E do not vary with N, one may replace (27) with the simpler choice $k(N) := \max\left\{k \in \mathbb{N} \cap (0, N] : k\, m_{-2k}^{1/k} \leq N\right\}$. This will effect our bound only by a multiplicative constant in the exponent. The specific examples follow easily.

5.4 Tiny persistence: proof of Corollary 5

Here we prove Corollary 5 from Theorem 3. Let $w(\lambda)$ be the density of the absolutely continuous component of the spectral measure. Condition (4) becomes $w \geq \nu \mathbb{I}_E$ for some E of positive measure and $\nu > 0$. The proof will follow that of Corollary 4, however this time choose $E = [1, x_N]$ and a suitable ν_N which both depend on N. We set $k = k(N)$ as in (27) and $\ell = \ell(N)$ as in (25). We

note that, by Remark 1, our estimates are valid for all $N > 1$ (provided that $x_N \geq 2$). It remains to optimize the choice of x_N.

Part 1. Since $w(\lambda) = 0$ for $|\lambda| \leq 1$, we have $m_{-k} \leq 1$ for all $k > 0$. Also we have $w(\lambda) \geq x_N^{-\alpha}$ on $[1, x_N]$. Putting $\nu_N = x_N^{-\alpha}$, (27) becomes

$$c_1 c_0^{1/k} k x_N^{-\alpha/k} \leq N \quad \text{and} \quad k \leq N.$$

By choosing $x_N = \exp(\frac{c}{\alpha} N)$, for an appropriate universal constant c, and $k = N$ we satisfy these inequalities. We may therefore conclude by (26) that

$$P_f(N) \leq 5 N x_N \exp(-C x_N \cdot N \cdot \tfrac{N}{4}) = \exp(-e^{C_1 N}).$$

Part 2. In this case we have $k = 1$ and $\nu = x_N^{-\alpha}$. Thus (27) becomes $c x_N^{\alpha} \leq N$, for some constant $c > 0$. This is satisfied by the choice $x_N = \left(\frac{N}{c}\right)^{1/\alpha}$, which yields the bound $P_f(N) \leq \exp(-C N^{1+\frac{1}{\alpha}} \log N)$.

6 Proof of the analytic theorem

In this section we prove Theorem 4, first for continuous and then for discrete time.

6.1 Continuous time

Here we always assume $I = [-1, 1]$ (the general case follows by scaling). We will prove the following slightly stronger version of Theorem 4 in continuous time.

Proposition 1. *Let f be k-times differentiable with $\inf_I f^{(k)} > 0$. Suppose that $\int_{-0.9}^{0.9} f^{(k)} \geq k!$. Then $\sup_I |f| \geq \frac{1}{c^k}$, where $c > 0$ is a universal constant.*

The proof will use Chebyshev's polynomials of the first kind, defined through

$$T_k(x) = \cos(k \arccos x), \quad x \in [-1, 1].$$

Fix $k \in \mathbb{N}$, and let $x_j = \cos\left(\frac{(k-j)\pi}{k}\right)$, $j = 0, 1, \ldots, k$ be the $k + 1$ extremal points for $T_k(x)$ (hereafter called "Chebyshev extrema of order k"). Notice that $-1 = x_0 < x_1 < \cdots < x_n = 1$ and $T_k(x_j) = (-1)^{k-j}$.

Let the $L^\infty(\{x_j\})$-norm of a function $f : \{x_j\} \to \mathbb{R}$ be $\|f\| = \max |f(x_j)|$. A classical property of Chebyshev polynomials is the following.

Claim 6.1. *$a 2^{1-k} T_k$ has minimal $L^\infty(\{x_j\})$-norm among all polynomials of degree $k \in \mathbb{N}$ and leading coefficient $a > 0$. The value of this norm is $a 2^{1-k}$.*

Proof. Without loss of generality, we may assume that $a = 1$. Suppose P_k was a monic polynomial of degree k with $\|P_k\| < 2^{1-k}$. Recall that $2^{1-k} T_k(x_j) = (-1)^{k-j} 2^{1-k}$ on all points x_j. Thus the difference $w(x) = 2^{1-k} T_k(x) - P_k(x)$ alternates signs when evaluated on the points $\{x_j\}$. By the

intermediate value theorem, w has at least k roots. But this is impossible, since w is a polynomial of degree $\leq k - 1$. $\qquad\square$

Denote by $f[x_0, x_1, \ldots, x_n]$ the leading coefficient of the unique degree-n polynomial which interpolates f at the points $x_0, x_1, \ldots, x_n$. The next claim states that, under similar conditions to those of Proposition 1, this leading coefficient cannot be too small.

Claim 6.2. *Suppose that $f : I \to \mathbb{R}$ is $(k - 1)$-times differentiable, and that $f^{(k-1)}$ is piecewise differentiable with $f^{(k)} > 0$ and $\int_{-0.9}^{0.9} f^{(k)} \geq k!$. Then:*

$$f[x_0, x_1, \ldots, x_k] \geq M^{-k},$$

where $M > 0$ is a universal constant.

First let us see how Claim 6.2 may be used to conclude the proof of Proposition 1.

Proof of Proposition 1. Let $f : I \to \mathbb{R}$ be as in the premise, that is, $f^{(k)} > 0$ for all $x \in I$ and $\frac{1}{N}\int_{-0.9}^{0.9} f^{(k)} \geq k!$. Let P_k be the interpolation polynomial of f at the Chebyshev extremal points $\{x_j\}_{j=0}^k$, and let $a = f[x_0, \ldots, x_k]$ denote the leading coefficient of P_k. By Claim 6.2, we have $|a| \geq M^{-k}$ (where $M > 0$ is universal), so that by Claim 6.1 we deduce that

$$\sup_I |f| \geq \max_{0 \leq j \leq k} |f(x_j)| = \max_{0 \leq j \leq k} |P_k(x_j)| \geq |a| 2^{1-k} \geq (2M)^{-k}.$$

$\qquad\square$

It remains to prove Claim 6.2. To do so, we apply the following standard result from interpolation theory, known as the *Hermite-Genocchi formula* (see e.g. [22, Thm 4.2.3]):

Lemma 6.3 (The Hermite-Genocchi formula). *Let $\{x_j\}_{j=0}^k \subset I$ be distinct points, and let $f : I \to \mathbb{R}$ be $(k - 1)$-times differentiable with $f^{(k-1)}$ being weakly differentiable. Then:*

$$f[x_0, x_1, \ldots, x_k] = \int_{\Sigma_k} f^{(k)}\Big(t_0 x_0 + t_1 x_1 + \ldots t_k x_k\Big) d\sigma,$$

where

$$\Sigma_k = \{(t_0, \ldots, t_k) : \forall j \; t_j \geq 0 \text{ and } \sum_{j=0}^k t_j = 1\}, \tag{28}$$

and $d\sigma$ is the induced volume form on Σ_k.

Notice that Σ_k is a k-dimensional simplex (embedded in $\mathbb{R}^{k+1}$), thus $\text{Vol}_k(\Sigma_k) = 1/k!$. The following lemma enables us to bound efficiently the integral which appears in the Hermite-Genocchi formula.

Lemma 6.4. *Fix $k \in \mathbb{N}$, and let $\{x_j\}_{j=0}^k \subset I$ be the Chebyshev extrema points. Write Σ_k for the simplex in* (28). *Then there exists a non-negative function $g_k \in L^1(I)$ so that for any $F \in L^1(I)$ we have*

$$\int_{\Sigma_k} F\Big(t_0 x_0 + t_1 x_1 + \ldots t_k x_k\Big) d\sigma = \int_{-1}^1 F(s) g_k(s) ds. \tag{29}$$

Moreover, there exists $L > 1$ such that for all $k \in \mathbb{N}$ and all $|s| \leq 0.9$, we have $g_k(s) \geq \frac{1}{k!L^k}$.

Proof. The function

$$g_k(s) = \mathrm{Vol}_{k-1}\Big(\Big\{(t_0, t_1, \ldots, t_k) : \sum_j t_j x_j = s, \sum_j t_j = 1, \forall i \ t_i \geq 0\Big\}\Big)$$

satisfies (29) with any $F \in L^1(I)$, due to a change of variables (and Fubini's theorem). Since g_k is continuous it is also in $L^1(I)$.

For $s \in I = [-1, 1]$ define the set

$$A_s = \Big\{(t_0, \ldots, t_k) : \sum_{j=0}^k t_j = 1, \ \sum_{j=0}^k t_j x_j = s, \ (t_1, \ldots, t_{k-1}) \in \Big[0, \frac{1}{10(k-1)}\Big]^{k-1}\Big\}$$

Recalling that $x_0 = -1$ and $x_k = 1$ we notice that, given $(t_1, \ldots, t_{k-1}) \in [0, 1/(10(k-1))]^{k-1}$, there is a unique pair $(t_0, t_k) \in \mathbb{R}^2$ that satisfies the independent linear equations

$$\begin{cases} \sum_{j=0}^k t_j = 1, \\ \sum_{j=0}^k t_j x_j = s, \end{cases} \iff \begin{cases} t_0 + t_k = a \\ -t_0 + t_k = b, \end{cases} \tag{30}$$

where a, b are numbers (depending on $t_1, \ldots t_{k-1}$). Therefore, A_s is a $(k-1)$-dimensional manifold (embedded in $\mathbb{R}^{k+1}$), which may be viewed as the graph of a linear function on the domain $\Big[0, \frac{1}{10(k-1)}\Big]^{k-1}$. This implies that $\mathrm{Vol}_{k-1}(A_s) \geq \Big(\frac{1}{10(k-1)}\Big)^{k-1} \geq \frac{1}{k!L^k}$ for some $L > 0$.

It remains to check that if $|s| \leq 0.9$, for any $(t_1, \ldots, t_{k-1}) \in \Big[0, \frac{1}{10(k-1)}\Big]^{k-1}$ the solution to (30) obeys $t_0 \geq 0$, $t_k \geq 0$ (allowing us to conclude that $g_k(s) \geq \mathrm{Vol}_{k-1}(A_s)$). Solving explicitly we have

$$t_0 = \frac{1}{2}\Big(1 - s - \sum_{j=1}^{k-1} t_j(1 - x_j)\Big),$$

so that $t_0 \geq 0$ if and only if $\sum_{j=1}^{k-1} t_j(1 - x_j) \leq 1 - s$. We verify this as follows:

$$\sum_{j=1}^{k-1} t_j(1 - x_j) \leq \frac{1}{10} - \frac{1}{10(k-1)} \sum_{j=1}^{k-1} x_j = \frac{1}{10} \leq 1 - s,$$

where we used that $|s| \leq 0.9$. The verification of $t_k \geq 0$ is obtained in a symmetric way. $\qquad \square$

Now we return to prove Claim 6.2.

Proof of Claim 6.2. By using Lemma 6.3 and then applying Lemma 6.4 with $F = f^{(k)}$, we get that:

$$
\begin{aligned}
f[x_0, x_1, \ldots, x_k] &= \int_{\Sigma_k} f^{(k)}\Big(t_0 x_0 + t_1 x_1 + \ldots t_k x_k\Big) dt \\
&\geq \frac{1}{k! L^k} \int_{-0.9}^{0.9} f^{(k)}(s) ds \\
&\geq \frac{1}{L^k}.
\end{aligned}
$$

where $L > 0$ is a universal constant. $\square$

6.2   Discrete time

In this section we prove Theorem 4 in discrete time using an interpolation method. We denote the discrete derivative operator by Δ that is,

$$(\Delta f)(n) = f(n+1) - f(n). \tag{31}$$

When applied iteratively k times we write Δ^k.

A somewhat stronger version of Theorem 4 over $\mathbb{Z}$ (similar to Proposition 1) is the following.

Proposition 2. *Let $k, N \in \mathbb{N}$ such that $N \geq k$. Let $f : \mathbb{Z} \to \mathbb{R}$ be such that $\Delta^{(k)} f(n) > 0$ for every $n \in [-N, N]$. Denote $I_N = [-\frac{9}{10}N, \frac{9}{10}N] \cap \mathbb{Z}$ and suppose that $\frac{1}{N} \sum_{n \in I_N} f^{(k)}(n) \geq \frac{k!}{N^k}$. Then $\sup_{[-2N, 2N]} |f| \geq \frac{1}{c^k}$, where $c > 0$ is a universal constant.*

Thm 4 and its application were changed accordingly (what follows from $N \to 2N$ here)

Proof. Let $B_0(x) = \mathbb{1}_{[-\frac{1}{2}, \frac{1}{2}]}(x)$ and $B_k(x) = (B_0)^{*k}(x)$, where $(B_0)^{*k}$ is the convolution of B_0 with itself k times. Define the continuous time function:

$$F(x) = \sum_n f(n) B_{k+1}(x - n).$$

Observe that B_k takes values in $[0, 1]$, is finitely supported, is a piecewise degree-k polynomial, and $\sum_n B_k(x - n) = 1$ for all $x \in \mathbb{R}$. The derivative of F is given by the following relation (see [5, eq. (10.3)]):

$$F'(x) = \frac{d}{dx}\left(\sum_{n \in \mathbb{Z}} f(n) B_{k+1}(x - n)\right) = \sum_{n \in \mathbb{Z}} (\Delta f)(n) B_k(x - n),$$

where Δ is the discrete derivative (recall (31)). By repeating this k times we get:

$$F^{(k)}(x) = \sum_{n \in \mathbb{Z}} \Delta^k f(n) B_1(x - n).$$

Since $\Delta^k f(n) > 0$ for all $n \in [-N, N] \cap \mathbb{Z}$, we have $F^{(k)}(x) > 0$ for all $x \in [-N + 1, N - 1]$.

Moreover, we have,

$$\frac{1}{N} \int_{-0.9N}^{0.9N} F^{(k)} \geq \frac{1}{2N} \sum_{n \in [-0.9N, 0.9N] \cap \mathbb{Z}} \Delta^k f(n) \geq \frac{1}{2} \frac{k!}{N^k}.$$

Therefore, by Proposition 1 we deduce that

$$\sup_{n \in [-2N, 2N] \cap \mathbb{Z}} f(n) \geq \sup_{n \in [-N-k, N+k] \cap \mathbb{Z}} f(n) \geq \sup_{x \in [-N+1, N-1]} F(x) \geq \frac{1}{c^k},$$

for some universal number c, as required. This concludes the proof. $\square$

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