Persistence of Gaussian Stationary Processes

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Joint work with

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Gaussian stationary processes (GSP)

For $T \in \{\mathbb{R}, \mathbb{Z}\}$, a random function $f : T \mapsto \mathbb{R}$ is a GSP if it is

- <u>Gaussian:</u> $(f(x_1), \dots f(x_N)) \sim \mathcal{N}_{\mathbb{R}^N}(0, \Sigma_{x_1,\dots,x_N})$,
- <u>Stationary</u> (shift-invariant): $(f(x_1+s), \dots f(x_N+s)) \overset{d}{\sim} (f(x_1), \dots f(x_N))$,

for all $N \in \mathbb{N}$, $x_1, \dots, x_N, s \in T$.

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Motivation:

- Background noise for radio / cellular transmissions
- Ocean waves
- Vibrations of bridge strings / membranes
- Brain transmissions
- internet / car traffic
- ...

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Covariance function

$$r(s,t) = \mathbb{E}(f(s)f(t)) = r(s-t) \qquad t, s \in T.$$

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Spectral measure

By Bochner's theorem there exists a finite, non-negative, symmetric measure ρ over T^* ($\mathbb{Z}^* \simeq [-\pi,\pi]$ and $\mathbb{R}^* \simeq \mathbb{R}$) s.t.

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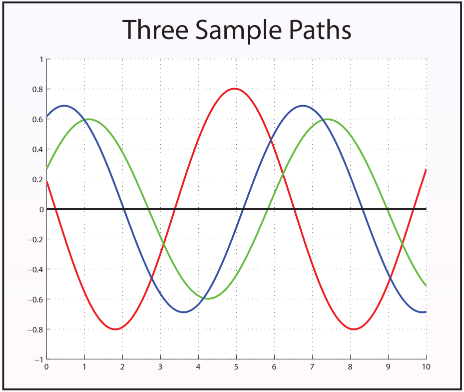
Assumption: $\int |\lambda|^\delta d\rho(\lambda) < \infty$ for some $\delta > 0$.
("finite polynomial moment" $\Rightarrow$ r is Hölder contin.)

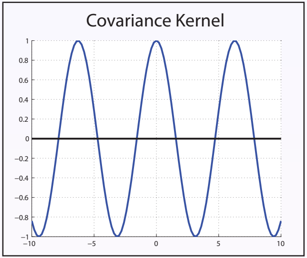
ζ_j i.i.d. $\mathcal{N}(0,1)$

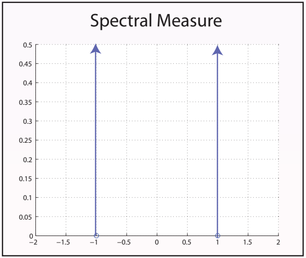
$$f(x) = \zeta_0 \sin(x) + \zeta_1 \cos(x)$$

$$r(x) = \cos(x)$$

$$\rho = \frac{1}{2}(\delta_1 + \delta_{-1})$$



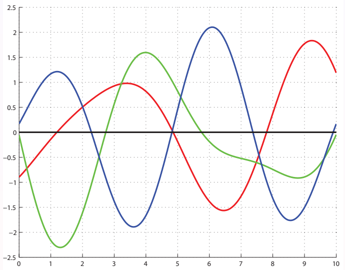




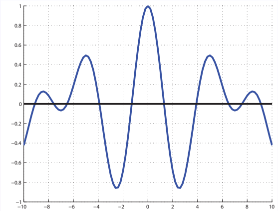
Toy-Example Ib - Almost periodic wave

$$
\begin{aligned}
f(x) = &\zeta_0 \sin(x) + \zeta_1 \cos(x) \\
&+ \zeta_2 \sin(\sqrt{2}x) + \zeta_3 \cos(\sqrt{2}x) \\
r(x) = &\cos(x) + \cos(\sqrt{2}x) \\
\rho = &\tfrac{1}{2}\left(\delta_1 + \delta_{-1} + \delta_{\sqrt{2}} + \delta_{-\sqrt{2}}\right)
\end{aligned}
$$

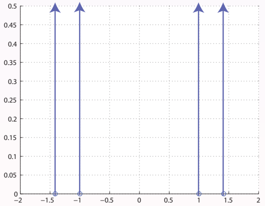
Three Sample Paths



Covariance Kernel



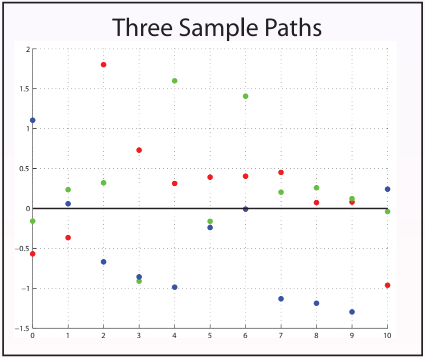
Spectral Measure

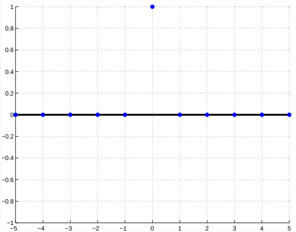


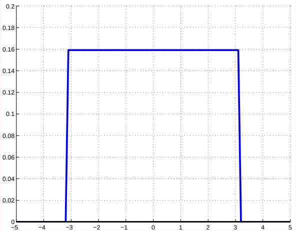
$$f(n) = \zeta_n$$

$$r(n) = \delta_{n,0}$$

$$d\rho(\lambda) = \frac{1}{2\pi}\mathbb{1}_{[-\pi,\pi]}(\lambda)d\lambda$$





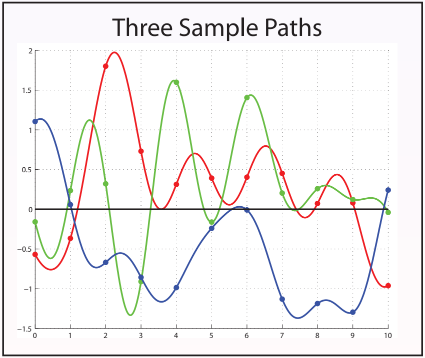


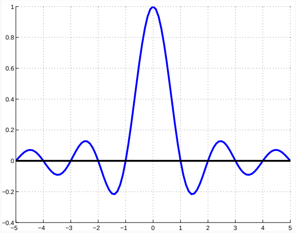
Example IIb - Sinc kernel

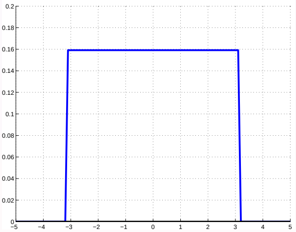
$$f(n) = \sum_{n \in \mathbb{N}} \zeta_n \operatorname{sinc}(x - n)$$

$$r(n) = \frac{\sin(\pi x)}{\pi x} = \operatorname{sinc}(x)$$

$$d\rho(\lambda) = \frac{1}{2\pi} \mathbb{1}_{[-\pi,\pi]}(\lambda) d\lambda$$







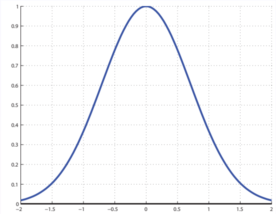
Example III - Gaussian Covariance (Fock-Bargmann)

$$f(x) = \sum_{n \in \mathbb{N}} \zeta_n \frac{x^n}{\sqrt{n!}} e^{-\frac{x^2}{2}}$$

$$r(x) = e^{-\frac{x^2}{2}}$$

$$d\rho(\lambda) = \sqrt{\pi} e^{-\frac{\lambda^2}{2}} d\lambda$$

Covariance Kernel



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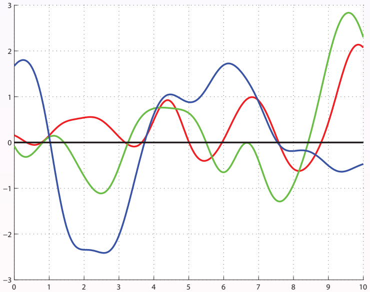
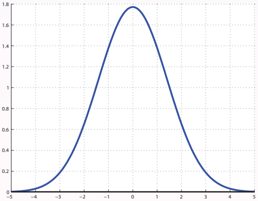


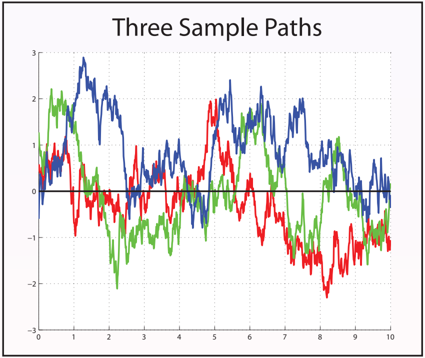
Spectral Measure

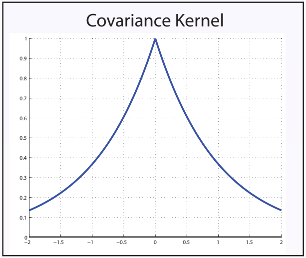


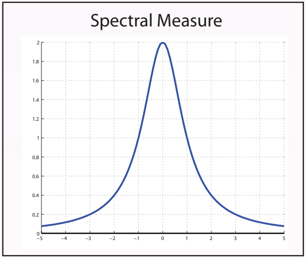
Example IV - Exponential Covariance (Ornstein-Uhlenbeck)

$$r(x) = e^{-|x|}$$

$$d\rho(\lambda) = \frac{2}{\lambda^2 + 1} d\lambda$$







Persistence

The **persistence probability** of a stochastic process f over a level $\ell \in \mathbb{R}$ in the time interval $(0, N]$ is:

$$P_f(N) := \mathbb{P}\Big(f(x) > \ell, \forall x \in (0, N]\Big).$$

Picture of persistence

Persistence Probability

Persistence (above the mean)

The **persistence probability** of a **centered** stochastic process f in the time interval $(0, N]$ is:

$$P_f(N) := \mathbb{P}\Big(f(x) > 0, \forall x \in (0, N]\Big).$$

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Question: For a GSP f, what is the behavior of $P_f(N)$ as $N \to \infty$?

Guess: "typically" $P(t) \asymp e^{-\theta t}$.

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Toy Examples

$(X_n)_{n\in\mathbb{Z}}$ i.i.d.	$P_X(N) = 2^{-N}$
$Y_n = X_{n+1} - X_n$	$P_Y(N) = \frac{1}{(N+1)!} \asymp e^{-N\log N}$
$Z_n \equiv Z_0$	$P_Z(N) = \mathbb{P}(Z_0 > 0) = \dfrac{1}{2}.$

Engineering and Applied Mathematics (1940–1970)

- 1944 Rice - “Mathematical Analysis of Random Noise”.
  - Mean number of level-crossings (Rice formula)
  - Behavior of $P(t)$ for $t \ll 1$ (**short range**).
- 1962 Slepian - “One-sided barrier problem”.
  - Slepian’s Inequality: $r_1(x) \geq r_2(x) \Rightarrow P_1(N) \geq P_2(N)$.
  - specific cases
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  - If $r(x) \to 0$ as $x \to \infty$, then $P(N) = o(N^{-\alpha})$ for any $\alpha > 0$.
  - If $|r(x)| < ax^{-\alpha}$ then $P(N) \leq \begin{cases} e^{-CN} & \text{if } \alpha > 1 \\ e^{-CN/\log N} & \text{if } \alpha = 1 \\ e^{-CN^{\alpha}} & \text{if } 0 < \alpha < 1 \end{cases}$
  - examples for $P(t) > e^{-C\sqrt{N}\log N} \gg e^{-CN}$ $(r(x) \asymp x^{-1/2})$.

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Applicable mainly when r is non-negative or summable.

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Physics (1990–2010)

- GSPs used in models for electrons in matter, diffusion, spin systems
- Majumdar et al.: Heuristics explaining why $P_f(N) \asymp e^{-\theta N}$ "generically".

Probability and Anlysis(2000+)

- Hole probability for point processes
  - GAFs in the plane (Sodin-Tsirelson, Nishry), hyperbolic disc (Buckley et al.)
  - for sinc-kernel: $e^{-cN} < P(N) < 2^{-N}$ (Antezana-Buckley-Marzo-Olsen, '12)

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- Non-negative correlations -Dembo & Mukherjee (2013, 2015)
  – motivated by random polynomials and diffusion processes.

- Lower bounds for GSP on $\mathbb{Z}$ - Krishna & Krishnapur (2016)
  – motivated by nodal lines of spherical harmonics.

First Spectral Result

Theorem 1 (Feldheim & F., 2013)

Suppose that on some interval $[-a, a]$ we have $d\rho = w(\lambda)d\lambda$ with $0 < m \leq w(x) \leq M$. Then

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$$Z_n \equiv Z_0 \Rightarrow P_Z(N) = \frac{1}{2} \qquad \rho = \delta_0$$

Main tool: spectral decomposition

Key Observation

$$\rho = \rho_1 + \rho_2 \Rightarrow f \overset{d}{=} f_1 \oplus f_2,$$

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Proof:

$$\begin{aligned}
&\mathrm{cov}((f_1 + f_2)(0), (f_1 + f_2)(x)) \\
&= \mathrm{cov}(f_1(0), f_1(x)) + \mathrm{cov}(f_2(0), f_2(x)) \\
&= \widehat{\rho_1}(x) + \widehat{\rho_2}(x) = \widehat{\rho_1 + \rho_2}(x) = \mathrm{cov}(f(0), f(x)). \qquad \square
\end{aligned}$$

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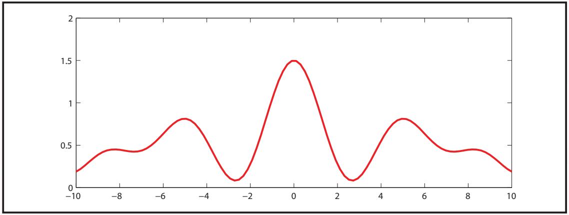
$$\rho = \rho_1 + \rho_2 \Rightarrow f \stackrel{d}{=} f_1 \oplus f_2,$$

Application:

$$\rho = m\mathbb{1}[-\frac{\pi}{k}, \frac{\pi}{k}] + \mu \Rightarrow f = S \oplus g$$

where $r_S(x) = c\ \mathrm{sinc}(\frac{x}{k})$, and g is some GSP.



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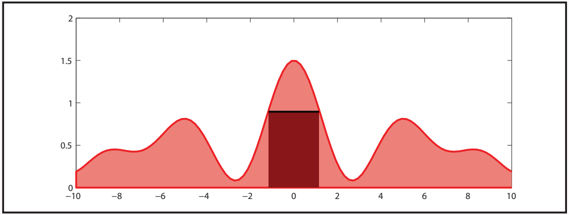

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Observation.

$(S(nk))_{n\in\mathbb{Z}}$ are i.i.d.

Proof: $\mathbb{E}[S(nk)S(mk)] = r_S((m-n)k) = 0$.

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$f = S \oplus g$, where $(S(nk))_{n \in \mathbb{Z}}$ are i.i.d.

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Let us use this observation to obtain an upper bound on $P_f(N)$.

$$P_f(N) \leq \mathbb{P}\left(S \oplus g > 0 \text{ on } (0, N] \,\middle|\, \frac{1}{N}\sum_{n=1}^{N} g(n) < 1 \right) + \mathbb{P}\left(\frac{1}{N}\sum_{n=1}^{N} g(n) \geq 1 \right)$$

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Lemma 1 - average of a GSP.

$\frac{1}{N}\sum_{n=1}^{N} g(n) \sim \mathcal{N}_{\mathbb{R}}(0, \sigma_N^2)$, where $\sigma_N^2 \leq \frac{C_0}{N}$.

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- Here we use the upper bound M.
- Lemma 1 $\Rightarrow \mathbb{P}(\frac{1}{N}\sum_{n=1}^{N} g(n) \geq 1) \leq e^{-c_1 N}$.

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Proof:

$$\log \mathbb{P}(X_j \ge -b_j,\, 1 \le j \le N) = \log \prod_{j=1}^{N} \Phi(b_j)$$

$$= \sum_{j=1}^{N} \log \Phi(b_j) \le N \log \Phi\left(\frac{1}{N}\sum b_j\right) \le N \log \Phi(1).$$

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A corroloary to works by Talagrand, Shao-Wang (1994):

Lemma 3 - small ball.

Let g be a GSP whose spectral measure ρ has some finite δ-moment (i.e., $\int |\lambda|^\delta d\rho(\delta) < \infty$). Let $\varepsilon > 0$. Then $\mathbb{P}(|g| < \varepsilon \text{ on } (0, N]) \geq e^{-cN}$.

Motivation revisited: very recent progress

- Non-negative correlations -Dembo & Mukherjee (2013, 2015):
  - motivated by random polynomials and diffusion processes
  - If $r(x) \geq 0$, then $\exists \lim_{N\to\infty} \frac{-\log P(N)}{N} \in [0,\infty)$ (application of Slepian). In particular, if $r(x) \geq 0$ then $P(N) \geq e^{-\alpha N}$ for some $\alpha > 0$.
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  - on $\mathbb{Z}$, if $\rho_{AC} \neq 0$ then $P(N) \geq e^{-CN^2}$.
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Conjecture 1: explods $\Rightarrow P(N) \gg e^{-\alpha N}$, vanishes $\Rightarrow P(N) \ll e^{-\alpha N}$.

Conjecture 2: $P(N) \leq e^{-CN^2}$ when ρ **vanishes on an interval** around 0 ("spectral gap").

Persistence is largely determined by the spectral behavior near 0.

Theorem 2 (Feldheim, F., Nitzan, 2017)

Suppose that in $[-a,a]$ the spectral measure has density $w(\lambda)$ which satisfies $c_1\lambda^\gamma \le w(\lambda) \le c_2\lambda^\gamma$ for some $\gamma > -1$. Then:

$$\log P_f(N) \begin{cases} \asymp -N^{1+\gamma}\log N, & \gamma < 0 \\ \asymp -N, & \gamma = 0 \\ \lesssim -\gamma N \log N, & \gamma > 0. \end{cases}$$

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Corollaries:

- establish both conjectures (and more)
- remove the condition $r(t) \ge 0$ from Dembo-Mukherjee
- with Krishna-Krishnapure: matching upper and lower bounds over $\mathbb{Z}$
- achieve the first example of $P(N) \ll e^{-CN\log N}$

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Further improvements:

- $w(\lambda) \leq c_2\lambda^\gamma \Rightarrow$ upper bounds, $w(\lambda) \geq c_1\lambda^\gamma \Rightarrow$ lower bounds
- formulate using $\rho([0,\lambda])$ for $\lambda \ll 1$, provided that $\rho_{AC} \neq 0$
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Missing: lower bound over $\mathbb{R}$ when $\gamma > 0$!

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Let $T = \mathbb{R}$. If the spectral measure vanishes on an interval containing 0, and on $[1, \infty)$ it has density $w(\lambda)$ such that $w(\lambda) \geq \lambda^{-100}$, then

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- heavy tail $\Rightarrow$ f is "rough" $\Rightarrow$ tiny persistence.
- light tail $\Rightarrow$ f is smooth $\Rightarrow$ matching lower bounds as over $\mathbb{Z}$ **[in progress]**

Ideas from the proof.

Exploding spectrum: upper bound

$f = S \oplus g$, where $(S(nk))_{n\in\mathbb{Z}}$ are i.i.d.

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An analytic lemma (degree p)

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If $\{f > 0\} \cap G$ occoured, by **the analytic lemma** there is a large set $R \subseteq [0, N]$ ($|R| \geq \frac{N}{2}$) such that $|f| < \frac{2\ell}{N}$ on R.
By the **spectral decomposition**, $f(kn) = Z_n \oplus g_n$ where Z_n are i.i.d.
By **Anderson's lemma**,

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Suppose for simplicity that $w(\lambda) \le \lambda^2$ for $|\lambda| \le a$.

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$\ell = \sqrt{N \log N} \Rightarrow$ both sides are $e^{-CN \log N}$.

Further directions

- other levels
- other dimensions
- the mysterious discontinuities in $\log P_f(N)$
- singular measures
- existence of limiting exponent (e.g. $\lim_{N\to\infty} \frac{\log P_f(N)}{N}$)
- non-stationary processes

Thanks!

Thank you.

"Persistence can grind an iron beam down into a needle."

– – Chinese Proverb.