Drawing outerplanar graphs

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August 2, 2012

Abstract

It is shown that for any outerplanar graph G there is a one to one mapping of the vertices of G to the plane, so that the number of distinct distances between pairs of connected vertices is at most three. This settles a problem of Carmi, Dujmovic, Morin and Wood. The proof combines (elementary) geometric, combinatorial, algebraic and probabilistic arguments.

1 Introduction

A *linear* embedding of a graph G is a mapping of the vertices of G to distinct points in the plane. The image of every edge uv of the graph is the open interval between the image of u and the image of v. The length of that interval is called the *edge-length* of uv in the embedding. A *degenerate drawing* of a graph G is a linear embedding in which the images of all vertices are distinct. A *drawing* of G is a degenerate drawing in which the image of every edge is disjoint from the image of every vertex. The *distance-number* of a graph is the minimum number of distinct edge-lengths in a drawing of G, the *degenerate distance-number* is its counterpart for degenerate drawings.

An *outerplanar* graph is a graph that can be embedded in the plane without crossings in such a way that all the vertices lie in the boundary of the unbounded face of the embedding. In [1], Carmi, Dujmovic, Morin and Wood ask if the degenerate distance-number of outerplanar graphs are uniformly bounded. We answer this positively by showing that the degenerate distance number of outerplanar graphs is at most 3. This result is derived by explicitly constructing a degenerate drawing for every such graph.

Theorem 1. *For almost every triple $a, b, c \in (0, 1)$, every outerplanar graph has a degenerate drawing using only edge-lengths a, b and c.*

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For matters of convenience, throughout the paper we consider all linear embeddings as mapping vertices to the complex plane.

2 Background and Motivation

While the distance-number and the degenerate distance-number of a graph are two natural notions in the context of representing a graph as a diagram in the plane, this was not the sole motivation to their introduction.

Both notions were introduced by Carmi, Dujmovic, Morin and Wood in [1], in order to generalize several well studied problems. Indeed, Erdős introduced in [2] the problem of finding the minimal number of distinct distances between n points in the plane. This problem can be rephrased as finding the degenerate distance-number of K_n, the complete graph on n vertices. Recently, Larry Guth and Nets Katz made a ground-breaking advancement in solving this problem, proving in [3] a lower-bound of $cn/\log n$ on this number, which almost matches the $n/\sqrt{\log n}$ upper-bound due to Erdős. Another problem, which was considered by Szemerédi (See Theorem 13.7 in [5]), is that of finding the minimum possible number of distances between n non-collinear points in the plane. This problem can be rephrased as finding the distance-number of K_n. One interesting consequence of the known results on these questions is that the distance-number and degenerate distance-number of K_n are not the same, thus justifying the two separate notions. For a short survey of the history of both problems, including some classical bounds, the reader is referred to the background section [1].

Another notion which is generalized by the degenerate distance-number is that of a unit-distance graph, that is, a graph that can be embedded in the plane so that two vertices are at distance one if and only if they are connected by and edge. Observe that all unit-distance graphs have degenerate distance-number 1 while the converse is not true. Constructing "dense" unit-distance graphs is a classical problem. The best construction, due to Erdős [2], gives an n-vertex unit-distance graph with $n^{1+c/\log\log n}$ edges, while the best known upper-bound, due to Spencer, Szemerédi and Trotter [6], is $cn^{4/3}$ (A simpler proof for this bound was found by Székely, see [7]). Note that this implies that the k most frequent interpoint distances between n points occur in total no more than $ckn^{4/3}$ times, and thus that a graph with degenerate distance-number k may have no more than $ckn^{4/3}$ edges. Katz and Tardos gave in [4] another bound on the frequency of interpoint distances between n points in the plane, which yields that a graph with distance-number k may have no more than $cn^{1.46}k^{0.63}$ edges.

After introducing the notions of distance-number and degenerate distance-number, Carmi, Dujmovic, Morin and Wood, studied in [1] the behavior of graphs with bounded degree with respect to these notions. They show that graphs with bounded degree greater or equal to five can have degenerate distance-number arbitrarily large, giving a polynomial lower-bound for graphs with bounded degree greater or equal to seven. They also gave a logarithmic upper-bound of $\log(n)$ to the distance-number of bounded degree graphs with bounded treewidth. In the same paper, the authors ask

whether this bound can be improved for outerplanar graphs, and in particular whether such graphs have a uniformly bounded degenerate distance-number, the problem which we resolve here.

3 Preliminaries

Outerplanarity, Δ-trees and T^*. An *outerplanar* graph is a graph that can be embedded in the plane without crossings so that all its vertices lie in the boundary of the unbounded face of the embedding. The edges which border this unbounded face are uniquely defined, and are called the *external* edges of the graph; the rest of the edges are called *internal*.

Let Δ be the triangle graph, that is, a graph on three vertices v_0, v_1, and v_2, whose edges are v_0v_1, v_0v_2 and v_2v_1. A graph is said to be a Δ-*tree* if it can be generated from Δ by iterations of adding a new vertex and connecting it to both ends of some external edge other than v_0v_1. This results in an outerplanar graph whose bounded faces are all triangles. The adjacency graph of the bounded faces of such a graph is a binary tree, that is – a rooted tree of maximal degree 3. In fact, all Δ-trees are subgraphs of an infinite graph T^*. All bounded faces of T^* are triangles, and the adjacency graph of those faces is a complete infinite binary tree. The root of T^* is denoted by T^*_{root}. An illustration of a Δ-tree can be found in the left hand side of figure 3.

It is a known fact, which can be proved using induction, that the triangulation of every outerplanar graph is a Δ-tree. All outerplanar graphs are therefore subgraphs of T^*, a fact which reduces Theorem 1 to the following:

Proposition 1. *For almost every triple $a, b, c \in (0, 1)$, the graph T^* has a degenerate drawing using only edge-lengths a, b and c.*

The rhombus graph H, Covering T^* by rhombi. In order to prove the above proposition, we construct an explicit embedding of T^* in $\mathbb{C}$. To do so we introduce a covering of T^* by copies of a particular directed graph H which we call a *rhombus*. We then embed T^* into $\mathbb{C}$, one copy of H at a time.

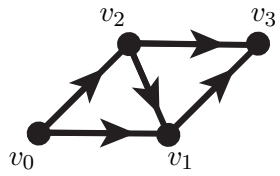

Figure 1: The rhombus graph H.

The *rhombus* directed graph H, is defined to be the graph satisfying $V_H = \{v_0, v_1, v_2, v_3\}$ and $E_H = \{v_0v_1, v_0v_2, v_2v_3, v_1v_3, v_2v_1\}$. We call v_0 the base vertex of H.

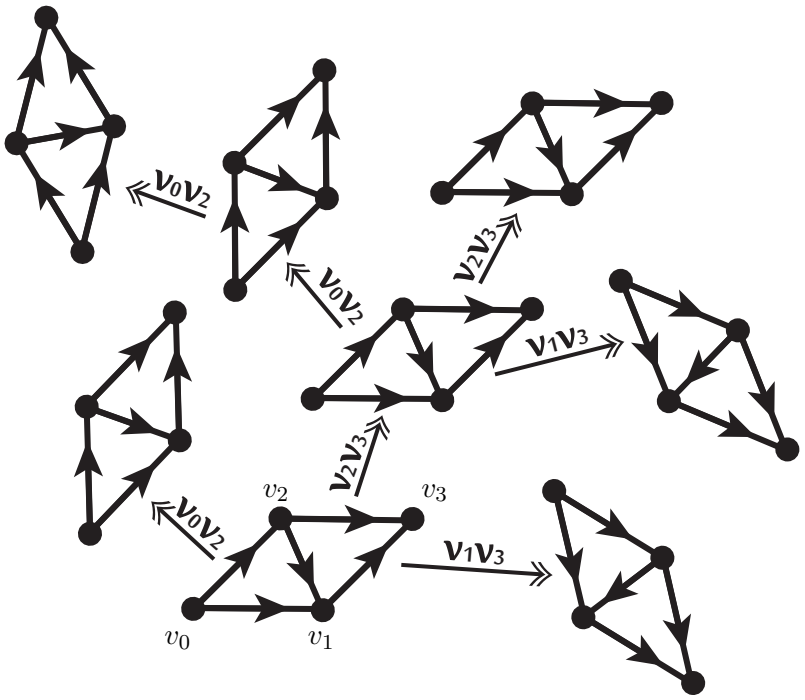

Figure 2: A portion of H^*, including the names of the vertices of the node H^*_{root} and the labels on the arcs.

We further define H^* to be the infinite directed trinary tree whose nodes are copies of H, labeling the three arcs emanating from every node by v_0v_2, v_2v_3 and v_1v_3. We write $L(a)$ for the label of an arc a. Let N be a node of H^*, and let $v_iv_j \in E_H$; we call a pair (N, v_i) a vertex of H^*, and a pair (N, v_iv_j), an edge of H^*. Notice the distinction between arcs of H^* and edges of H^*, and the distinction between nodes and vertices. The root of H^* is denoted by H^*_{root}. A portion of H^* is depicted in figure 2.

There exists a natural map π from the vertices of H^* to the vertices of T^* which maps each node of H^* to a pair of adjacent triangles of T^*. π is defined in such a way that H^*_{root} is mapped to T^*_{root} and to one of its neighboring triangles, and every directed arc MN of H^*, satisfies $\pi((M, L(MN))) = \pi((N, v_0v_1))$ (in the sense of mapping origin to origin and destination to destination). In the rest of the paper we extend π naturally to edges and subgraphs, and abridge $\pi((N, v))$ to $\pi(N, v)$. A portion of T^* and its covering by H^* through π are depicted in figure 3.

Encoding the rhombi. In order to embed T^* into $\mathbb{C}$, rhombus-by-rhombus, a way to refer to every node $N \in H^*$ is called for. We encode N by the sequence of labels on the path from H^*_{root} to N. This trinary sequence is denoted by S_N. The map $N \to S_N$ is a bijection.

One may think of each label in S_N as a direction, "left", "right" or "forward", in which one must

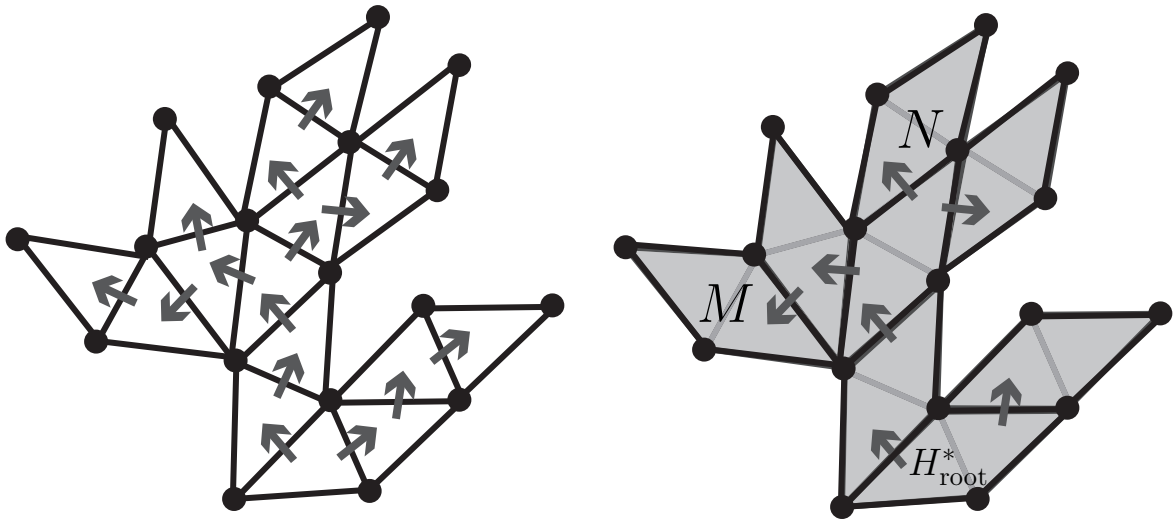

Figure 3: A portion of T^* and the corresponding covering by H^*. The orientation of the edges is omitted to simplify the drawing. The nodes M and N are QR-encoded by $S_M = (v_0v_2, v_2v_3, v_0v_2, v_0v_2)$, $\mathrm{QR}(M) = ((0,1,0,0),(0,0,0),3)$ and $S_N = (v_0v_2, v_2v_3, v_2v_3)$, $\mathrm{QR}(N) = ((0,2),(0),1)$ respectively.

descend H^*, until finally arriving at N. To simplify our proofs, we further encode S_N, by describing this sequence by "how many forward steps to take between each turn left or right" and "is the i-th turn left or right".

Formally, we do this by further encoding S_N using a triple $(\{q_i(N)\}_{i=1}^{m(N)+1}, \{\rho_i(N)\}_{i=1}^{m(N)}, m(N))$. We set $q_i(N)$ to be the number of v_2v_3-s between the $(i-1)$-th non-v_2v_3 label in S_N and the i-th one (for $i = 1$ and for $i = m(N)+1$, the number of v_2v_3-s before the first non-v_2v_3 label in S_N and after the last non-v_2v_3 label in S_N, respectively). We set $\rho_i(N)$ to be 0 if the i-th non-v_2v_3 element is v_0v_2 and 1 if it is v_1v_3. We call the triple $(\{q_i(N)\}, \{\rho_i(N)\}, m(N))$ the *QR-encoding* of N denoting it by $\mathrm{QR}(N)$.

In accordance with our informal introduction, a QR-encoding $(\{q_i\}, \{\rho_i\}, m)$, should be interpreted as taking q_1 steps forward, then turning left or right according to ρ_1 being 0 or 1 respectively, then taking another q_2 steps forward in the new direction and so on and so forth. The QR-encoding of each node is unique.

Encoding the vertices of T^*. The encoding of the nodes of H^* naturally extends to an encoding of the vertices of T^* by defining $\mathrm{QR}(u) = \{\mathrm{QR}(N) \ : \ \pi(N, v_0) = u\}$ for $u \in T^*$. This is indeed an encoding of all the vertices of T^*, as for every vertex $u \in T^*$ there exists at least one node N such that $\pi(N, v_0) = u$. However, it is not unique, as an infinite number of nodes encode each vertex. As a unique encoding of every vertex is desirable for our purpose, we make the following observation.

Observation 1. *Let $u \in T^*$, there exists a unique node N such that $QR(N) = (\{q_i\}, \{\rho_i\}, m) \in QR(u)$, satisfying $q_{m+1} = 0$ and either $q_m > 0$ or $m = 1$. We call such an encoding the proper encoding of u.*

Proof. It is not difficult to observe that the only proper encodings of $\pi(H^*_{\text{root}}, v_0)$ and $\pi(H^*_{\text{root}}, v_1)$ are $((0,0),(0),1)$ and $((0,0),(1),1)$ respectively.

For every vertex $u \in T^*$, except from $\pi(H^*_{\text{root}}, v_0)$ and $\pi(H^*_{\text{root}}, v_1)$, there exists a unique node $N_u \in H^*$ satisfying that $\pi(N_u, v_i) = u$ for some $i \in \{2, 3\}$. Let $\sim$ denote the concatenation operation between sequences. Using this notation we have that either $S(N_u) \sim v_2v_3 \sim v_0v_2$ or $S(N_u) \sim v_2v_3 \sim v_1v_3$ encode a node whose base vertex is mapped by π to u. One may verify from the definition of QR-encodings that S_N ending with either v_2v_3, v_0v_2 or with $v_2v_3 \sim v_1v_3$ is equivalent to $q_{m+1} = 0$ and $q_m > 0$. □

Polynomial embeddings. A *d-polynomial embedding of a graph G using k edge-lengths* is a <u>one-to-one</u> mapping $\psi : V_G \to \mathbb{C}[x_1, \ldots, x_d]$ where $\mathbb{C}[x_1, \ldots, x_d]$ is the space of complex polynomials in d variables, such that for every fixed $x \in \mathbb{T}^d = \{(x_1, \ldots, x_d) \in \mathbb{C}^d : \forall i \in \{1, \ldots, d\}, |x_i| = 1\}$ the map $v \mapsto \psi(v)(x) = \psi_x(v)$ is a linear embedding using only k non-zero edge-lengths.

The importance of d-polynomial embeddings to our purpose stems from the following proposition:

Proposition 2. *If ψ is a d-polynomial embedding of a graph G with k edge-lengths, then for almost every $x = (x_1, \ldots, x_d) \in \mathbb{T}^d$, ψ_x is a degenerate drawing of G with k edge-lengths.*

Proof. For any $v, w \in V_G$, the polynomials $\psi(v)(x)$ and $\psi(w)(x)$ may coincide only on a set of measure 0 in T^d. Taking union over all the pairs v_1, v_2, we get that outside an exceptional set of measure zero in $\mathbb{T}^d$, the map ψ_x is one-to-one. □

4 Three Distances Suffice for Degenerate Drawings

In this section we prove Proposition 1 and thus Theorem 1. To do so, for $x_0, x_1 \in \mathbb{T}$, we introduce in section 4.1 a 2-polynomial embedding $\psi = \psi(x_0, x_1) = \psi_{(x_0,x_1)} : T^* \to \mathbb{C}$. In section 4.2 we then write an explicit formula for the image of every vertex v under ψ. This we do using the QR-encoding introduced in the preliminaries section. In section 4.3 we prove that ψ is one-to-one. Finally, in section 4.4 we conclude the proof of Proposition 1.

4.1 The definition of ψ

In this section we define ψ. An outline of our construction is as follows: we start by presenting $\psi_H(x)$, a 1-polynomial embedding of H which embeds the rhombus graph onto a rhombus of side length 1 with angle x (identifying the complex number x with its angle on the unit circle). We then use a boolean function Ty on the nodes of T^* to decide whether each rhombus is mapped to a

translated and rotated copy of $H(x_0)$ or of $H(x_1)$. Finally, we define ψ in the only way that respects both the covering π and the function Ty. The image of several subsets of T^* through $\psi(x_0, x_1)$ is depicted in figure 5.

We set $\psi_H(x)(v_0) = 0$, $\psi_H(x)(v_1) = 1$, $\psi_H(x)(v_2) = x$ and $\psi_H(x)(v_3) = x + 1$. This is indeed a polynomial drawing, mapping the rhombus graph to a rhombus of edge length 1, whose $v_1v_0v_2$ angle is x. Figure 4 illustrates the image of H under ψ_H.

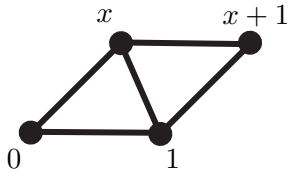

Figure 4: The image of H under ψ_H. Observe how x determines the $v_1v_0v_2$ angle of the rhombus.

We define an auxiliary function Ty. Let MN be an arc of H^*. We set

$$\mathrm{Ty}(N) = \begin{cases} \mathrm{Ty}(M) & L(MN) = v_2v_3 \\ \mathrm{Ty}(N) \oplus q_{m(M)+1}(M) \pmod 2 & L(MN) = v_0v_2 \\ \mathrm{Ty}(N) \oplus q_{m(M)+1}(M) \oplus 1 \pmod 2 & L(MN) = v_1v_3 \end{cases}, \tag{1}$$

where $\oplus$ represents addition modulo 2. We set $\mathrm{Ty}(H^*_{\mathrm{root}}) = 0$.

Set $\psi(\pi(H^*_{\mathrm{root}})) = \psi_H(x_0)(H)$. Let $M, N \in H^*$ be a pair of nodes such that MN is an arc of H^*, and assume that ψ is already defined on the vertices of $\pi(M)$. By π's definition, this implies that $\psi(\pi(N, v_0))$ and $\psi(\pi(N, v_1))$ are already defined. We then define $\psi(\pi(N, v_2)), \psi(\pi(N, v_3))$ so that $\psi(\pi(N, v_0)), \psi(\pi(N, v_1)), \psi(\pi(N, v_2)), \psi(\pi(N, v_3))$ form a translated and rotated copy of $H(x_{\mathrm{Ty}(N)})$.

As the image of every edge in T^* is isometric to some edge of either $H(x_0)$ or $H(x_1)$, we get

Observation 2. *Every edge of T^* is mapped through ψ to an interval of length 1, $|x_0-1|$, or $|x_1-1|$.*

While this definition of $\psi(x_0, x_1)$ is complete, an explicit formula for every vertex in T^* under $\psi(x_0, x_1)$ is required for proving that ψ is indeed a polynomial. We devote the next section to develop this formula.

4.2 The image of ψ

In this section we state a formula for $\psi \circ \pi$ of every base vertex.

Let $u \in T^*$ and let $N \in H^*$, such that $\mathrm{QR}(N) = (\{q_k\}, \{\rho_k\}, m)$ is the proper encoding of u. The first i elements of $\{q_k\}, \{\rho_k\}$ encode a node in T^* which is denoted by N_i (where $N_0 = H^*_{\mathrm{root}}$ which corresponds to the null sequence). Naturally, $N_m = N$. From (1) we get

$$\mathrm{Ty}(N_i) = \mathrm{Ty}(N_{i-1}) \oplus q_i \oplus \rho_i \pmod 2. \tag{2}$$

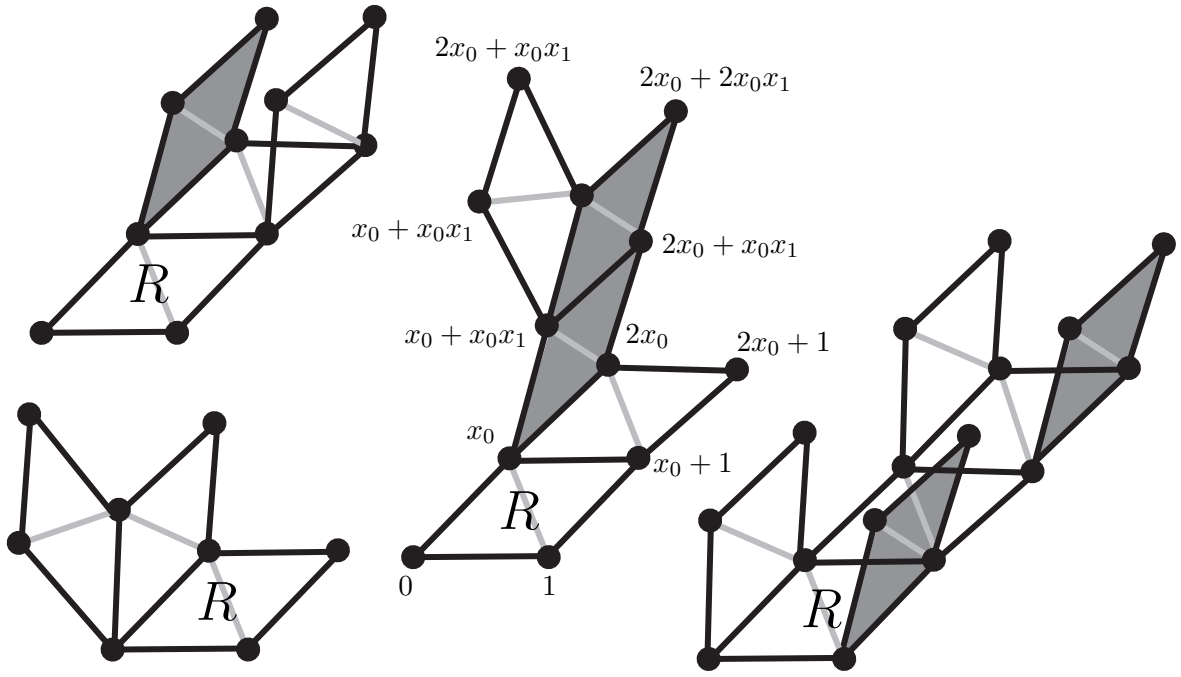

Figure 5: The image of several subgraphs of T^* under ψ. Explicit values are given for several vertices. In each graph, the image of $\pi(H^*_{\text{root}})$ under ψ is marked by R. Rhombi of angle x_1 are dark.

Observe that in the embedding of every H^* node through ψ, the edges v_0v_2, v_1v_3 are parallel, as are the edges v_0v_1, v_2v_3. Next, we define $P_i(x_0, x_1)$ to be a unit vector in the direction of the edges $(v_0, v_1), (v_2, v_3)$ in $\psi(\pi(N_i))$ which, for $i > 0$, is the same as the direction of $(v_0, v_2), (v_1, v_3)$ in $\psi(\pi(N_{i-1}))$.

Formally

$$P_i^u(x_0, x_1) = P_i(x_0, x_1) = \psi(\pi(N_i, v_1)) - \psi(\pi(N_i, v_0)) = \psi(\pi(N_{i-1}, v_2) - \psi(\pi(N_{i-1}, v_0)),$$

where the last equality holds for $i > 0$. Notice that $P_0(x_0, x_1) = 1$.

With this in mind, it is possible to follow the change in P_i between one N_i and the next. This yields:

$$P_i(x_0, x_1) = P_{i-1}(x_0, x_1) \cdot x_{\text{Ty}(N_{i-1})}. \tag{3}$$

For $0 \le i \le m$ write $Q_i^u(x_0, x_1) = Q_i(x_0, x_1) = \psi_{x_0,x_1}(\pi(N_i, v_0))$. Observe that $Q_0 = \psi(\pi(H^*_{\text{root}}, v_0)) = 0$. Let us describe how to get Q_i from Q_{i-1} using $(\{q_k\}, \{\rho_k\}, m)$. By definition,

$$Q_i(x_0, x_1) - Q_{i-1}(x_0, x_1) = \psi(\pi(N_i, v_0)) - \psi(\pi(N_{i-1}, v_0)).$$

Thus $Q_i(x_0, x_1) - Q_{i-1}(x_0, x_1)$ can be calculated from the labels of the edges along the path connecting (N_{i-1}, v_0) and (N_i, v_0). Each edge labeled v_2v_3 contributes to this difference P_i, and thus in total

such edge contribute $q_i \cdot P_i$. An edge with label v_1v_3 contributes $P_i/x_{\mathrm{Ty}(N_{i-1})} = P_{i-1}$, while an edge labeled v_0v_2 does not change the base vertex at all.

Applying this to the encoding, we get that

$$Q_i - Q_{i-1} = q_i \cdot P_i + \rho_i \cdot P_{i-1}.$$

Summing this over $1 \leq i \leq m$, we get:

$$\begin{aligned}\psi_{x_0,x_1}(u) = Q_m &= \sum_{i=1}^{m} (q_i P_i + \rho_i P_{i-1}) \\ &= \rho_1 + \sum_{i=1}^{m-1} (q_i + \rho_{i+1}) P_i + q_m P_m\end{aligned}$$

Equivalently, letting $\rho_m = 0$ and $q_0 = 0$ we have

$$\psi_{x_0,x_1}(u) = \sum_{i=0}^{m} (q_i + \rho_{i+1}) P_i(x_0, x_1) = \sum_{i=0}^{m} c_i P_i(x_0, x_1), \tag{4}$$

where

$$c_i = q_i + \rho_{i+1}. \tag{5}$$

Observe that for every $u \in T^*$, $\psi_{x_0,x_1}(u)$ is a polynomial in x_0 and x_1 (because P_i are monomials). Also observe that the total degree of P_i, which we denote by $\deg P_i$, obeys $\deg P_i = \deg P_{i-1} + 1$. Therefore $\{c_i\}$ may be regarded as the coefficients of the polynomial $\psi_{x_0,x_1}(u)$.

Note that in particular, using the above notations, Observation 1 and the fact that $(\{q_k\}, \{\rho_k\}, m)$ is proper yield

$$c_m = q_m > 0. \tag{6}$$

4.3 Showing that ψ is a polynomial embedding

In this section we show that the image of the vertices of T^* under ψ are all distinct. Relation (4) and Observation 2 imply that if this is the case, then ψ is a polynomial embedding of T^* using three edge lengths.

The main proposition of this section is the following:

Proposition 3. *Let $u, w \in T^*$ be two distinct vertices. Then $\psi_{x_0,x_1}(u)$ and $\psi_{x_0,x_1}(w)$ are distinct polynomials.*

Proof. Let $(\{q_i^u\}, \{\rho_i^u\}, m)$,$(\{q_i^w\}, \{\rho_i^w\}, n)$ be the proper QR-encoding sequences for u, w respectfully, and let N_k^u and N_k^w be the nodes encoded by the first k elements of those sequences respectively. We write $\nu_i^u = \mathrm{Ty}(N_i^u)$, $\nu_i^w = \mathrm{Ty}(N_i^w)$ for all i.

Notice that by Observation 1 the two sequences are distinct. The fact that $\pi(H^*_{\text{root}}, v_0)$ and $\pi(H^*_{\text{root}}, v_1)$ have unique images under ψ is straightforward, as these are the only vertices whose image is a polynomial of total degree 0. We can therefore assume $m > 1$.

Assume for the sake of obtaining a contradiction that $\psi_{x_0,x_1}(u) \equiv \psi_{x_0,x_1}(w)$ as functions of (x_0, x_1), and thus in particular $c^u_i = c^w_i$ for all i.

Combining this with (6) and Observation 1, we get $m = n$.

Let j be the first index to satisfy $(q^u_j, \rho^u_j) \neq (q^w_j, \rho^w_j)$. By (2) and (3) this implies

$$\forall i \leq j : P^u_i = P^w_i \text{ and } \nu^u_{i-1} = \nu^w_{i-1}. \tag{7}$$

Moreover, since $q^u_{j-1} = q^w_{j-1}$ and $c^u_{j-1} = c^w_{j-1}$ we get by (5) that $\rho^u_j = \rho^w_j$. We deduce that $q^u_j \neq q^w_j$. Since $c^u_j = c^w_j$ we have $q^u_j - q^w_j = \rho^w_{j+1} - \rho^u_{j+1} \in \{-1, 0, 1\}$. As we have assumed this difference to be non-zero, we may assume without loss of generality

$$q^u_j - q^w_j = \rho^w_{j+1} - \rho^u_{j+1} = 1. \tag{8}$$

Applying (7) for $i = j + 1$ and the last relation to (2), we get

$$\nu^u_j = \nu^u_{j-1} \oplus q^u_j \oplus \rho^u_j = \nu^w_{j-1} \oplus q^w_j \oplus \rho^w_j \oplus 1 = \nu^w_j \oplus 1.$$

By (3) we get

$$\frac{P^u_{j+1}}{P^w_{j+1}} = \frac{x_{\nu^u_j}}{x_{\nu^w_j}} \neq 1,$$

which implies $c^u_{j+1} = c^w_{j+1} = 0$. This in turn implies that $q^u_{j+1} = q^w_{j+1} = 0$ and $\rho^u_{j+2} = \rho^w_{j+2} = 0$. Using now relation (2) for $i = j + 2$ and recalling (8), we get

$$\nu^u_{j+2} = \nu^u_{j+1} \oplus 0 \oplus \rho^u_{j+1} = (\nu^w_{j+1} \oplus 1) \oplus 0 \oplus (\rho^w_{j+1} - 1) = \nu^w_{j+2}.$$

Again by (3) we have

$$\frac{P^u_{j+2}}{P^w_{j+2}} = \frac{P^u_{j+1}}{P^w_{j+1}} \cdot \frac{x_{\nu^u_{j+2}}}{x_{\nu^w_{j+2}}} = \frac{P^u_{j+1}}{P^w_{j+1}} \neq 1,$$

which implies $c^u_{j+2} = c^w_{j+2} = 0$. Continuing by induction, we conclude that $c^u_{j+k} = c^w_{j+k} = 0$ for all $k > 1$. Thus $j = m$, and so by (6), $q^u_j = c^u_j = c^w_j = q^w_j$, a contradiction to (8). □

4.4 Three Distances Suffice for Degenerate Drawings

We are now ready to write the proof of Proposition 1, and thus conclude the proof of Theorem 1.

Proof of Proposition 1. By Proposition 3, ψ is a 2-polynomial embedding of every finite subgraph $G \subseteq T^*$, using 3 edge-lengths. By Proposition 2 and Observation 2, the set

$$\{(x_0, x_1) \ : \text{ s.t. } x_0, x_1 \in \mathbb{T}^2 \text{ and } \psi(x_0, x_1) \text{ is a degenerate drawing}\}$$

is of full measure, and each of these degenerate drawings uses only side lengths 1, $|x_0-1|$ and $|x_1-1|$. Let $a \in (0,1)$, the embedding $a \cdot \psi$, i.e. the composition of a multiplication by a on ψ, is thus a degenerate drawing of G for almost every x_0, x_1 using the side lengths $a, a|1-x_0|, a|1-x_1|$. The result follows. □

4.5 Open problems

Several interesting problems concerning graphs with low (degenerate) distance number remain unanswered. In this section we state those of greater interest to us. The first and most natural one is:

Problem 1. *Do outerplanar graphs have a uniformly bounded distance number?*

While we believe we may be able to answer this problem positively, our construction is rather complicated and is thus postponed to a future paper. It will be interesting to see a simple construction which can be easily described.

The general problem which, in our mind, extends this work most naturally is:

Problem 2. *Which families of graphs have a uniformly bounded (degenerate) distance number?*

Observe that planarity, for example, is far from being suitable – as the complete bi-partite graph $K_{n,2}$ is an example of a planar graph whose degenerate distance number is of order $\sqrt{(n)}$.

Finally, our result implies that the maximal degenerate distance number of an outerplanar graph is at most three. It is easy to see that there are outerplanar graphs whose degenerate distance number is two. Are there any outerplanar graphs whose degenerate distance number is indeed three?

Problem 3. *Is it true that the maximum possible degenerate distance number of an outerplanar graph is two?*

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