Three Edge Lengths Suffice For Drawing Outerplanar Graphs

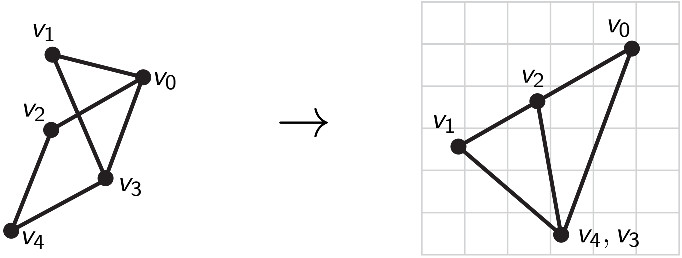
Noga Alon   Ohad N. Feldheim

Tel-Aviv University

IMU, 2012

Mapping Graphs to the Plane

Let $G = (V_G, E_G)$, $\pi : V_G \to \mathbb{C}$,

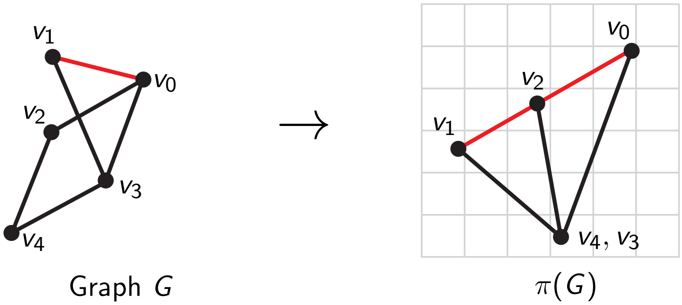


Graph G $\pi(G)$

Mapping Graphs to the Plane

Let $G = (V_G, E_G)$, $\pi : V_G \to \mathbb{C}$, $e = (v_0, v_1)$.

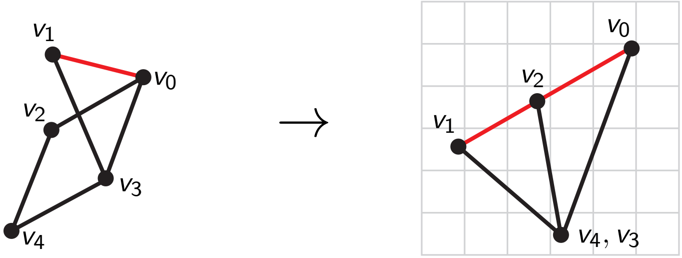
We set: $\pi(e) := (\pi(v_0), \pi(v_1))$



Mapping Graphs to the Plane

Let $G = (V_G, E_G)$, $\pi : V_G \to \mathbb{C}$, $e = (v_0, v_1)$.

We set: $\pi(e) := (\pi(v_0), \pi(v_1))$ $\mathrm{len}_\pi(e) := |\pi(v_0) - \pi(v_1)|$



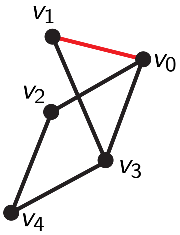
Graph G $\pi(G)$

Mapping Graphs to the Plane

Let $G = (V_G, E_G)$, $\pi : V_G \to \mathbb{C}$, $e = (v_0, v_1)$.

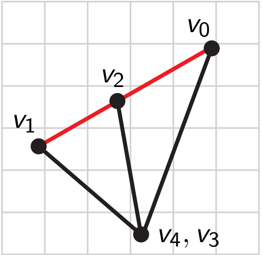
We set: $\pi(e) := (\pi(v_0), \pi(v_1))$ $\mathrm{len}_\pi(e) := |\pi(v_0) - \pi(v_1)|$

$$\mathrm{lens}(\pi) := |\{\mathrm{len}_\pi(e) : e \in E_G\}|$$



Graph G

$\longrightarrow$

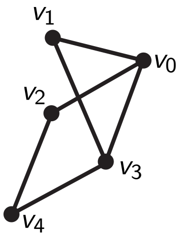


$\pi(G)$

Drawings

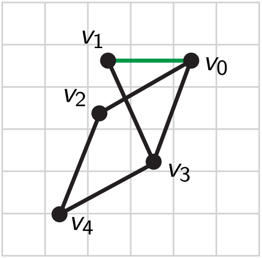
Let $G = (V_G, E_G)$, $\pi : V_G \to \mathbb{C}$,

Drawing: $\forall v, v' \in V_G$, $\forall e \in E_G$: $\pi(v) \neq \pi(v')$, $\pi(v) \notin \pi(e)$



Graph G

$\longrightarrow$



$\pi(G)$

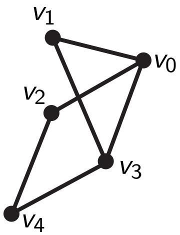
Drawings

Let $G = (V_G, E_G)$, $\pi : V_G \to \mathbb{C}$,

Drawing: $\forall v, v' \in V_G$, $\forall e \in E_G$: $\pi(v) \neq \pi(v')$, $\pi(v) \notin \pi(e)$

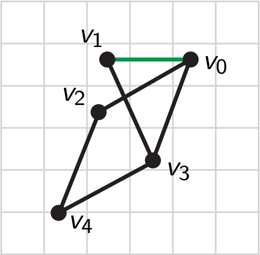
Distance Number

$$\mathrm{dn}(G) := \min\{\mathrm{lens}(\pi) \ : \ \pi \text{ is a drawing of } G\}$$



Graph G

$\longrightarrow$

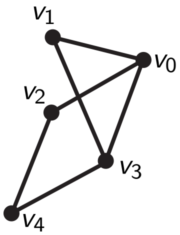


$\pi(G)$

Degenerate Drawings

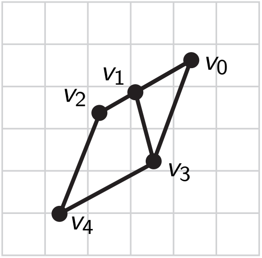
Let $G = (V_G, E_G)$, $\pi : V_G \to \mathbb{C}$,

Deg. Drawing: $\forall v, v' \in V_G$: $\pi(v) \neq \pi(v')$



Graph G

$\rightarrow$



$\pi(G)$

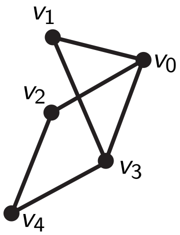
Degenerate Drawings

Let $G = (V_G, E_G)$, $\pi : V_G \to \mathbb{C}$,

Deg. Drawing: $\forall v, v' \in V_G$: $\pi(v) \neq \pi(v')$

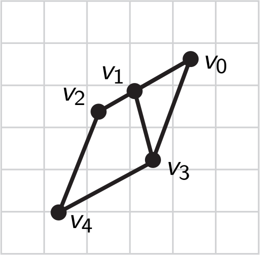
Degenerate Distance Number

$$\mathrm{ddn}(G) := \min\{\mathrm{lens}(\pi) \ : \ \pi \text{ is a deg. drawing of } G\}$$



Graph G

$\longrightarrow$



$\pi(G)$

Properties of the Distance Number

Properties of dn and ddn

Let $G \subset H$.

- $\mathrm{dn}(G) \leq \mathrm{dn}(H)$
- $\mathrm{ddn}(G) \leq \mathrm{ddn}(H)$
- $\mathrm{ddn}(G) \leq \mathrm{dn}(G)$

Distance Number - Previous Results

What bounds can we get on dn, ddn? Are they ever different?

Distance Number - Previous Results

What bounds can we get on dn, ddn? Are they ever different?

Results on K_n

Guth-Katz('11)/Erdős('46): $\frac{c_1 n}{\log n} \leq \mathrm{ddn}(K_n) \leq \frac{c_2 n}{\sqrt{\log n}}$

Distance Number - Previous Results

What bounds can we get on dn, ddn? Are they ever different?

Results on K_n

Guth-Katz('11)/Erdős('46): $\frac{c_1 n}{\log n} \leq \mathrm{ddn}(K_n) \leq \frac{c_2 n}{\sqrt{\log n}}$

Szemerédi('95)/Erdős('51): $\frac{n-1}{3} \leq \mathrm{dn}(K_n) \leq \frac{n}{2}$

Distance Number - Previous Results

What bounds can we get on dn, ddn? Are they ever different?

Results on K_n

Guth-Katz('11)/Erdős('46): $\frac{c_1 n}{\log n} \leq \text{ddn}(K_n) \leq \frac{c_2 n}{\sqrt{\log n}}$

Szemerédi('95)/Erdős('51): $\frac{n-1}{3} \leq \text{dn}(K_n) \leq \frac{n}{2}$

Results on graphs with bdd. degree Δ, n vertices

Carmi, Dujmović, Morin, Wood('08):

For $\Delta \geq 5$: ddn is not uniformly bdd.

For $\Delta \geq 7$: exist G_n with $\text{ddn}(G) = \Omega(n^c)$ for $c(\Delta) < C < 1$.

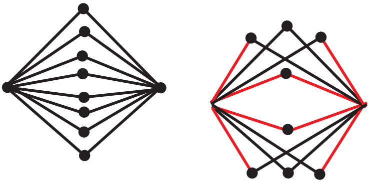
$\text{dn}(G) = O(\Delta^4 \log n)$ if G_n's treewidth is uniformly bdd.

Distance Number - Planar Graphs

Planar Graph: Has a drawing without crossings.

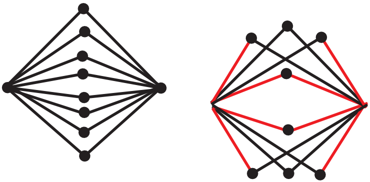
Distance Number - Planar Graphs

Planar Graph: Has a drawing without crossings.
$\exists$ planar G_n with $\mathrm{ddn}(G) = \Omega(\sqrt{n})$.



Distance Number - Planar Graphs

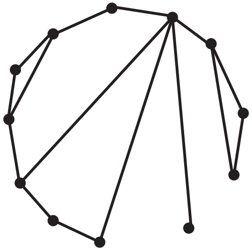
Planar Graph: Has a drawing without crossings.
$\exists$ planar G_n with $\mathrm{ddn}(G) = \Omega(\sqrt{n})$.



Carmi, Dujmovic, Morin, Wood('08): Do **outerplanar** graphs have uniformly bounded (degenerate) distance number?

Outerplanar Graphs

Outerplanar Graph: $\exists$ drawing, without crossings, s.t. unbounded face contains all vertices.

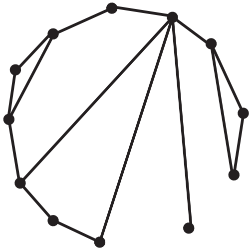


Outerplanar graph

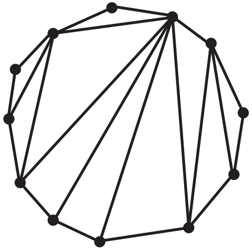
Outerplanar Graphs

Outerplanar Graph: $\exists$ drawing, without crossings, s.t. unbounded face contains all vertices.

Triangulated Outerplanar Graph: Outerplanar graph where all bounded faces are triangles.



Outerplanar graph



Trinagulated outerplanar graph

Our Results

Carmi, Dujmovic, Morin, Wood('08): Do outerplanar graphs have uniformly bounded (degenerate) distance number?

Our Results

Carmi, Dujmovic, Morin, Wood('08): Do outerplanar graphs have uniformly bounded (degenerate) distance number?

Theorem (Alon, F.)

For almost every **triple** $a, b, c \in (0, 1)$, every outerplanar graph has a **degenerate drawing** using only edge-lengths a, b and c.

Our Results

Carmi, Dujmovic, Morin, Wood('08): Do outerplanar graphs have uniformly bounded (degenerate) distance number?

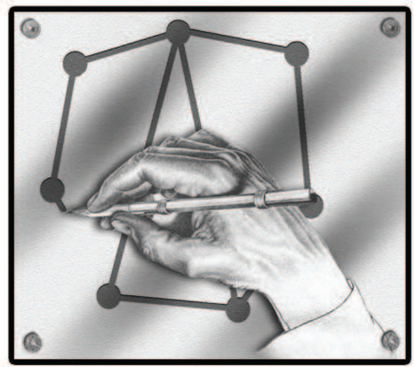
Theorem (Alon, F.)

For almost every **triple** $a, b, c \in (0, 1)$, every outerplanar graph has a **degenerate drawing** using only edge-lengths a, b and c.

Work in progress...

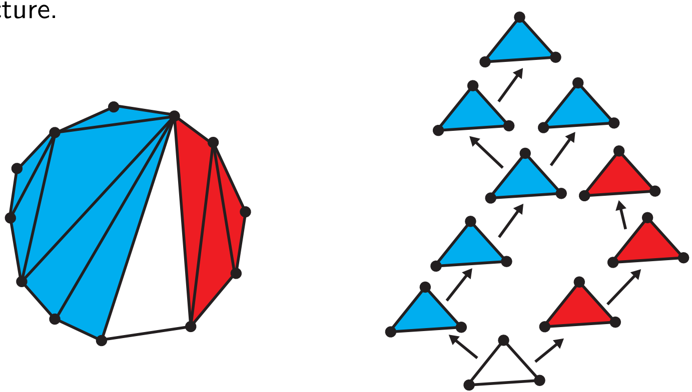
For almost every **nine** values $a_0, \ldots, a_8 \in (0, 1)$, every outerplanar graph has a **drawing** using only edge-lengths $a_0, \ldots, a_8$.

Drawing Outerplanar Graphs



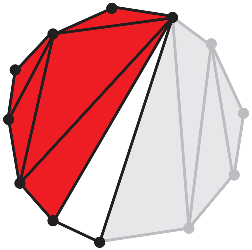
Triangle Trees

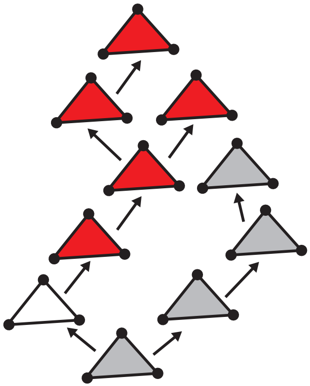
Bdd. faces of triangulated outerplanar graphs have a binary tree structure.



Triangle Trees

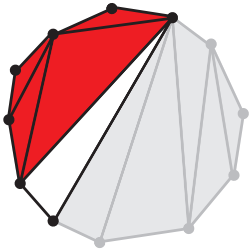
Bdd. faces of triangulated outerplanar graphs have a binary tree structure.

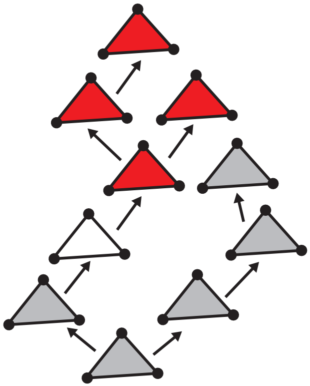




Triangle Trees

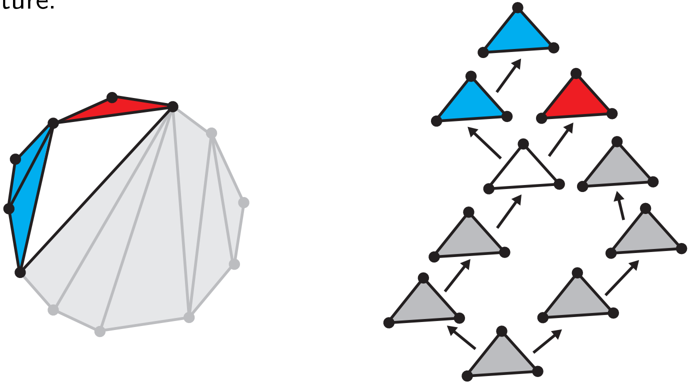
Bdd. faces of triangulated outerplanar graphs have a binary tree structure.





Triangle Trees

Bdd. faces of triangulated outerplanar graphs have a binary tree structure.



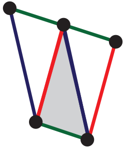
First Attempt

It would therefore be enough to draw every triangle tree G.

First Attempt

It would therefore be enough to draw every triangle tree G.

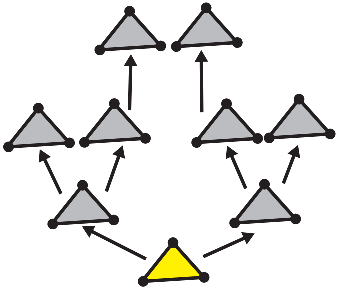
Naïve attempt: Take a general triangle T. Map every face of G to a copy of T preserving orientation.

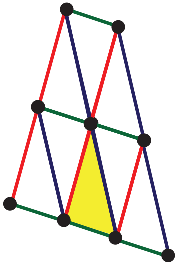


First Attempt

It would therefore be enough to draw every triangle tree G.

Naïve attempt: Take a general triangle T. Map every face of G to a copy of T preserving orientation.
We get a lattice...

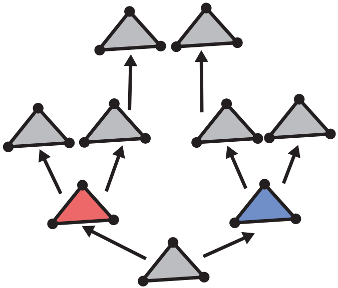


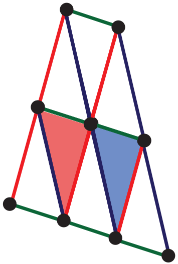


First Attempt

It would therefore be enough to draw every triangle tree G.

Naïve attempt: Take a general triangle T. Map every face of G to a copy of T preserving orientation.
We get a lattice...

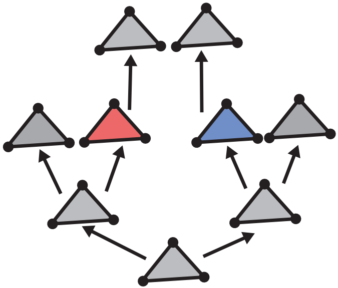


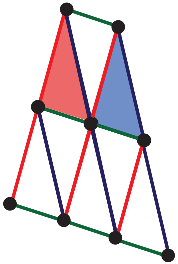


First Attempt

It would therefore be enough to draw every triangle tree G.

Naïve attempt: Take a general triangle T. Map every face of G to a copy of T preserving orientation.
We get a lattice...

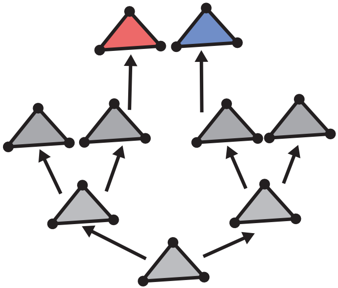


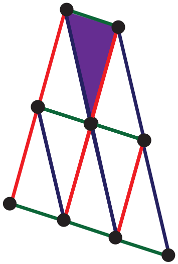


First Attempt

It would therefore be enough to draw every triangle tree G.

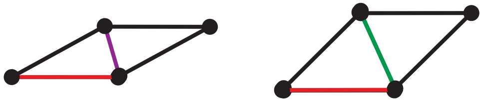
Naïve attempt: Take a general triangle T. Map every face of G to a copy of T preserving orientation.
We get a lattice... ...and fail due to commutativity.



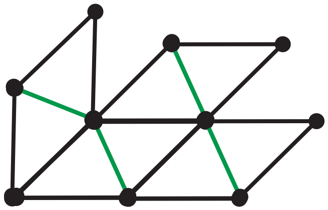


Rhombus

Rhombus graph, **base edge** in red.

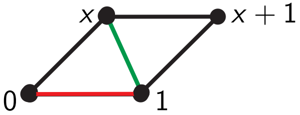


It is simpler to work with rhombi then with triangles.
Every triangle tree is covered by a tree of rhombi.



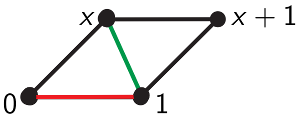
Embedding the Rhombi

External edges are always of length 1. $x \in \mathbb{T}$ the unit circle.

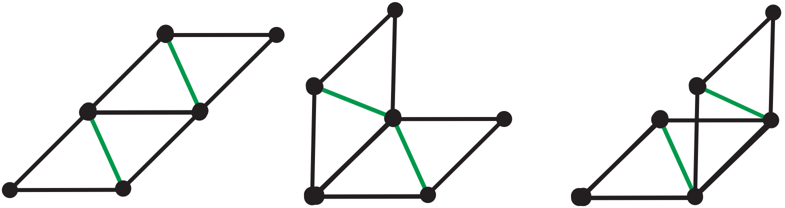


Embedding the Rhombi

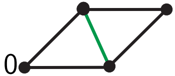
External edges are always of length 1. $x \in \mathbb{T}$ the unit circle.



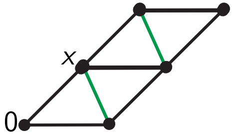
We embed adjacent rhombi always turning to the left.



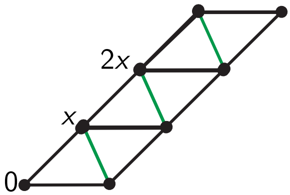
How does it look with one rhombus type



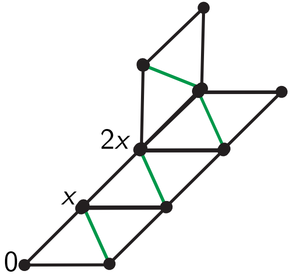
How does it look with one rhombus type



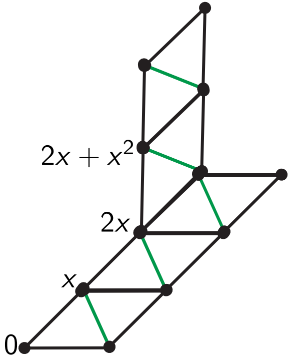
How does it look with one rhombus type



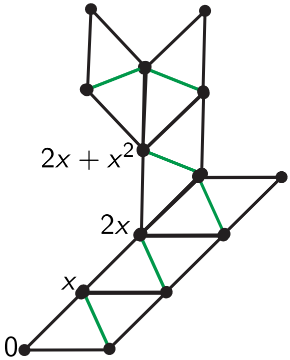
How does it look with one rhombus type



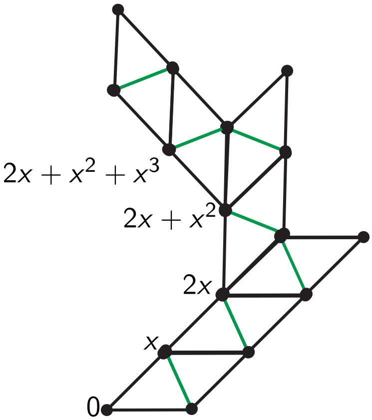
How does it look with one rhombus type



How does it look with one rhombus type



How does it look with one rhombus type



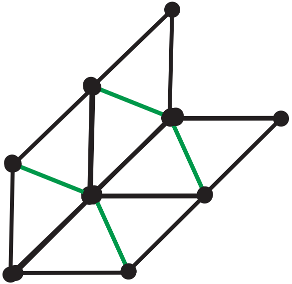
The position of a vertex is thus $\sum k_i x^i$.

Embedding the Rhombi - 2

It is enough to show that different vertices are mapped to different polynomials.

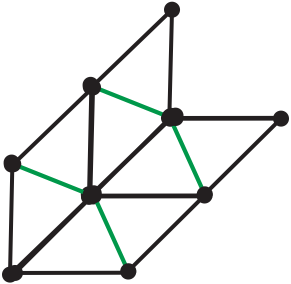
Embedding the Rhombi - 2

It is enough to show that different vertices are mapped to different polynomials. Local obstruction:



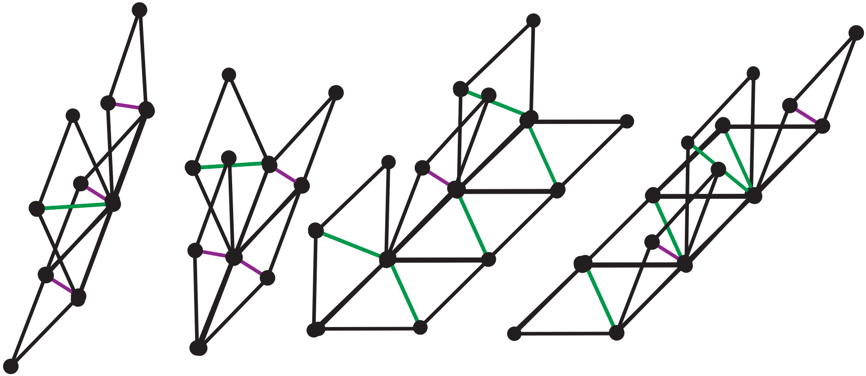
Embedding the Rhombi - 2

It is enough to show that different vertices are mapped to different polynomials. Local obstruction:



Solution: using two kinds of rhombi. The position of a vertex is thus $\sum_i \sum_j k_{ij} x^i y^j$.

Our Construction



What's not on this talk

What's not on this talk?

- The actual proof

What's not on this talk

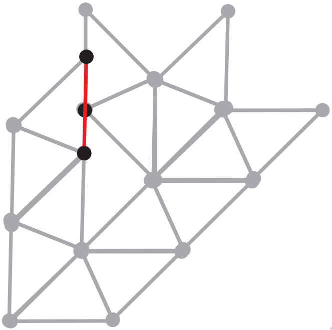
What's not on this talk?

- The actual proof
- Why the construction yields **degenerate** drawings

What's not on this talk

What's not on this talk?

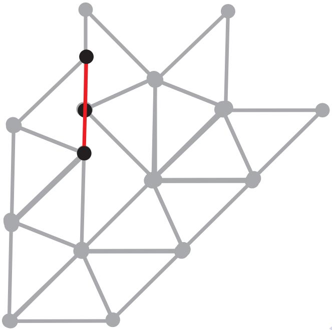
- The actual proof
- Why the construction yields **degenerate** drawings



What's not on this talk

What's not on this talk?

- The actual proof
- Why the construction yields **degenerate** drawings
- How do we think this can be overcome



Open Problems

Open Problems

- Tighten the bounds. In particular –

Open Problems

Open Problems

- Tighten the bounds. In particular –
- Is 2 the degenerate distance number of every outerplanar graph?

Open Problems

Open Problems

- Tighten the bounds. In particular –
- Is 2 the degenerate distance number of every outerplanar graph?
- Except outerplanarity and bdd. degree + bdd. treewidth, what other properties bound the asymptotics of the (degenerate) distance number?


Thank you