Long-range order in random 3-colorings in high dimensions

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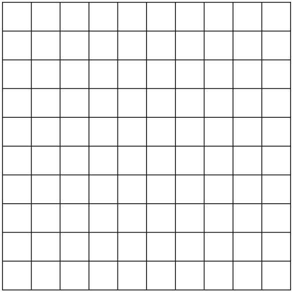
Joint work with Yinon Spinka

IMA, University of Minnesota

June 15, 2015

Setup and terminology

Consider a finite set $\Lambda \subset \mathbb{Z}^d$.



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0	2	0	1	0	2	0	1	0	2
1	0	1	0	1	0	1	2	1	0
0	1	2	2	2	1	2	1	0	2
1	0	0	1	0	2	1	0	2	0
0	1	2	1	2	1	0	1	0	1
1	0	1	2	1	0	1	2	1	0
0	1	0	1	0	2	0	1	0	2
2	0	2	0	1	0	2	0	2	1
1	1	0	2	0	1	0	1	1	2

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0	1	2	2	2	1	2	1	0	2
1	0	0	1	0	2	1	0	2	0
0	1	2	1	2	1	0	1	0	1
1	0	1	2	1	0	1	2	1	0
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1	0	1	0	2	0	1	0	1	0
0	2	0	2	1	2	0	1	0	2
2	0	1	0	2	0	1	0	2	1
1	2	0	2	0	1	2	1	0	2
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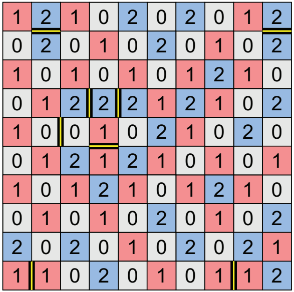
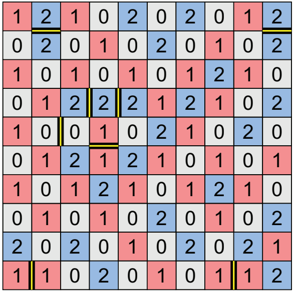
- Denote # of singularities in f:

$$N(f) := |\{u \sim v \ : \ f(u) = f(v)\}|.$$



The Potts Model

We consider a random 3-coloring f of $\Lambda \subset \mathbb{Z}^d$ with one parameter β called *inverse temperature*.



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0	1	2	2	2	1	2	1	0	2
1	0	0	1	0	2	1	0	2	0
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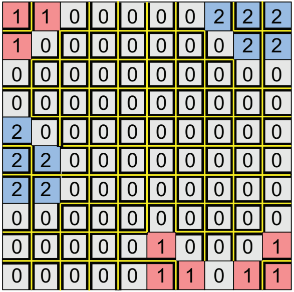

$\beta \ll 0$

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0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$\beta = -\infty$

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1	0	1	0	1	0	1	2	1	0
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1	0	0	1	0	2	1	0	2	0
0	1	2	1	2	1	0	1	0	1
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0	2	0	2	1	2	0	1	0	2
2	0	1	0	2	0	1	0	2	1
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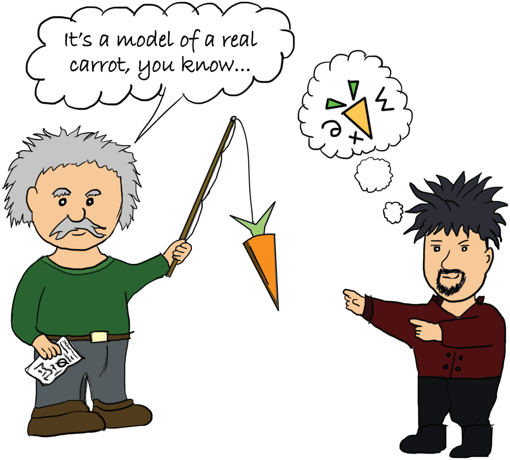
- q-states Potts is defined similarly

1	0	5	0	1	0	5	2	1	2
0	5	4	5	0	5	0	1	4	5
4	0	1	0	2	0	1	0	5	0
0	2	0	4	5	2	0	4	0	1
5	0	1	1	2	5	5	0	2	0
0	2	0	1	4	5	2	1	0	4
2	0	5	0	4	0	2	0	5	0
0	1	4	1	0	5	0	5	0	1
1	4	5	0	2	0	1	2	2	2
2	1	0	2	0	1	0	0	5	0

$\beta \gg 0, q = 5$

Motivation

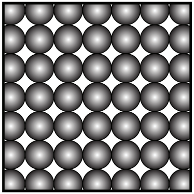


Motivation from statistical mechanics

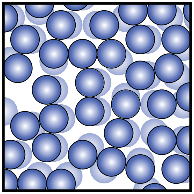
Equilibrium statistical mechanics = research of phases

Motivation from statistical mechanics

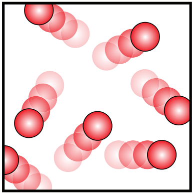
Phases of matter.



Solid



Liquid

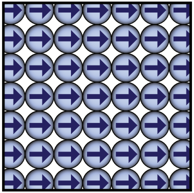


Gas

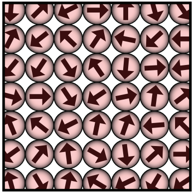
Phase depends on temperature and pressure.

Motivation from statistical mechanics

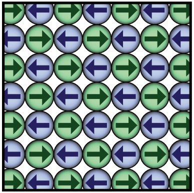
Magnetic phases.



Magnet



Paramagnet

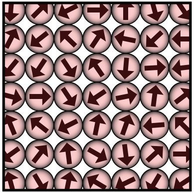


Anti-Ferromagnet

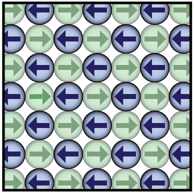
Phase depends on temperature and external field.

Motivation from statistical mechanics

Magnetic phases.



Diamagnet



Ferrimagnet

Phase depends on temperature and external field.

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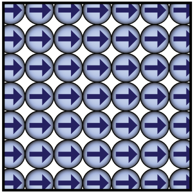

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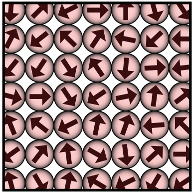

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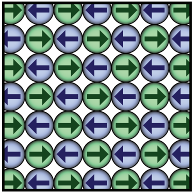

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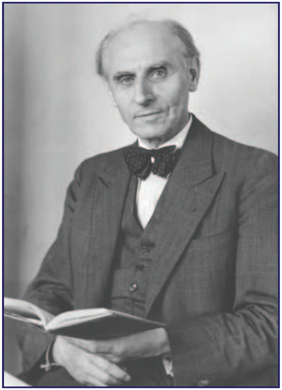
Goal: explain phases through microscopic mechanics

Wolfgang Paulli



Wilhelm Lenz

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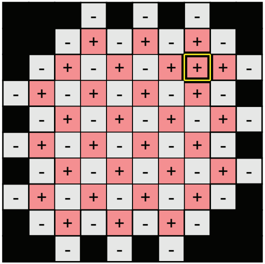

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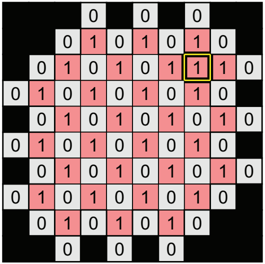

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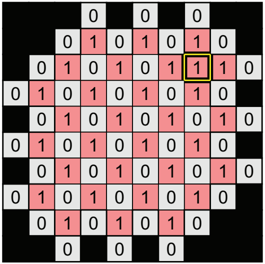

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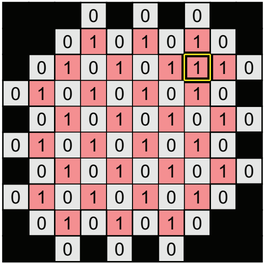

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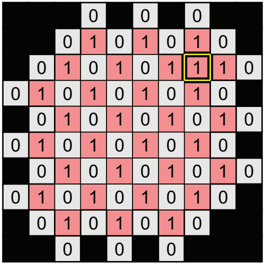
- Ferromagnet $(\beta < 0)$ and anti-ferromagnet $(\beta > 0)$ are equivalent.



Thermodynamical limit

Thermodynamical questions deal with large volume systems. That is **fixed** d, with $\mathbf{n} \to \infty$ (*thermodynamical limit*).

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Order vs. Disorder: dependence on boundary conditions.

- Λ large domain.
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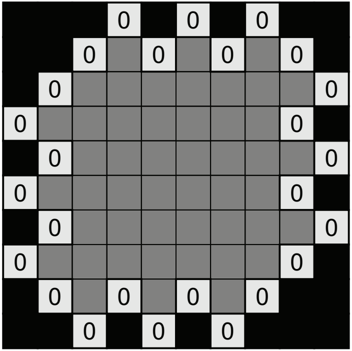

Even zero boundary conditions

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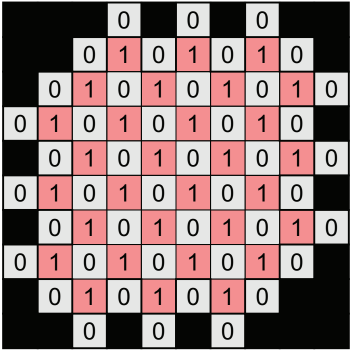

Sample with 0-boundary conditions on even domain

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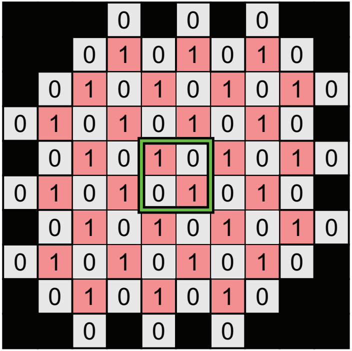

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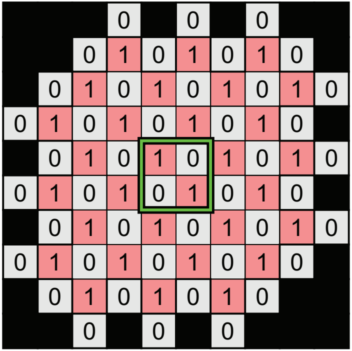

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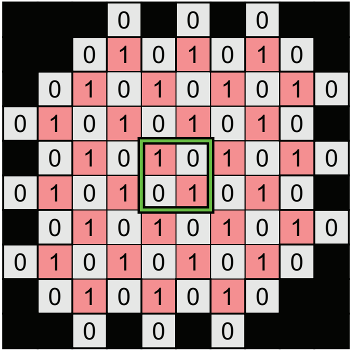
- Mature notions:
  Gibbs measures & pure phases.



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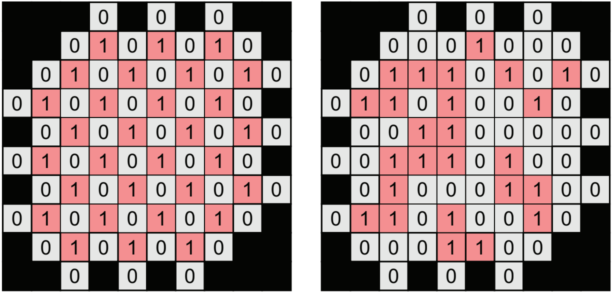
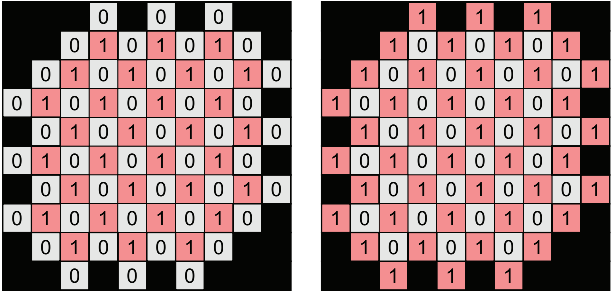


Questions about the model

Basic Questions:

- In which d does a phase transition occur?
- What does a typical $\beta \gg 0$ sample look like?



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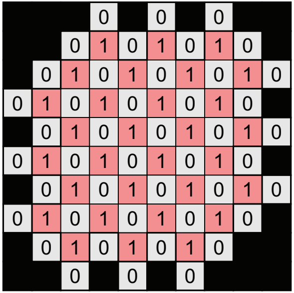
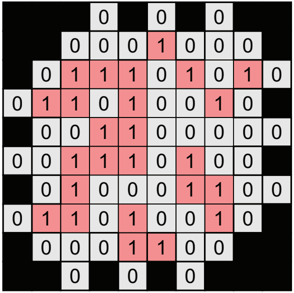
- Behavior at/near criticality?





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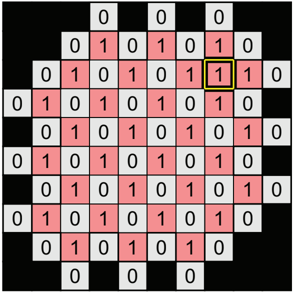

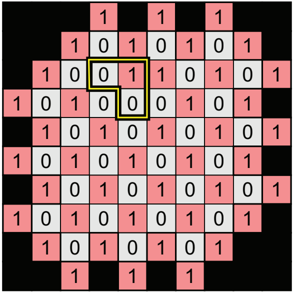

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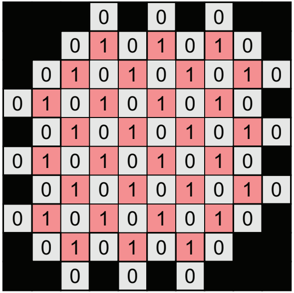
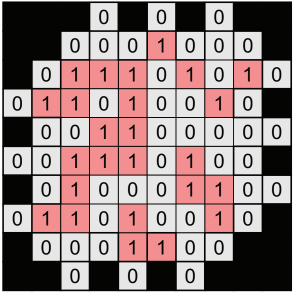
Advanced questions:

- Behavior at/near criticality?
- Rapid/Torpid mixing?
- How fast do correlations decay?





Using zero-boundary conditions

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More specific strategy for $\beta \gg 0$.

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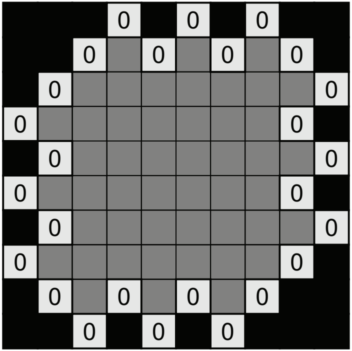

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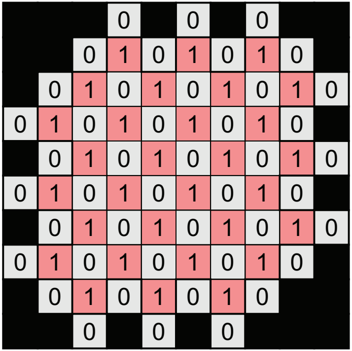


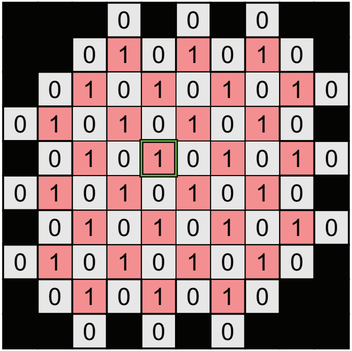
Sample with 0-boundary conditions on even domain

Using zero-boundary conditions

How to demonstrate multiple pure phases?

More specific strategy for $\beta \gg 0$.

- Λ large even domain.
- Condition on $f(v) = 0$ for all v on the boundary.
- Show that the frequencies on even and odd sublattice are unbalanced.



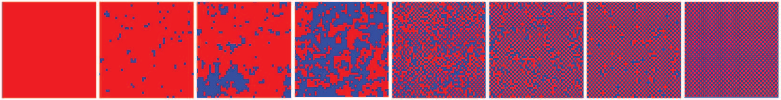
sample with 0-boundary conditions on even domain

Properties of the Ising model

Answers to these questions are now known for the Ising model ($q = 2$):

- In all $d \geq 2$ there is a critical temperature $1/\beta_c = \Theta(d)$ (error terms are known).
- $\beta < \beta_c$ implies a unique pure state.
- $\beta > \beta_c$ implies two pure states.
- In $\beta > \beta_c$ one sublattice is biased towards $+$ and the other towards $-$.

Ising $2d$ ferromagnets and anti-ferromagnets:

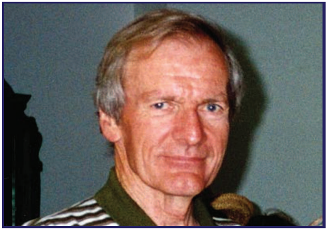


$\beta =$ $-\infty$ $\ll 0$ $F-Crit$ <0 >0 $AF-Crit$ $\gg 0$ ∞

Beyond Ising

Clock and Potts models.

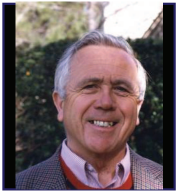
Cyril Domb



Renfrey Potts

Kotecky Conjecture

Baxter (1982): $d = 2, q = 3$ Potts AF - critical at $\beta = \infty$.



Rodney Baxter

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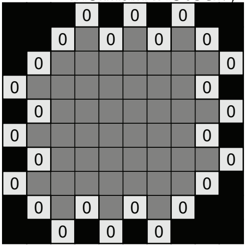

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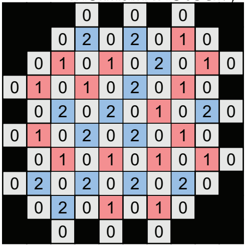
- For $\beta > \beta_c$: six pure states (phase co-existence).
- Each state corresponds to one color dominant on one sublattice and nearly absent from the other.



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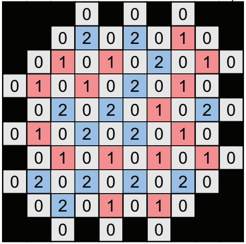

- For $\beta > \beta_c$: six pure states (phase co-existence).
- Each state corresponds to one color dominant on one sublattice and nearly absent from the other.
- For $\beta < \beta_c$: one disordered pure phase, correlations decay exponentially fast.

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3rd law: *the entropy of a perfect crystal at absolute zero is zero.*

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1	2	0	2	0	1	2	1	0	2
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The remaining entropy is called **residual entropy**.

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1	0	1	2	1	0	1	0	1	2
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1	0	1	0	2	0	1	0	1	0
0	2	0	2	1	2	0	1	0	2
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1	2	0	2	0	1	2	1	0	2
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This is very encouraging, but fixed n is irrelevant for thermodynamical limits.

Zero Temperature through other model

Galvin and Kahn(2004): $d \gg 0$ hard-core (independent set) model has a phase transition.

David Galvin Jeff Kahn

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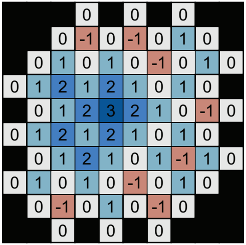
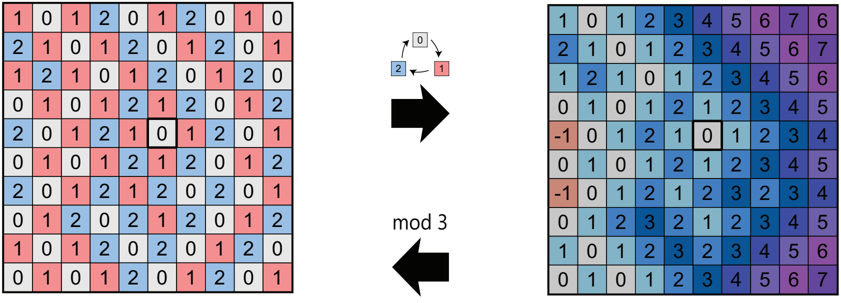
Peled(2010): $d \gg 0$ $\hom(\mathbb{Z}^d, \mathbb{Z})$ with zero boundary conditions fluctuate mainly between ± 1.



Ron Peled

Homomorphism height functions and 3-colorings

There is a natural bijection between 3-colorings and $\mathrm{hom}(\mathbb{Z}^d, \mathbb{Z})$.



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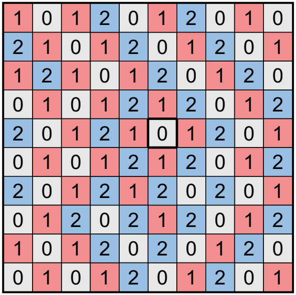

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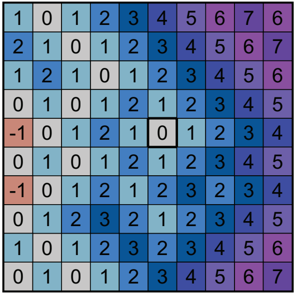
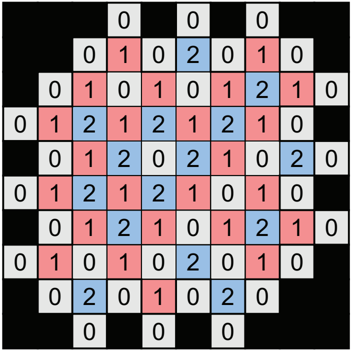


Pointed HHFs

HHF values between $\pm 1 \Rightarrow$ Coloring values of even 0, odd 1,2.

Zero-temperature case of the Kotecky conjecture.

...and hence for $\beta = \infty$ the conjecture has been verified:



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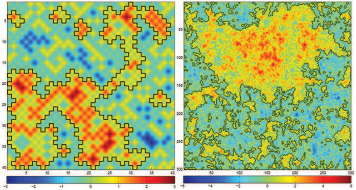
- Zero-temperature has no physical meaning.

Peled's method for $\beta = \infty$

The main proposition in Peled's method is that external level line of length L around a vertex are $\exp(-cL/d\log^2 d)$ unlikely.



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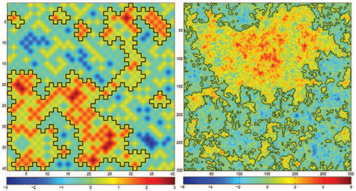

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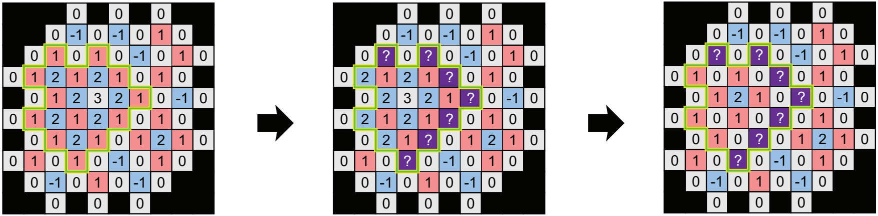

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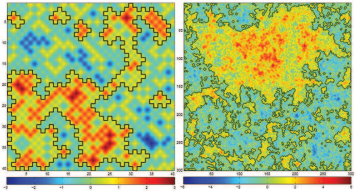
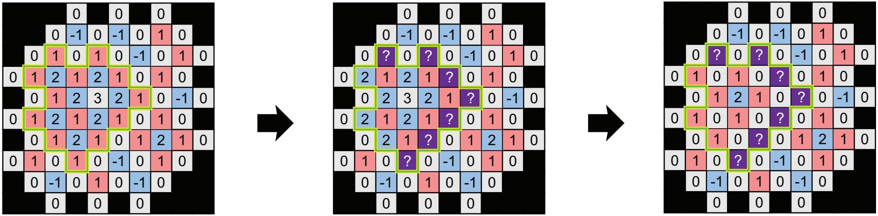


Level lines from Peled's paper

The main ingredient is the shift-minus transformation, whose entropy gain is $\frac{L}{2d}$.



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Peled's method and the special case of 3-states

Write F_L for colorings with contour of length L around v. We thus map: each $f \in F_L$, to $2^{L/2d}$ other colorings.
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Non-trivial. Hard to estimate in-degree, and requires either

- (Peled) altering the map to avoid high in-degree.
- (Galvin *& al.*) probabilistic biasing (flow method).

Beyond proper colorings of $\mathbb{Z}^d$

It is non-trivial to extend this result even to colorings of the torus:

0	1	0	1	0	2	0	1	0	1	0
1	0	1	2	1	0	1	0	1	0	1
2	1	0	1	0	1	2	1	0	1	2
0	2	1	0	1	2	0	2	1	2	0
2	1	2	1	2	1	2	1	0	1	2
1	2	1	0	1	0	1	0	1	2	1
0	1	0	1	2	1	0	1	0	1	0
2	0	1	2	1	0	1	2	1	0	2
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1	0	1	0	2	0	1	2	1	2	1
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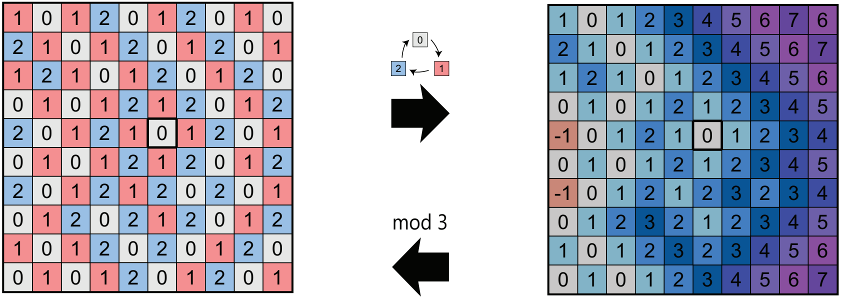
Periodic boundary conditions

Beyond proper colorings of $\mathbb{Z}^d$

The bijection does not extend to the torus.



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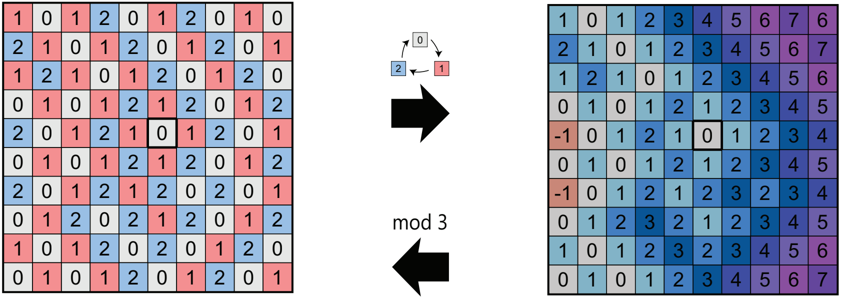

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However, algebraic topology says that it nearly does.

Beyond zero-temperature

Periodic boundary rigidity at zero-temperature (F. & Peled 2013)

In high dimension, a typical uniformly chosen proper 3-coloring with periodic boundary conditions is nearly constant on either the even or odd sublattice.

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In high dimension, a typical uniformly chosen proper 3-coloring with periodic boundary conditions is nearly constant on either the even or odd sublattice.

- This is a first step beyond the HHF structure.

Positive temperature

Positive temperature

Finding contours in positive temperature is quite problematic...

1	2	1	0	2	0	2	0	1	2
0	2	0	1	0	2	0	1	0	2
1	0	1	0	1	0	1	2	1	0
0	1	2	2	2	1	2	1	0	2
1	0	0	1	0	2	1	0	2	0
0	1	2	1	2	1	0	1	0	1
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2	0	2	0	1	0	2	0	2	1
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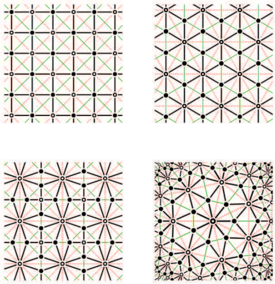
$\beta \gg 0$ sample

Remark - Asymmetric case.

The 3-state AF Potts model has recently been studied on asymmetric planar lattices.

Kotecky, Sokal and Swart (2013):
In such lattices there is a phase transition at positive temperature, with 3 pure states.



Lattices from KSS paper

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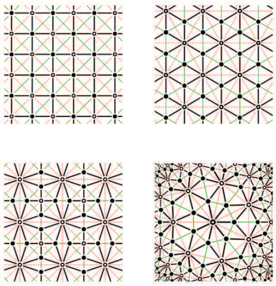
The proof uses the asymmetry to define and exploit better the phase interface.



Lattices from KSS paper

Positive temperature on $\mathbb{Z}^d$

To implement the idea of Peled's proof we require:

- alternative for contours,
- alternative for the transformation,
- better method for using the entropy,
- method to bound the in-degree of a coloring.

Breakup

A key definition in approaching positive temperature is that of a **Breakup** (w.r.t. to a vertex v_1), in lieu of Peled's sublevel components.

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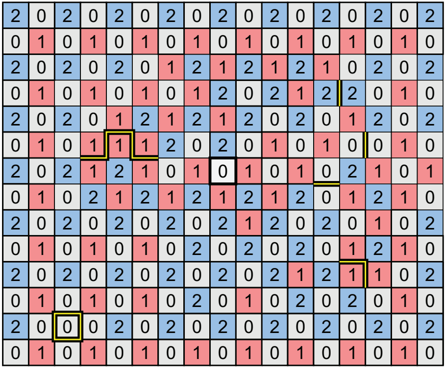
We start by defining four phases for vertices:



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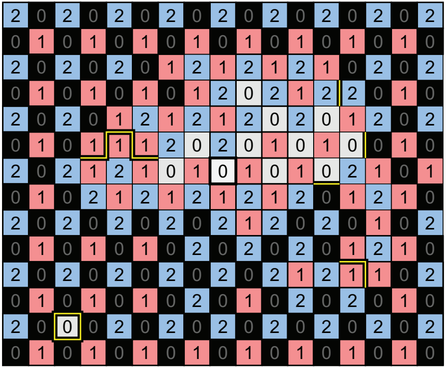
Phase $0 :=$ even 0



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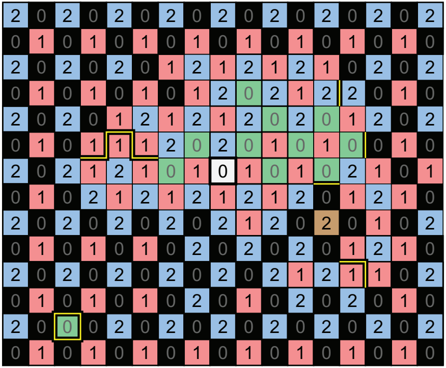

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Phase 1 := odd 1, even 2

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0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
2	0	2	0	2	0	1	2	1	2	1	2	1	0	2	0	2
0	1	0	1	0	1	0	1	2	0	2	1	2	2	0	1	0
2	0	2	0	1	2	1	2	1	2	0	2	0	1	2	0	2
0	1	0	1	1	1	2	0	2	0	1	0	1	0	0	1	0
2	0	2	1	2	1	0	1	0	1	0	1	0	2	1	0	1
0	1	0	2	1	2	1	2	1	2	1	2	0	1	2	1	0
2	0	2	0	2	0	2	0	2	1	2	0	2	0	1	0	2
0	1	0	1	0	1	0	2	0	2	0	2	0	1	2	1	0
2	0	2	0	2	0	2	0	2	0	2	1	2	1	1	0	2
0	1	0	1	0	1	0	2	0	1	0	2	0	2	0	1	0
2	0	0	0	2	0	2	0	2	0	2	0	2	0	2	0	2
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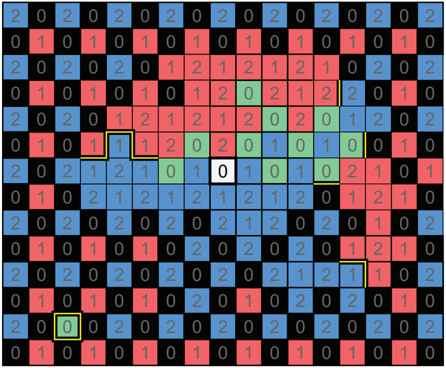

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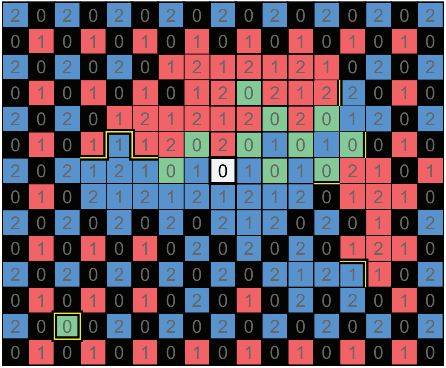

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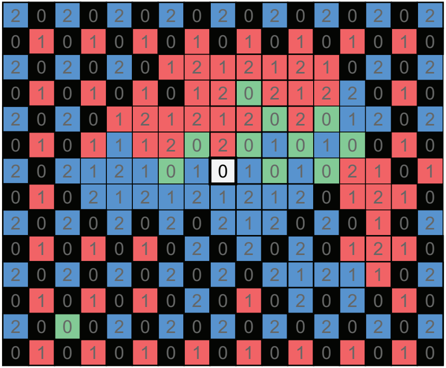
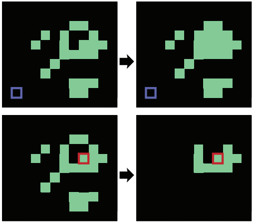


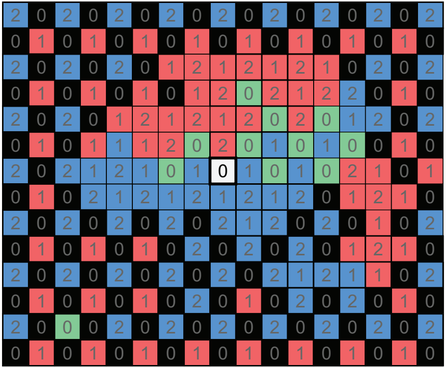
Breakup

The first ingredient in our proof is a notion of a **Breakup** w.r.t. an odd vertex v_1. This - in lieu of Peled's sublevel components.

We now repeatedly take *co-connected closures*:

complement → conn. component → complement





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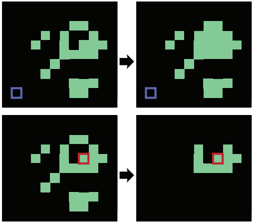

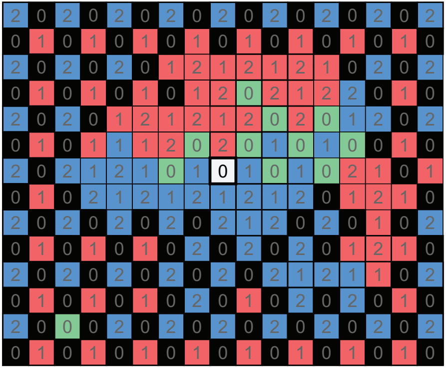

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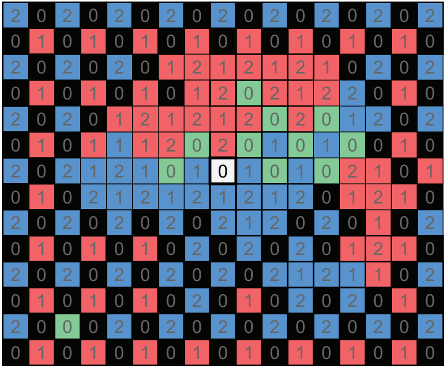

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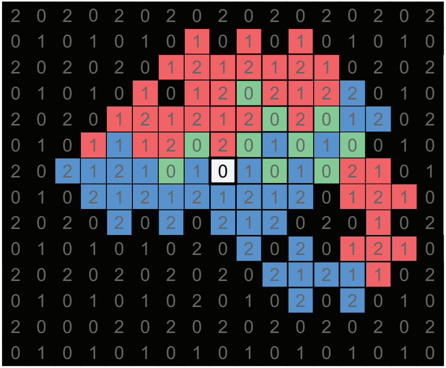

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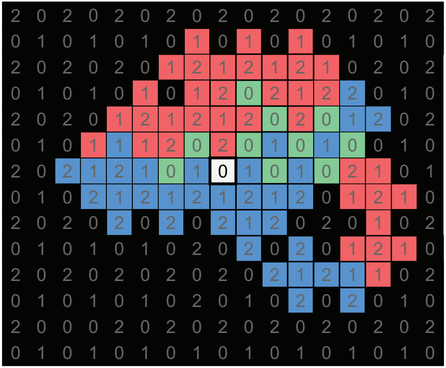

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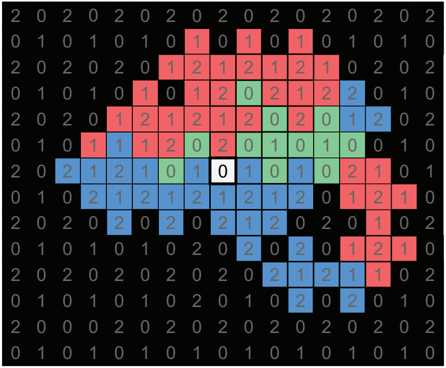

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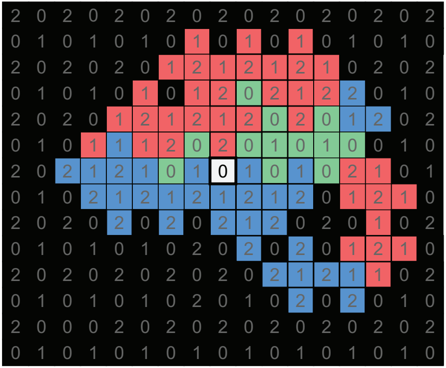

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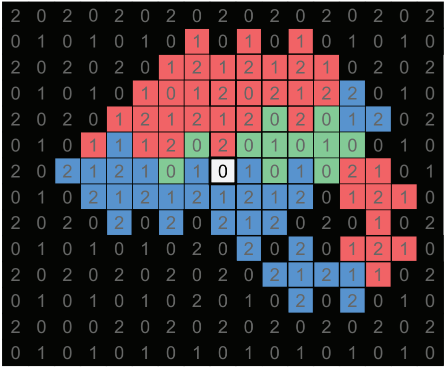

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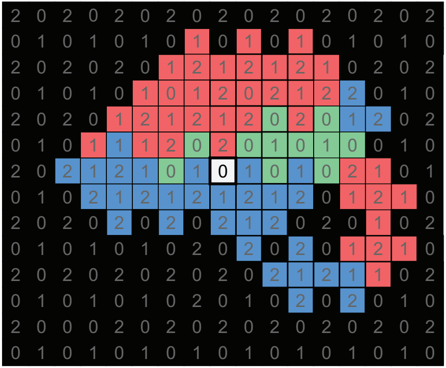

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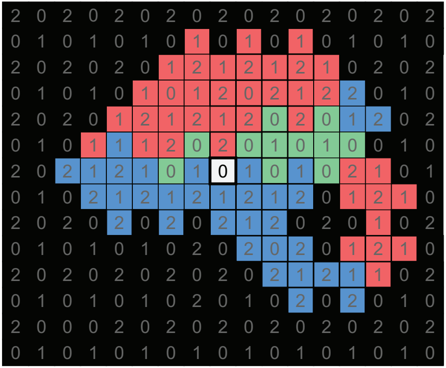

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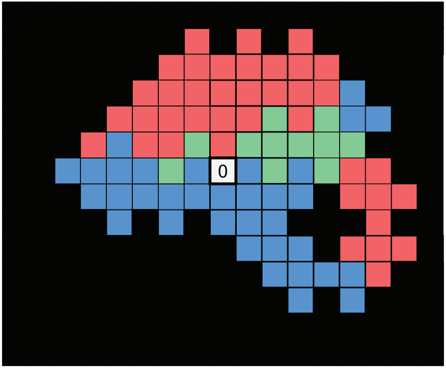
The result is the **Breakup.**



Transformation family

We now extend Peled's transformation to breakups.

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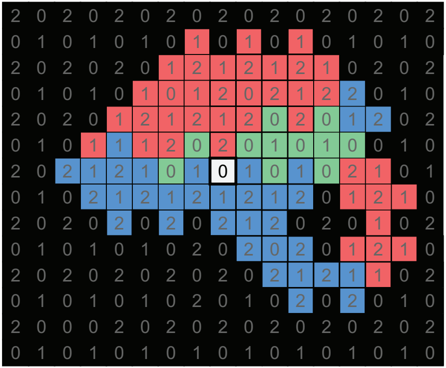

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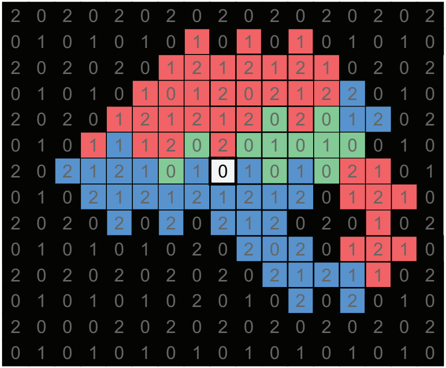

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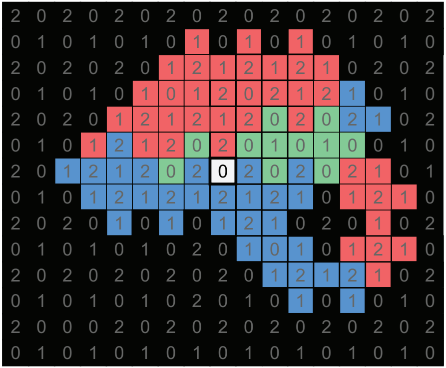

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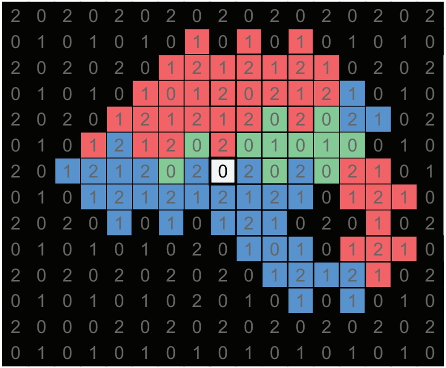

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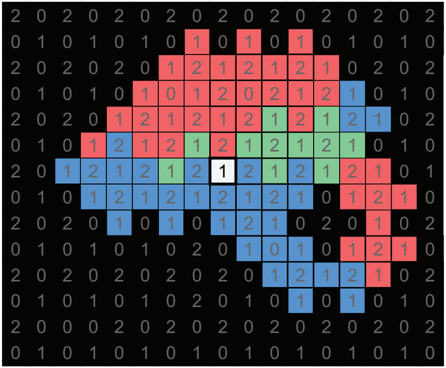

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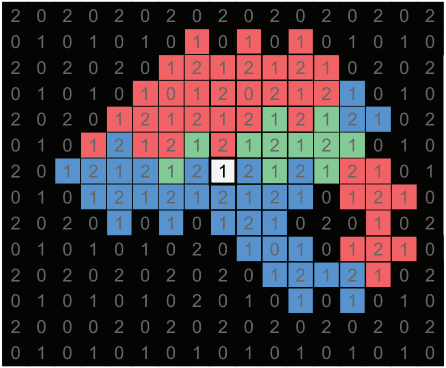

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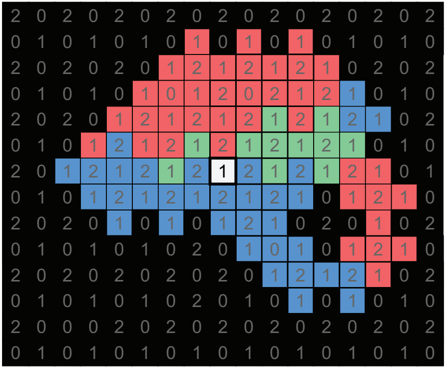

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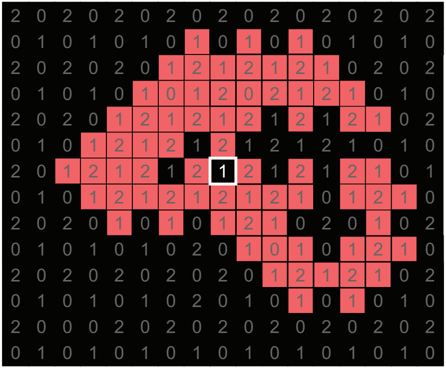

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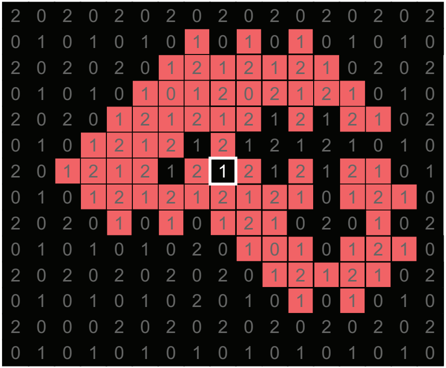

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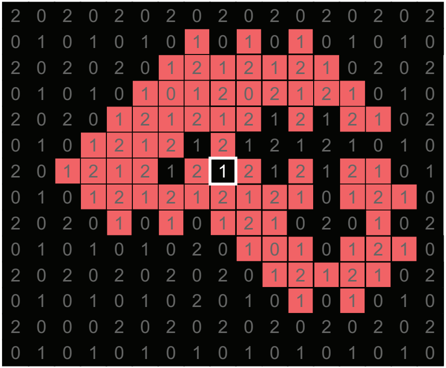

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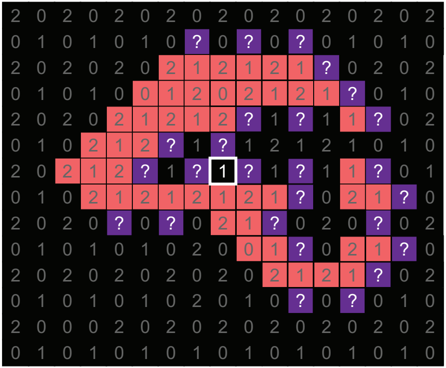

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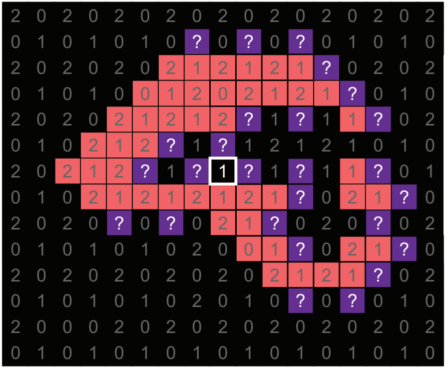

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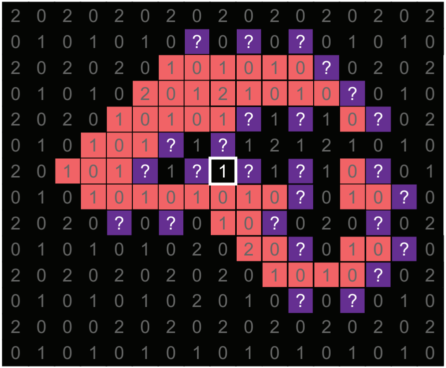

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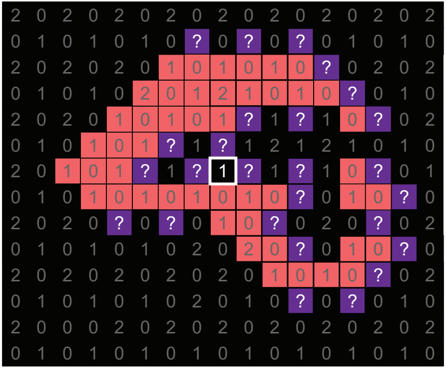

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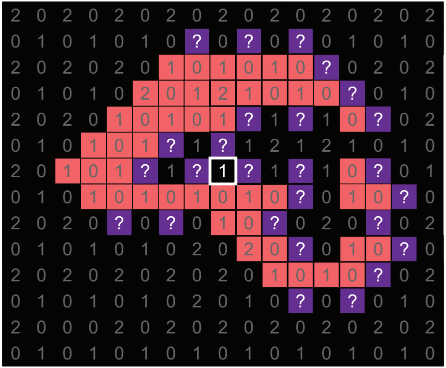

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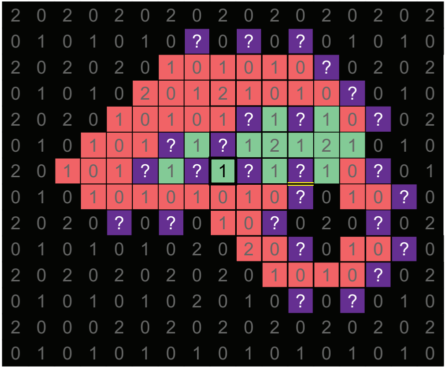

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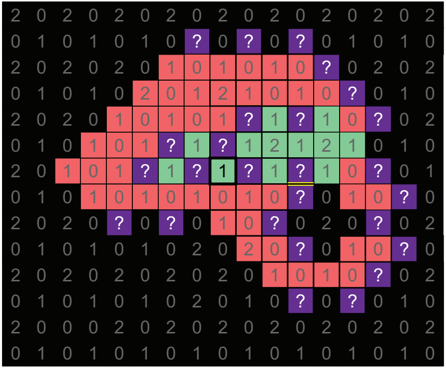

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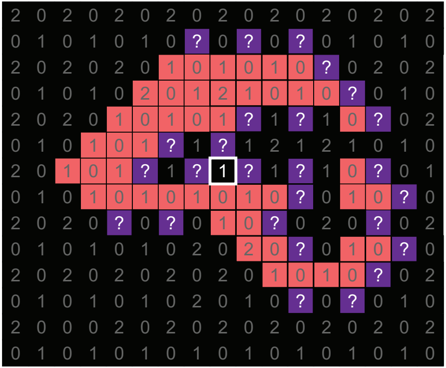

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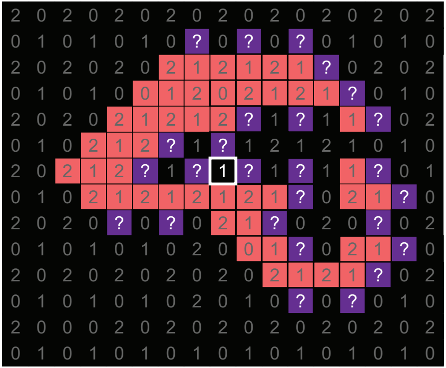

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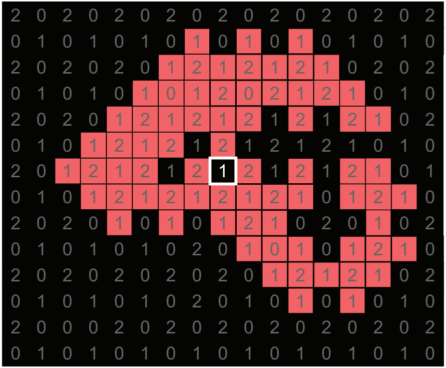

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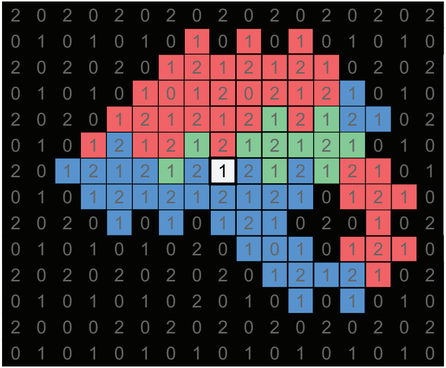

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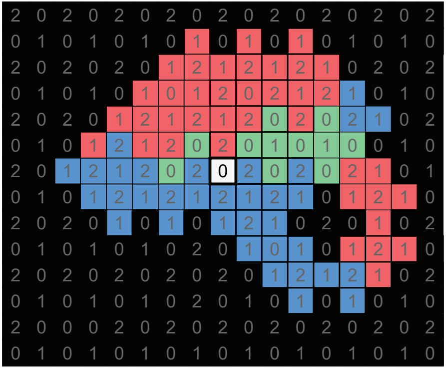

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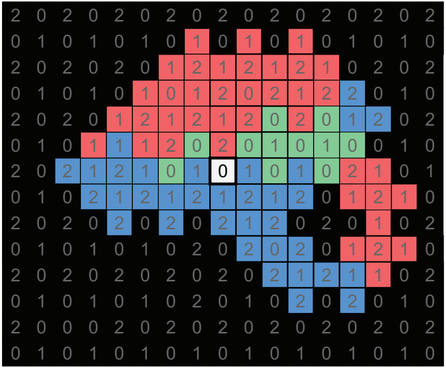

Results

Properties of breakups:

- When the coloring is proper it coincides with Peled's contours.
- We can show, using improved flow methods, that breakups with long boundary are unlikely.

Results

0-boundary rigidity at positive temperature (F. & Spinka 2015+)

For every d high enough, there exists β_0 such that in a typical sample of the 3-state AF Potts with 0-boundary conditions and $\beta > \beta_0$, nearly all the even vertices take the color 0.

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- Open: show that β_0 decreases with d.

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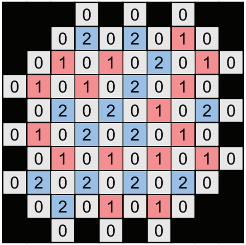

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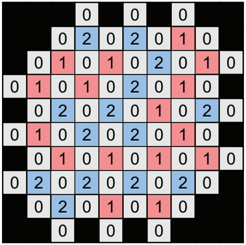
- We also have a "structural" theorem which provides similar bounds on other deviations from the pure state.



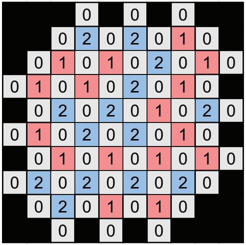
Result

0-boundary rigidity at positive temperature (F. & Spinka 2015+)

For every d high enough, there exists β_0 such that in a sample of the 3-state AF Potts with 0-BC and $\beta > \beta_0$, we have

1. $\mathbb{P}(f(v) \neq 0) < e^{-cd}$ for all even v,
2. $\mathbb{P}(f(u) = 0) < e^{-cd}$ for all odd u,
3. $\mathbb{P}(f(u) = f(v)) < e^{-cd-\beta}$ for all $u \sim v$.

- We also have a "structural" theorem which provides similar bounds on other deviations from the pure state.
- Our results allow us to prove convergence to an infinite-volume measure under 0-boundary conditions.



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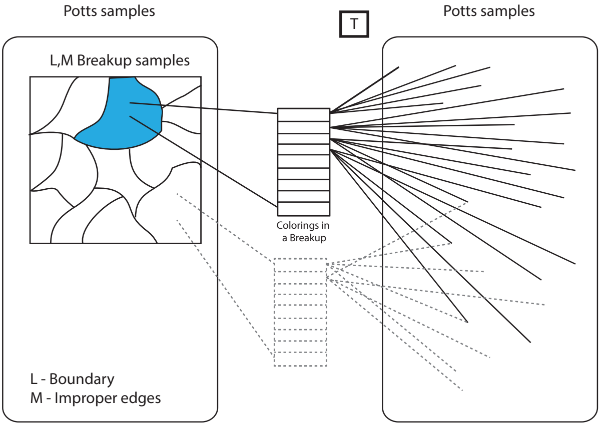
- how to bound the indegree of every configuration.

Bounding the indegree

Flow one measure unit from every coloring.



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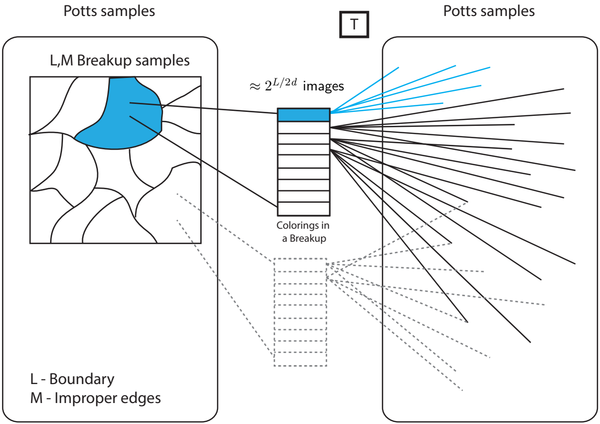

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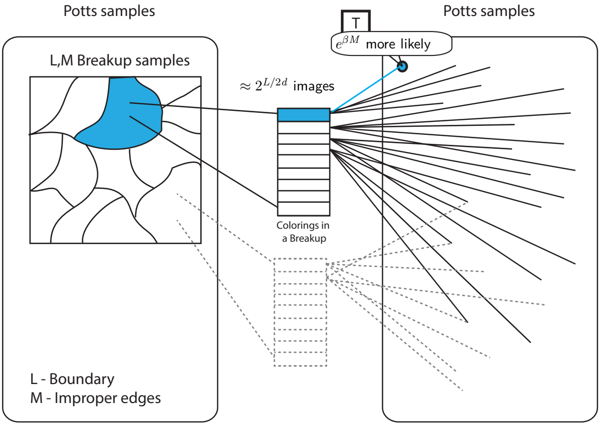

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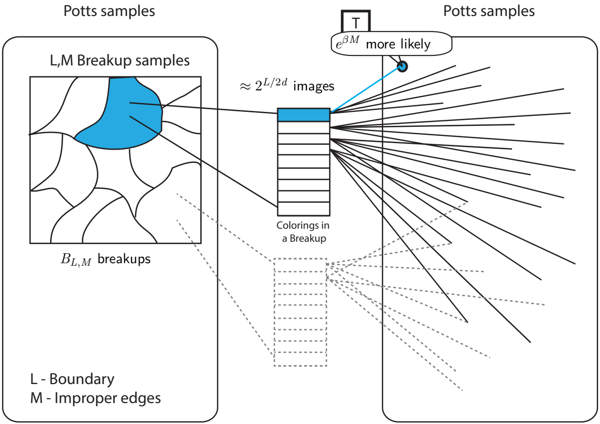

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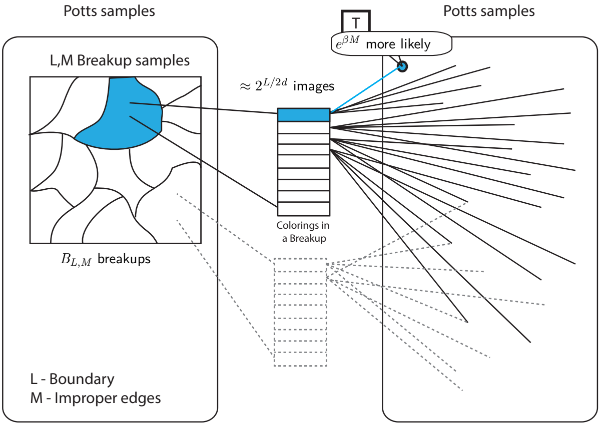

$$\mathbb{P}(\exists B \in B_{L,M} \;:\; f \in B) \le |B_{L,M}| \cdot e^{-\beta M} \cdot 2^{-L/2d}$$

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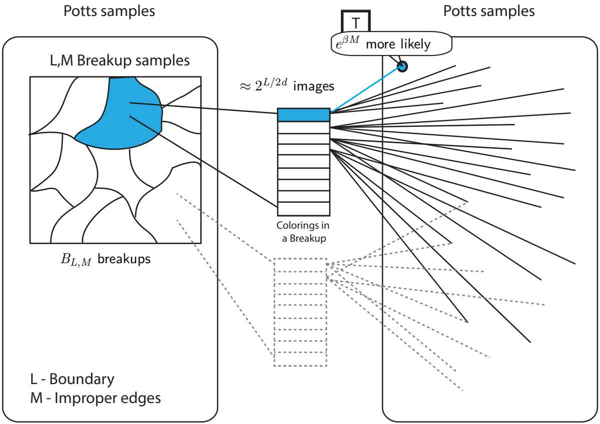

Not good enough!

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Flow

Flow principle:

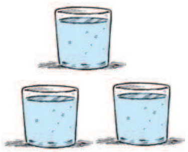
Let S, D be two finite sets.
Given a flow $\nu\colon S \times D \to [0,1]$, such that

for every $s \in S$, we have $\sum_{d \in D} \nu(s,d) \geq 1$ and

for every $d \in D$, we have $\sum_{s \in S} \nu(s,d) \leq p$,

we can deduce

$$|S| \leq p|D|.$$



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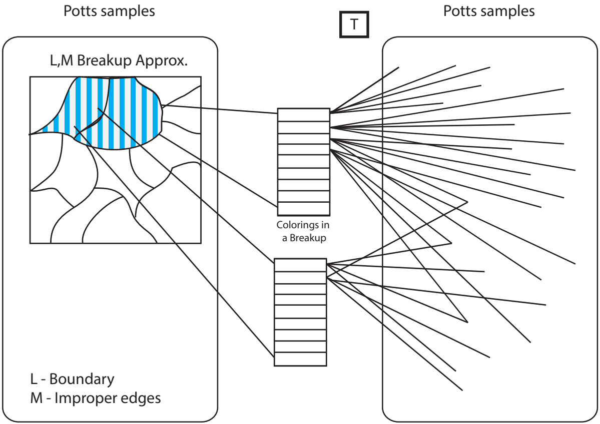

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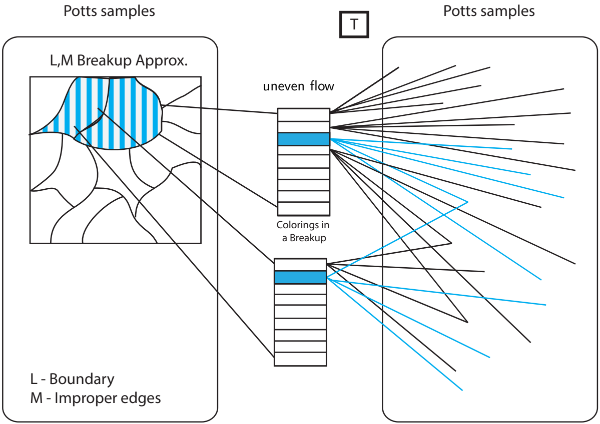

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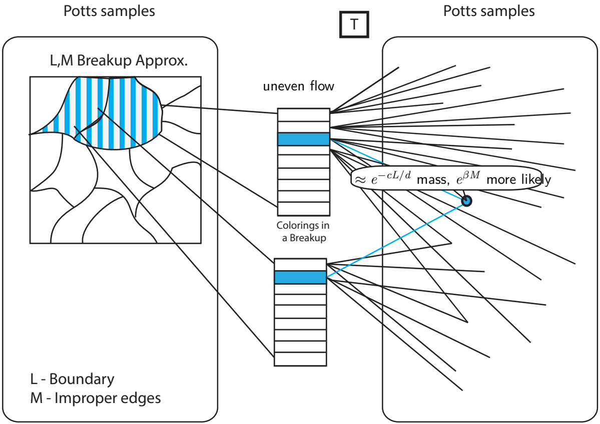

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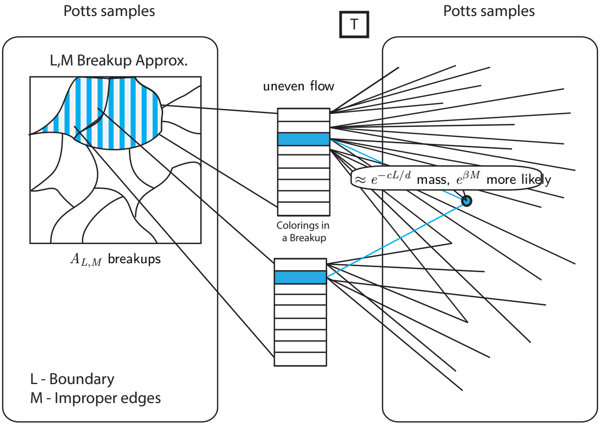

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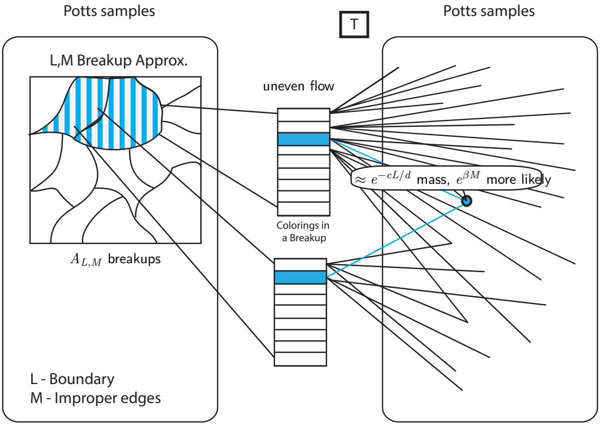

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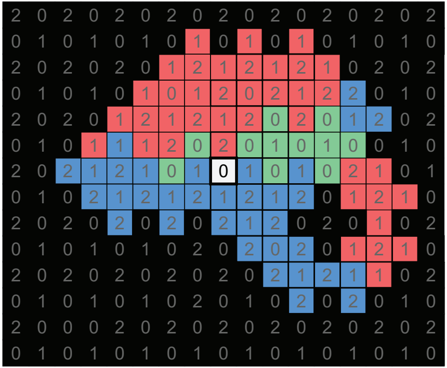
Approximation

A key step inspired by previous methods is to obtain a small family of approximations for the Breakup.



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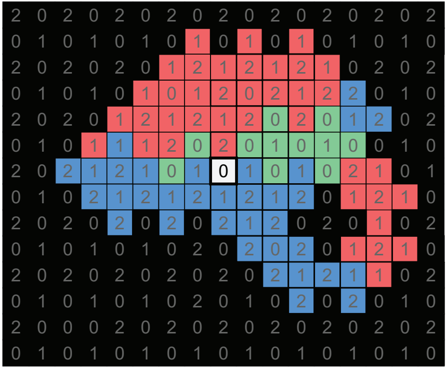

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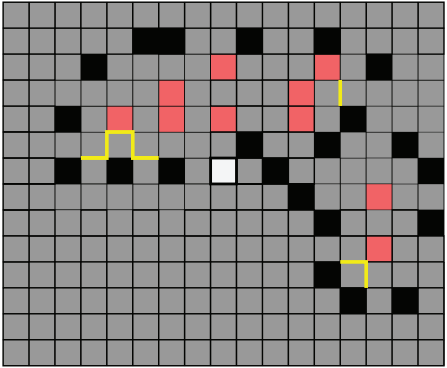

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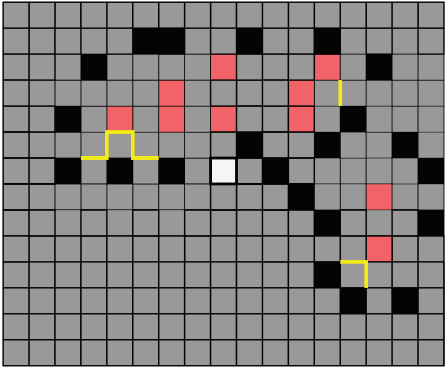

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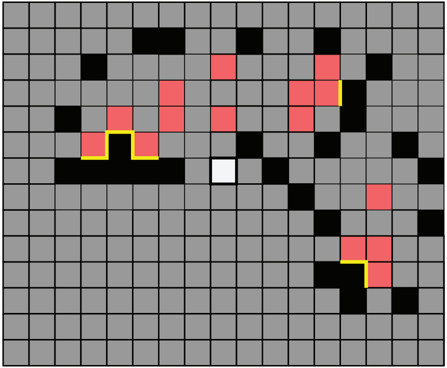

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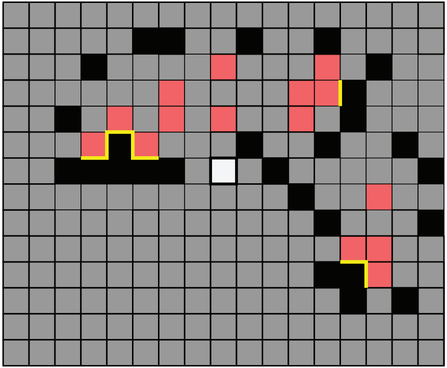

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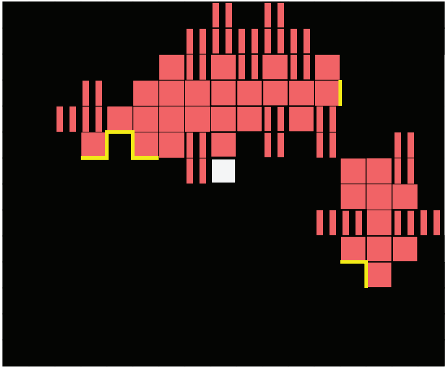

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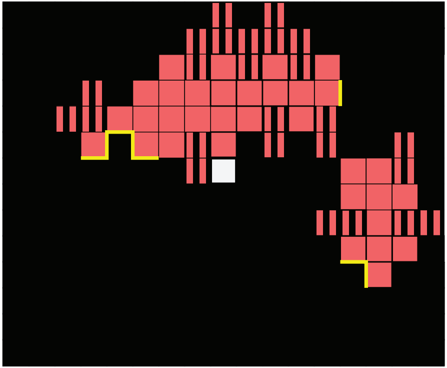

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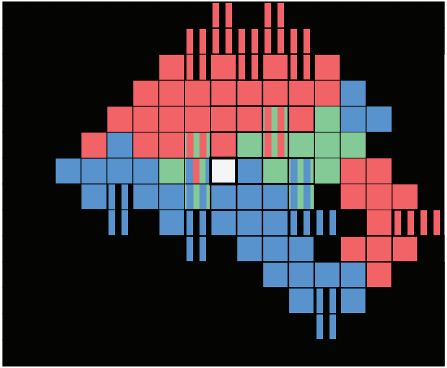

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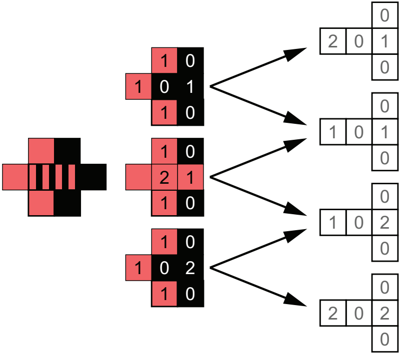

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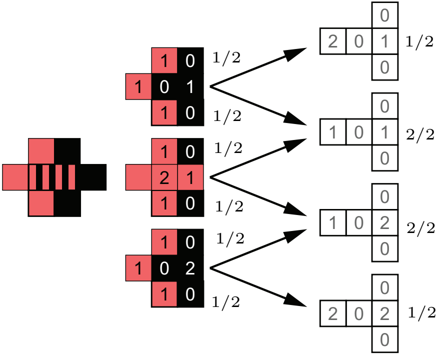
If we use our entropy uniformly we get:



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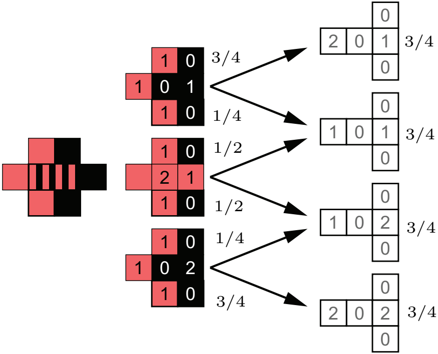
However if we use it more carefully we get:



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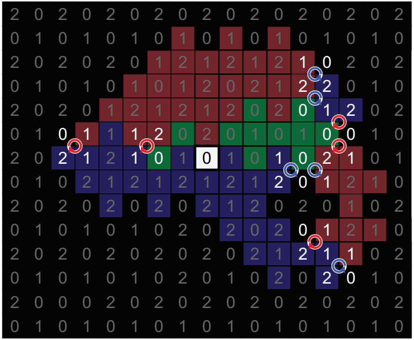
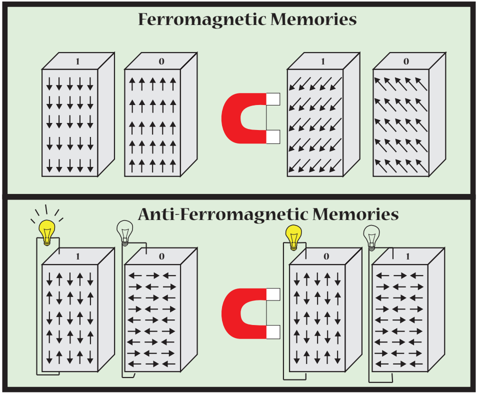


Thank you!



Room temperature AF memory resistor (Marti et al.)