3/2 firefighters are not enough

Ohad N. Feldheim[*]   Rani Hod[†]

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Abstract

The firefighter problem is a monotone dynamic process in graphs that can be viewed as modeling the use of a limited supply of vaccinations to stop the spread of an epidemic. In more detail, a fire spreads through a graph, from burning vertices to their unprotected neighbors. In every round, a small amount of unburnt vertices can be protected by firefighters. How many firefighters per turn, on average, are needed to stop the fire from advancing?

We prove tight lower and upper bounds on the amount of firefighters needed to control a fire in the Cartesian planar grid and in the strong planar grid, resolving two conjectures of Ng and Raff.

1 Introduction

The firefighter problem is the following dynamic problem introduced by Hartnell [8]. Given an undirected graph $G = (V, E)$, a fire initially breaks out at a nonempty subset of vertices $\varnothing \subset S \subset V$. In every round t, $f(t)$ firefighters are available to be positioned at vacant and unburnt vertices of G. These firefighters remain on their assigned vertices for the entire process, protecting them from the fire. At the end of each round, the fire spreads to all unprotected vertices adjacent to at least one burnt vertex.

For infinite graphs, two scenarios are possible:

(i) In finite time, the fire is controlled (i.e., is unable to spread further) and thus all but a finite number of vertices remain unburnt and unprotected.

(ii) The fire spreads indefinitely.

Natural questions that can be asked are whether the fire can be controlled, and, if so, how fast; a related question is how many vertices can we save: absolute number for finite graphs, measure (defined properly) for infinite graphs.

The firefighter problem was considered for a variety of families of graphs, including infinite grids [3, 5, 15, 16, 17, 19], finite grids [13, 19], and trees [6, 8].

In this paper we focus on two infinite grids: the Cartesian grid $\mathbb{Z} \,\square\, \mathbb{Z}$, which is the 4-regular graph on the vertex set $\mathbb{Z} \times \mathbb{Z}$ in which the neighbors of every vertex form a sphere of radius 1 with respect to the ℓ_1 metric, and the strong grid $\mathbb{Z} \boxtimes \mathbb{Z}$, which is the 8-regular graph on the vertex set $\mathbb{Z} \times \mathbb{Z}$ in which the neighbors of every vertex form a sphere of radius 1 with respect to the ℓ_∞ metric. A third infinite grid, which we only briefly mention, is the 6-regular triangular grid $\mathbb{Z} \,\triangle\, \mathbb{Z}$ satisfying $\mathbb{Z} \,\square\, \mathbb{Z} \subset \mathbb{Z} \,\triangle\, \mathbb{Z} \subset \mathbb{Z} \boxtimes \mathbb{Z}$.

We refer henceforth to vertices of these grids as points.

[*]School of Mathematics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. E-mail: ohad_f@netvision.net.il. Research supported by an ERC advanced grant.

[†]School of Computer Science, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel. E-mail: rani.hod@cs.tau.ac.il. Research supported by an ERC advanced grant.

1.1 Previous results

Wang and Moeller [19] proved that when $f \equiv 1$, a single-source fire cannot be controlled even in the non-negative quadrant $\mathbb{N} \square \mathbb{N}$ of $\mathbb{Z} \square \mathbb{Z}$. With an additional firefighter ($f \equiv 2$) a single-source fire in $\mathbb{Z} \square \mathbb{Z}$ can be controlled within 8 turns and 18 burnt points. Fogarty [5] proved that with $f \equiv 2$ firefighters, any finite-source fire in $\mathbb{Z} \square \mathbb{Z}$ can be controlled. Messinger [16] proved that for any $n \in \mathbb{N}$, a single-source fire in $\mathbb{Z} \square \mathbb{Z}$ can be controlled using the periodic function

$$f(t) = \begin{cases} 2, & t \bmod (2n+1) \text{ is zero or odd;} \\ 1, & t \bmod (2n+1) \text{ is even and nonzero,} \end{cases}$$

whose average is $(3n+2)/(2n+1) = 3/2 + O(1/n)$. Ng and Raff [18] proved that any periodic function f whose average exceeds $3/2$ allows the firefighters to control any finite-source fire in $\mathbb{Z} \square \mathbb{Z}$.

Develin and Hartke [3] proved that, for $d \geq 3$, a single-source fire in $\mathbb{Z}^{\square d} = \mathbb{Z} \square \cdots \square \mathbb{Z}$ cannot be controlled using $f \equiv 2d-2$ firefighters (and is controlled by $f \equiv 2d-1$ firefighters within just two turns). Moreover, they showed that for any fixed m, $f \equiv m$ firefighters cannot control an m^2-source fire in $\mathbb{Z}^{\square d}$.

Fogarty [5] claimed that $f \equiv 2$ firefighters cannot control a single-source fire in the triangular grid $\mathbb{Z} \triangle \mathbb{Z}$ but her proof is not complete. Messinger [15] proved that slightly more firefighters can control it; namely, for any $n \in \mathbb{N}$ she describes a strategy using $f(t) = \begin{cases} 3, & t = 0 \bmod n; \\ 2, & t \neq 0 \bmod n \end{cases}$ firefighters.

Messinger [17] claimed that $f \equiv 3$ firefighters cannot control a single-source fire in the strong grid $\mathbb{Z} \boxtimes \mathbb{Z}$, or even to restrain it to a single quadrant, but here, too, the proof is not complete. She proved that slightly more firefighters can control it; that is, for any $n \in \mathbb{N}$ her scheme needs only $f(t) = \begin{cases} 4, & t = 0 \bmod n; \\ 3, & t \neq 0 \bmod n \end{cases}$ firefighters.

1.2 Our results

All of our results depend on properties of the cumulative sum $f^*(t) = \sum_{\tau=1}^{t} f(\tau)$ of the function f.

We show the following lower bound for the Cartesian grid $\mathbb{Z} \square \mathbb{Z}$, closing the gap between the existing lower bound 1 and the upper bound $3/2 + \epsilon$.

Theorem 1. *If $f^*(t)$ never exceeds $(3t+1)/2$ then no strategy using f firefighters can control a single-source fire in $\mathbb{Z} \square \mathbb{Z}$.*

Theorem 1 settles [18, Conjecture 1] when applied to the function $f(t) = 1 + (t \bmod 2)$ — that is, the sequence $2, 1, 2, 1, \ldots$. Moreover, Theorem 1 implies the lower bound 3 for the strong grid $\mathbb{Z} \boxtimes \mathbb{Z}$.

Corollary 2. *If $f^*(t)$ never exceeds $3t+1$ then no strategy using f firefighters can control a single-source fire in $\mathbb{Z} \boxtimes \mathbb{Z}$.*

We show a essentially matching upper bound for the strong grid.

Theorem 3. *If $\liminf f^*(t)/t > 3$ then for any finite-source fire in $\mathbb{Z} \boxtimes \mathbb{Z}$, there exists a strategy using f firefighters that can control it.*

Theorem 3 yields the following generalization of the known upper bound for the Cartesian grid $\mathbb{Z} \square \mathbb{Z}$, which allows for non-periodic functions. This settles [18, Conjecture 2].

Corollary 4. *If $\liminf f^*(t)/t > 3/2$ then for any finite-source fire in $\mathbb{Z} \square \mathbb{Z}$, there exists a strategy using f firefighters that can control it.*

Note that $\liminf$ is the correct measure for $f^*(t)/t$ rather than $\limsup$, since it is easy to build, for any $\epsilon > 0$, an example of a function f satisfying $\limsup f^*(t)/t = 4 - \epsilon$ (resp., $8 - \epsilon$) such that a single-source fire in $\mathbb{Z} \square \mathbb{Z}$ (resp., $\mathbb{Z} \boxtimes \mathbb{Z}$) cannot be controlled by f firefighters.

Our proofs can be easily adapted to show analoguous upper and lower bound for the triangular grid $\mathbb{Z} \triangle \mathbb{Z}$, in which the threshold is 2.

1.3 Related work

The firefighter problem is loosely connected with Conway's angel problem [1]. This is a game of pursuit in $\mathbb{Z} \boxtimes \mathbb{Z}$, in which the angel can move to any point within ℓ_∞-distance k and the devil can destroy one unoccupied point per turn, bearing similarities to the $f \equiv 1/k$ case of the firefighter problem. The two main differences between the angel problem and the firefighter problem are

1. The fire is *non-deterministic*, that is, it needs not choose its path in advance;

2. The firefighters play a *predetermined* strategy, that is, they cannot adapt their strategy to the fire's advancement.

It is known that for $1 \le k < 2$, where the fractional version is defined appropriately, the devil wins [12], and that for $k \ge 2$ the angel wins [2, 7, 11, 14]. Our results, when presented as a variant of the angel problem in which the fire is more powerful, show that the threshold is $1/3$ instead of 2.

The rest of the paper is organized as follows. In Section 2 we prove Theorem 1, in Section 3 we prove Theorem 3, and in Section 4 we show how these two theorems imply Corollaries 2 and 4.

Throughout the paper we denote the set of non-negative integers by $\mathbb{N}$ and the set of integers by $\mathbb{Z}$. For a sequence $s(t)$ we define $\liminf s(t) = \lim_{t_0 \to \infty} \inf \{s(t) : t \ge t_0\}$. By $\lceil x \rceil$ (resp., $\lfloor x \rfloor$) we denote the real number x rounded up (resp., down) to the closest integer.

2 Proof of Theorem 1

2.1 Time-line

Our proof of Theorem 1 makes use of several sequences, all of which are represented as some function measured at integer times t. To circumvent ambiguity that can arise due to timing subtleties, we define a time-line for the process as follows (here n is a positive integer).

Time t	What happens?
0	The grid is created, empty and void.
1/3	The initial set of points S is set on fire.
$n - 1/3$	The nth squad consisting of $f(n)$ firefighters is placed on the grid.
n	Nothing. Crickets chirp.
$n + 1/3$	The fire spreads to adjacent unprotected points.

2.2 Definitions and simple claims

Fix a strategy using f firefighters. In the following definitions t is a natural number representing time and $i, j \in \{\pm 1\}$ represent together a direction: north-east, north-west, south-west or south-east.

Although all objects we define are a function of time, we may omit t from the notation when the context allows.

Fire fronts, lengths and perimeter. The *fire front* $L_{i,j} = L_{i,j}(t)$ is the line

$$L_{i,j} = \{(x,y) \in \mathbb{Z} \times \mathbb{Z} : xi + yj = c_{i,j}\},$$

where $c_{i,j} = c_{i,j}(t)$ is the minimal natural number for which no point on $L_{i,j}$ is burning at time t.

The *length* $\rho_{i,j} = \rho_{i,j}(t)$ of a fire front $L_{i,j}$ is defined as the ℓ_∞ distance between $L_{i,j} \cap L_{i,-j}$ and $L_{i,j} \cap L_{-i,j}$.

The sum of the lengths of all four fire fronts is the *fire perimeter* at time t, which we denote by $\rho = \rho(t)$. Note that $\rho_{i,j} = \rho_{-i,-j} = \frac{1}{2}(c_{i,-j} + c_{-i,j})$ and thus $\rho = \sum_{i,j \in \{\pm 1\}} c_{i,j}$.

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Total and front potential. A point is *endangered* if it is unprotected and adjacent to burning point. We define the *total potential* $\phi = \phi(t)$ at time t as the number of endangered points on $L(t) = \bigcup_{i,j \in \{\pm 1\}} L_{i,j}(t)$; that is, the difference between the total number of points in $L(t)$ adjacent to burning points and the amount of firefighters protecting such points. For consistency, we define $\phi(0) = 1$ (that is, the fire source is the single endangered point).

Note that our choice of time-line dictates that all these ϕ endangered points catch fire at time $t + 1/3$.

Moreover, we define the *potential* $\phi_{i,j} = \phi_{i,j}(t)$ of a fire front $L_{i,j}$ as the contribution of points on $L_{i,j}$ to the potential. More precisely, an endangered point on a single $L_{i,j}$ contributes one to $\phi_{i,j}$ and an endangered point that belongs to two adjacent fire fronts contributes $1/2$ to the potential of each.[1]

Note that $\phi = \sum_{i,j \in \{\pm 1\}} \phi_{i,j}$.

Claim 5. For all $t \in \mathbb{N}$ and $i, j \in \{\pm 1\}$ we have $\phi_{i,j}(t) \leq \rho_{i,j}(t)$.

Proof. The length $\rho_{i,j}$ of the fire front $L_{i,j}$ must be able to accomodate all $\phi_{i,j}$ endangered points on $L_{i,j}$, which catch fire immediately. $\square$

Active and frozen fronts. The fire front $L_{i,j}$ is *active* at time $t \geq 0$ if $L_{i,j}(t+1) \neq L_{i,j}(t)$ and is *frozen* otherwise. Let $a_{i,j}(t) = c_{i,j}(t+1) - c_{i,j}(t)$; that is, the indicator variable $a_{i,j}(t)$ takes the value 1 if $L_{i,j}(t)$ is active and the value 0 if it is frozen.

We denote the number of active fire fronts at time t by $a(t) = \sum_{i,j \in \{\pm 1\}} a_{i,j}(t) \in \{0, 1, 2, 3, 4\}$. Note that by definition $a(t) = \rho(t+1) - \rho(t)$.

Claim 6. For all $t \in \mathbb{N}$ and $i, j \in \{\pm 1\}$ we have $a_{i,j}(t) = 0$ if and only if $\phi_{i,j}(t) = 0$.

Proof. Exactly $\phi_{i,j}(t)$ endangered points on $L_{i,j}(t)$ caught fire between time t and $t+1$ (specifically, at time $t + 1/3$). The fire front is active if and only if this number is positive. $\square$

Note that a reactivation of a frozen front can only occur when an adjacent active fire front endangers its corner, giving it a potential of $1/2$.

2.3 Bounding the potential

The following lemma bounds the potential from below by bounding the change in potential between consecutive times. Denote by $f_{i,j}(t)$ the number of firefighters placed on $L_{i,j}(t)$ until time t that were not counted in any $f_{i',j'}(\tau)$ for $\tau < t$ (this distinction is needed in order to avoid double-counting of firefighters on a frozen fire front) and let $f_{i,j}^*(t) = \sum_{\tau=1}^{t} f_{i,j}(\tau)$.

Lemma 7. *For all $t \in \mathbb{N}$ and $i, j \in \{\pm 1\}$ we have $\phi_{i,j}(t) \geq 1/4 + c_{i,j}(t) - f_{i,j}^*(t)$.*

Proof. If $L_{i,j}$ is active at time τ, then the $\phi_{i,j}(\tau)$ burning points on it have at least $1 + \phi_{i,j}(\tau)$ neighbors in $L_{i,j}(\tau+1)$, of which at most $f_{i,j}(\tau+1)$ are protected by time $\tau+1$. If $L_{i,j}$ is frozen at time τ, then all points on $L_{i,j}(\tau+1)$ adjacent to burning points are protected by time $\tau+1$. In any case, we have

$$\phi_{i,j}(\tau+1) \geq \phi_{i,j}(\tau) + a_{i,j}(\tau) - f_{i,j}(\tau+1).$$

Summing this for $\tau = 0, 1, \ldots, t-1$ yields

$$\phi_{i,j}(t) \geq \phi_{i,j}(0) - f_{i,j}^*(t) + \sum_{\tau=0}^{t-1} a_{i,j}(\tau) = \phi_{i,j}(t) \geq \phi_{i,j}(0) + c_{i,j}(t) - f_{i,j}^*(t),$$

as stated by the lemma. $\square$

The next two lemmata lay the foundations for the proof of Proposition 10.

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[1] As a special case, at time $t = 0$ we have $\phi_{i,j}(0) = 1/4$.

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Lemma 8. *If $\rho(t) \geq 2f^*(t) - 1$ then $\phi(t) > \rho(t)/2$.*

Proof. Summed over all directions $i, j \in \{\pm 1\}$, Lemma 7 yields $\phi(t) \geq 1 + \rho(t) - f^*(t) \geq 1/2 + \rho(t)/2$. □

Lemma 9. *If $\rho(t) \geq 2f^*(t) - 1$ then $\phi_{i,j}(t) + \phi_{-i,-j}(t) > 0$.*

Proof. We have $c_{i,j}(t) + c_{i,-j}(t) + c_{-i,j}(t) + c_{-i,-j}(t) = \rho(t)$ so at least one of the following cases is guaranteed to hold.

Case 1. If $c_{i,j}(t) + c_{-i,-j}(t) > \rho(t)/2$, then by applying Lemma 7 twice we get

$$\phi_{i,j}(t) + \phi_{-i,-j}(t) \geq 1/2 + c_{i,j}(t) + c_{-i,-j}(t) - f^*(t) > 1/2 + \rho(t)/2 - f^*(t) \geq 0.$$

Case 2. If $\rho_{i,j}(t) + \rho_{-i,-j}(t) = c_{i,-j}(t) + c_{-i,j}(t) \geq \rho(t)/2$ then by Claim 5 we have

$$\phi_{i,-j}(t) + \phi_{-i,j}(t) \leq \rho_{i,-j}(t) + \rho_{-i,j}(t) = \rho(t) - \rho_{i,j}(t) - \rho_{-i,-j}(t) \leq \rho(t)/2$$

and by Lemma 8 we get

$$\phi_{i,j}(t) + \phi_{-i,-j}(t) = \phi(t) - \phi_{i,-j}(t) - \phi_{-i,j}(t) \geq \phi(t) - \rho(t)/2 > 0. \qquad \square$$

The following proposition concludes the proof by showing that the fire expands indefinitely and thus cannot be controlled.

Proposition 10. *Assume that $f^*(t) \leq (3t + 1)/2$ for all $t \in \mathbb{N}$. Then $\rho(t) \geq 3t$ for all $t \in \mathbb{N}$.*

Proof. We prove this by induction on t. For $t = 0$ we have $\rho(0) = 0$. Assume $\rho(t) \geq 3t \geq 2f^*(t) - 1$. By Lemma 8 no two adjacent fire fronts can be frozen at time t, since the sum of the potential of the two others cannot exceed the sum of their lengths, which is the semi-perimeter. By Lemma 9 no two opposing fire fronts can be frozen at time t. Thus, $a(t) \geq 3$ and $\rho(t + 1) = \rho(t) + a(t) \geq 3t + 3$. □

3 Proof of Theorem 3

To make the proof easier, we make the following assumptions without loss of generality.

1. The fire breaks out in an ℓ_∞-ball of radius $r \geq 0$, i.e., an axes-parallel $(2r + 1) \times (2r + 1)$ square, centered at the origin.

2. There exist some $t_0 \in \mathbb{N}$ and $\epsilon > 0$ such that $f^*(t) \geq (3 + \epsilon)t$ for all $t \geq t_0$. This is because $\liminf f^*(t)/t > 3$ implies the existence of such t_0 and ϵ for which $\inf\{f^*(t)/t : t \geq t_0\} \geq 3 + \epsilon$.

3. We may assume $t_0 = 1$ since we may enlarge the initial fire by adding t_0 to r.

The only property of f we will use, which is a strengthened form of $f^*(t) > 3t$, is the following. Set $m = \lceil 1/\epsilon \rceil$. Then for all $k \in \mathbb{N}$ we have

$$f^*(mk + 1) \geq 3(mk + 1) + \epsilon(mk + 1) > 3mk + 3m + k.$$

Now we describe a strategy $\mathcal{S} = \mathcal{S}(m, r)$ that allows f firefighters to control a fire that breaks out in an ℓ_∞-ball of radius $r \geq 1$ centered at the origin.

Our strategy has four phases. In a terminology similar to the one used in Section 2, we are guaranteed to have at least $k - 1$ frozen fronts during the kth phase, hence when the fourth phase ends, all four fronts are frozen and the fire is controlled.

The following invariants are maintained:

- The shape of the fire at all times is an ℓ_∞-ellipse (that is, an axes-parallel rectangle).

- The firefighters are placed on the perimeter of an ℓ_∞-ellipse.

- Each firefighter is placed next to an already positioned firefighter (except for the first one, of course).

Time	Fire width	Fire height	Available firefighters	Frozen
1	$2r+1$	$2r+1$	≥ 4	-
$2r$	$6r-1$	$6r-1$	$\geq 6r+1$	North
$6rm+1$	$12rm+2r+1$	$6rm+4r$	$\geq 18rm+6r+4$	North, East
$6rm^2+10rm$	$6rm^2+16rm+2r$	$6rm^2+10rm+3r-1$	$\geq 18rm^2+36rm+10r$	N, E, W
$12rm^2+30rm$	$6rm^2+16rm+2r$	$12rm^2+30rm+3r-1$	$\geq 36rm^2+102rm+30r$	All

Table 1: Key times for the strategy $\mathcal{S}(m,r)$

First phase: northern front. This phase begins at time $t=1$ and ends at time $t=2r$. All firefighters are placed on the horizontal line $y=3r$ between $x_{\min}=1-3r$ and $x_{\max}=3r-1$. Note that $f^*(2r) \geq 6r+1$ and thus by the end of the phase, when the fire has grown to an ℓ_∞-ball of radius $3r-1$ and has reached the northern front, the front is just long enough so that the fire is not able to spread north anymore.

Second phase: eastern front. This phase begins at time $t=2r+1$ and ends at time $t=6rm+1$. While maintaining the west end of the northern front just out of the fire's reach, the firefighters continue the northern front eastwards until $x_{\max}=6rm+r+1$ and build an eastern front on this vertical line, starting at the corner $y_{\max}=3r$ and going south until $y_{\min}=-r-6rm$. Note that $f^*(6rm+1) \geq 18rm+6r+4$ and thus by the end of the phase, when the fire has grown to an ℓ_∞-ellipse of height $6rm+4r$ and width $12rm+2r+1$ and has reached the eastern front, the front is just long enough so that the fire is not able to spread east anymore.

Third phase: western front. This phase begins at time $t=6rm+2$ and ends at time $t=6rm^2+10rm$. While maintaining the south end of the eastern front just out of the fire's reach, the firefighters continue the northern front westwards until $x_{\min}=1-r-10rm-6rm^2$ and build a western front on this vertical line, starting at the corner $y_{\max}=3r$ and going south until $y_{\min}=1-10rm-6rm^2$. Note that $f^*(6rm^2+10rm) \geq 18rm^2+36rm+10r$ and thus by the end of the phase, when the fire has grown to an ℓ_∞-ellipse of height $6rm^2+10rm+3r-1$ and width $6rm^2+16rm+2r$ and has reached the western front, the front is long enough (by $2r$ or so) so that the fire is not able to spread east anymore.

Fourth phase: southen front. This phase begins at time $t=6rm^2+10rm+1$ and ends at time $t=12rm^2+30rm$. While maintaining the south end of the eastern front just out of the fire's reach, the firefighters continue the western front southwards until $y_{\min}=1-30rm-12rm^2$ and build a southern front on this horizontal line, starting at the corner $x_{\min}=1-r-10rm-6rm^2$ and going east until the eastern front is met at the corner $x_{\max}=6rm+r+1$. This actually happens about $20r$ rounds before the end of the phase, since $f^*(12rm^2+30rm) \geq 36rm^2+102rm+30r$ and thus by the end of the phase, when the fire has grown to an ℓ_∞-ellipse of height $12rm^2+30rm+3r-1$ and width $6rm^2+16rm+2r$, it is fully surrounded.

4   Proof of Corollaries 2 and 4

Using the following proposition, Theorem 1 implies Corollary 2 and Theorem 3 implies Corollary 4.

Proposition 11. *If f firefighters can control a fire that breaks out in a ball of radius $r \geq 0$ in the strong grid $\mathbb{Z} \boxtimes \mathbb{Z}$, then g firefighters can control a fire that breaks out in a ball of radius $2r$ in the Cartesian grid $\mathbb{Z} \square \mathbb{Z}$, where the function g is defined by* $g(t) = \begin{cases} \lfloor f(k)/2 \rfloor, & \text{if } t=2k-1; \\ \lceil f(k)/2 \rceil, & \text{if } t=2k. \end{cases}$

Proof. Without loss of generality, the fire center is the origin in both grids. Let $\mathcal{S}$ be the strategy used by the firefighters in $\mathbb{Z} \boxtimes \mathbb{Z}$ to control the fire, and assume that at time t, firefighters are placed in a set P_t of $|P_t| = f(t)$ points.

We exploit the connection between the metrics ℓ_1 and ℓ_∞ on the plane $\mathbb{R}^2$ to convert $\mathcal{S}$ to a strategy $\mathcal{S}'$ for $\mathbb{Z} \square \mathbb{Z}$. Specifically, we use the injective mapping $\curlywedge\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $\curlywedge(x,y) = (x+y, x-y)$. Partition the set P_t arbitrarily to two sets P'_t and P''_t of respective sizes $g(2t-1)$ and $g(2t)$. It is possible as $|P_t| = f(t) = g(2t-1) + g(2t)$. The strategy $\mathcal{S}'$ places firefighters in $\curlywedge(P'_t)$ at time $2t-1$ and in $\curlywedge(P''_t)$ at time $2t-1$.

Note that $\mathcal{S}'$ only places firefighters at even points; that is, points (x,y) such that $x+y$ is even. Recall that the graph $\mathbb{Z} \square \mathbb{Z}$ is bipartite, and the initial fire boundary consists of even points only. Therefore, at odd times the fire can only spreads to odd points (which are never protected) and at even times the fire can spread only to unprotected even points. It makes sense thus to consider the state of the process only at even times $t = 2k$. But behold — the square of the graph $\mathbb{Z} \square \mathbb{Z}$ restricted to even points is isomorphic to $\mathbb{Z} \boxtimes \mathbb{Z}$ using the isomorphism $\curlywedge$, and the initial fire, the ℓ_1-ball of radius r, is mapped by $\curlywedge$ to an ℓ_∞-ball of radius $2r$.

Since the strategy $\mathcal{S}$ is able to control the fire in $\mathbb{Z} \boxtimes \mathbb{Z}$ in some finite time T, the strategy $\mathcal{S}'$ will control the fire in the even part of $\mathbb{Z} \square \mathbb{Z}$. This establishes the result as $\mathbb{Z} \square \mathbb{Z}$ is bipartite. $\qquad\square$

Acknowledgements

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