A Planar 3-Convex Set is Indeed a Union of Six Convex Sets

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Abstract

Suppose S is a planar set. Two points a, b in S **see each other** via S if $[a, b]$ is included in S . F. Valentine proved in 1957 that if S is closed, and if for every three points of S , at least two see each other via S , then S is a union of three convex sets. The pentagonal star shows that the number three is best possible. We drop the condition that S is closed and show that S is a union of (at most) six convex sets. The number six is best possible.

1 Introduction

There are three common measures for evaluating the "non-convexity" of a set $X \subset \mathbb{R}^d$:
$\alpha(X) -$ The largest size of a visually independent subset of X.
$\beta(X) -$ the smallest size of a collection of seeing subsets of X that covers X, or, in other words, the chromatic number of the invisibility graph of X.
$\gamma(X) -$ the smallest size k such that X is a union of k convex sets.

Much effort has been devoted to bounding γ in terms of α. In general, there is no such bound, since there exist planar sets X with $\alpha(X) = 3$ but

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with $\gamma(X) = \infty$, and there exist closed sets $S \subset \mathbb{R}^4$ with $\alpha(S) = 2$ and $\gamma(S) = \infty$ (even $\beta(S) = \infty$). In the specific case of closed sets in the plane, the situation is different.

Valentine [1957] proved that for closed $S \subset \mathbb{R}^2$, $\alpha(S) = 2$ implies $\gamma(S) \leq 3$. Eggleston [1974] proved that for compact $S \subset \mathbb{R}^2$, $\alpha(S) < \infty$ implies $\gamma(S) < \infty$. Breen and Kay [1976] were the first to find an upper bound for γ in terms of α. They proved that for closed $S \subset \mathbb{R}^2$, if $\alpha(S) = m$ then $\gamma(S) \leq m^3 \cdot 2^m$. Later on, Perles and Shelah [1990] improved this upper bound to m^6, and Matoušek and Valtr [1999] obtained the best upper bound know today, $18m^3$. In the same paper, M. and V. give examples of closed planar sets with $\gamma(S)$ about $= cm^2$

There has also been some success in bounding γ in terms of α for certain cases of planar sets X that are not necessarily closed. Breen [1974] claims that for $X \subset \mathbb{R}^2$, $\alpha(X) = 2$ implies $\gamma(X) \leq 6$. Another result is of Matoušek and Valtr [1999] who proved that for $X \subset \mathbb{R}^2$ with $\alpha(X)$ finite, if X is starshaped then $\gamma(X) \leq 2(\alpha(X))^2$. They also proved that for $X \subset \mathbb{R}^2$, if $\mathbb{R}^2 \setminus X$ has no isolated points then $\gamma(X) \leq (\alpha(X))^4$.

In this work we shall focus on the case of $X \subset \mathbb{R}^2$ with $\alpha(X) = 2$. We wish to complete the work of Breen [1974], and give a detailed proof of the theorem claimed by Breen ($\alpha = 2 \Rightarrow \gamma \leq 6$). We intend to determine the maximum possible value of $\gamma(X)$ (assuming $X \subset \mathbb{R}^2$ and $\alpha(X) = 2$) under a variety of side conditions, pertaining to the location (within $\text{cl}X$) of the points of $\text{cl}X \setminus X$. We produce examples for all cases under discussion, showing that the bounds obtained are tight.

2 Definitions and Notations

Given $X \subseteq \mathbb{R}^2$, we say that two points $u, v \in \mathbb{R}^2$ **see each other** via X if the open interval (u, v) is included in X. (This applies even if the points u, v are not in X)

A is a **seeing subset** of X if $A \subseteq X$ and every two points of A see each other via X.

A subset of X is **visually independent** if no two of its points see each other via X.

Define the **invisibility graph** of X as the graph $G(X)$ with vertex set X and with $u, v \in X$ connected by an edge iff $[u, v] \nsubseteq X$.

We now define the three most common "measures of non-convexity" of X: (These are the notations found in the literature which we prefer.)

$\alpha(X)$ – The supremum of cardinalities of all visually independent subsets of X. That is, the clique number of the graph $G(X)$.

$\beta(X)$ – The chromatic number of $G(X)$. In other words, the smallest cardinality of a collection of seeing subsets of X that covers X.

$\gamma(X)$ – the smallest cardinality k such that X can be expressed as the union of k convex sets.

It is easy to see that $\alpha(X) \leq \beta(X) \leq \gamma(X)$.

The following notations will be used throughout this paper: For $X \subset \mathbb{R}^2$, define $S = \text{cl}X$. We shall write $M = S \setminus X$ (M is the set of points of S **missing** in X). We split M into two parts $M = M_b \cup M_i$, where $M_i = M \cap \text{ int}S$ and $M_b = M \cap \text{bd}S$.

S is **locally convex** at a point x if $x \in S$ and x has a neighborhood U such that $S \cap U$ is convex. We denote by Q (= lnc$S$ the set of points of local non-convexity (lnc points) of S. These are the points where S fails to be locally convex.

We say that S is **2-dimensional** at a point p if $p \in \text{cl}(\text{int}S)$.

$A \subset \mathbb{R}^d$ is an L_2-set if every two points of A can be connected by a polygonal line of at most 2 edges within A.

Given a subset $S_0 \subseteq S$, we say that S_0 is **convex relative to** S if for every $x, y \in S_0$, $[x, y] \subseteq S$ implies $[x, y] \subseteq S_0$.

3 Results

Throughout the following theorems we assume that X is a planar set, $\alpha(X) \leq 2$, and that $S = \text{cl}X$.

Main Theorem 1. $max\{\gamma(X) : X \subseteq \mathbb{R}^2, \alpha(X) \leq 2\} = 6$

We disassemble Main Theorem 1 into several independents theorems-Theorem A to Theorem G:

Theorem A. *If X is not an L_2-set (in particular, if X is not connected), then $\gamma(X) = 2$. (In this theorem, $\mathbb{R}^2$ can be replaced by an arbitrary real vector space.)*

Theorem B. *If S is not 2-dimensional at some point $p \in S$, then $\gamma(X) \leq 2$.*

Theorem C. *If $|M_i| > 1$, then $\gamma(X) \leq 3$. The number three is best possible, even when S is convex. If, in addition, $M_b = \phi$ or $M_b =$bd$S$, then $\gamma(X) = 2$.*

Theorem D. *If $|M_i| = 1$ and $M_b = \phi$ or $M_b =$ bd$S$, then $\gamma(X) \leq 4$. The number four is best possible.*

Theorem E. *If $M_i = \phi$ then $\gamma(X) \leq 3$. The number three is best possible, even when S is convex.*

Theorem F. *If $|M_i| = 1$ then $\gamma(X) \leq 6$. The number six is best possible.*

Theorem G. *If $|M_i| = 1$ and S is convex then $\gamma(X) \leq 4$. The number four is best possible. If, in addition, $M_b = \phi$ or $M_b =$ bd$S$ then $\gamma(X) = 2$.*

Table 1 summarizes all the cases above. In each box appears $max\, \gamma(X)$ under the conditions of that box. The number in parentheses is $max\, \gamma(X)$ under the conditions of the box together with the extra assumption that S is convex.

Much of the material contained in this paper can be summarized in the following extension of Valentine's Theorem:

Theorem 3.1. *If $X \subseteq \mathbb{R}^2$, $\alpha(X) \leq 2$, and the complement $\mathbb{R}^2 \setminus X$ has no one-pointed components, then $\gamma(X) \leq 3$.*

We also present two results and and a conjecture involving the measure β:

Main Theorem 2. $max\{\gamma(X) : X \subseteq \mathbb{R}^2, \beta(X) = 2\} = 4.$

It seems that Main Theorem 2 has been known for many years.

	$M_b = \phi$ or $M_b = bdS$	M_b unrestricted
$\lvert M_i \rvert > 1$	2 (2)	3 (3)
$\lvert M_i \rvert = 0$	3 (1)	3 (3)
$\lvert M_i \rvert = 1$	4 (2)	6 (4)

Table 1

Example 8. *We present a bounded set $X \subset \mathbb{R}^2$ with $\alpha(X) = 2$ and $\beta(X) = 4$.*

Conjecture. $max\{\beta(X) : X \subseteq \mathbb{R}^2, \alpha(X) = 2\} = 4$.

4 Proof of Theorem A

As X is not an L_2-set, there are two points $a, b \in X$ that cannot be connected by a polygonal line of fewer than 3 edges within X. In other words, there is no point in X that sees both a and b. Define $A = st(a) = \{x \in X : [a, x] \subseteq X\}$, $B = st(b)$. Notice that the sets A and B are disjoint. We show now that A is convex: Take $p, q \in A$, where $p = a + u$, $q = a + v$. For every $0 < \theta \leq 1$, $[a+\theta u, a+\theta v] \subseteq X$, because otherwise $\{a+\theta u, a+\theta v, b\}$ would be a visually independent set. Hence the full triangle $[a, p, q]$ is included in X, so a sees via X every point in $[p, q]$, which means that $[p, q] \subset A$, so A is convex. Similarly, B is convex. Now, for every $x \in X$, $x \in A \cup B$, because otherwise $\{a, b, x\}$ would be a visually independent set. Thus X is the union of two disjoint convex sets.

5 Proof of Theorem B

S is a closed set in the plane and therefore, according to Valentine [1957], is a union of at most three convex sets: $S = \cup_{i=1}^{n} C_i$, where $1 \leq n \leq 3$. As S is closed, we can assume that for each i, C_i is closed. In addition, we will assume that none of these convex sets is included in the union of the others.

If each C_i is 2-dimensional then S is 2-dimensional. Assume therefore, w.l.o.g., that C_1 is not 2-dimensional. If $\dim C_1 = 0$, then S is not connected and therefore X is not connected, so by theorem A, $\gamma(X) \leq 2$. Otherwise, $\dim C_1 = 1$, in which case C_1 is part of a line L. There is a point $p \in C_1$ such that $p \notin C_2 \cup C_3$. Since C_2, C_3 are closed, there is a neighborhood U of p that misses $C_2 \cup C_3$, and therefore $U \cap S = U \cap C_1$ is a segment. Define L to be the line containing this Denote by L_+, L_- the open half-planes determined by L.

Define $C = \mathrm{conv}(X \setminus L)$. We wish to show that $C \subseteq X$. Every two points in $X \cap L_+$ do not see p via X. Therefore, since $\alpha(X) = 2$, they see each other via $X \cap L_+$. Hence, $X \cap L_+$ is convex. By the same argument, $X \cap L_-$ is convex, so $C = \mathrm{conv}(X \setminus L) = \mathrm{conv}((X \cap L_+) \cup (X \cap L_-))$.

If $X \cap L_+ = \phi$ (or $X \cap L_- = \phi$) then $C = X \cap L_- \subseteq X$ (or $C = X \cap L_+ \subseteq X$). If both $X \cap L_+$ and $X \cap L_-$ are nonempty, then $C = \cup\{[a,b] : a \in X \cap L_+, b \in X \cap L_-\}$. The point p does not see any $a \in L_+ \cap X$ or $b \in L_- \cap X$. Therefore, again, as $\alpha(X) = 2$, for any such a, b, $[a,b] \subset X$. This implies that $C \subset X$.

It remains to deal with the set $L \cap X$. Since $\alpha(X) = 2$ and L is convex, $\alpha(X \cap L) \leq 2$. If $X \cap L$ is convex, we are done. Otherwise, $X \cap L$ is the disjoint union of two nonempty convex sets A, B, where, say, $p \in A$. If $C = \phi$ then we are done, so assume $C \neq \phi$.

In order to complete the proof, we would like to show that $\mathrm{conv}(B \cup C) \subset X$. Since both B and C are convex, $\mathrm{conv}(B \cup C) = \cup\{[b,c] : b \in B, c \in C\}$. Suppose $b \in B$ and $c \in C$:

Case 1: If $c \notin L$ then $[p,c] \nsubseteq X$ and $[p,b] \nsubseteq X$, hence $[b,c] \subset X$.

Case 2: If $c \in L$ then $c \in [c_+, c_-]$, where $c_+ \in X \cap L_+$ and $c_- \in X \cap L_-$. The points c_+, c_-, b do not see p via X, therefore, and since $\alpha(X) = 2$,

$[c_+, b] \subset X$ and $[c_-, b] \subset X$. Now, each point in $[c, b)$ is in the convex hull of a point in $[a_+, b)$ and a point in $[a_-, b)$ and therefore is in X (again, these two points do not see p, and $\alpha(X) = 2$).

This establishes that $\mathrm{conv}(B \cup C) \subset X$, which implies that X is the union of two convex sets: A, the component of p in $L \cap X$, and $\mathrm{conv}(B \cup C)(= B \cup C)$.

6 Proof of Theorem C

Coming to prove theorem C, we shall first show that if $|M_i| > 1$, then M_i contains a segment. Suppose $x, y \in M_i$, $x \neq y$, and let L be the line spanned by x, y. As $x, y \in \mathrm{int}S$, both have circular neighborhoods U_x, U_y in S. The intersection of $L \setminus \{x, y\}$ with these two neighborhoods consists of 4 segments. These segments lie in the three components of $L \setminus \{x, y\}$, and therefore at least one of them is disjoint from X. Therefore M_i includes a segment, call it I, and $L = \mathrm{aff}I$.

Denote by L_+, L_- the open half-planes determined by L. Define $X_+ = X \cap L_+, X_- = X \cap L_-$ and $S_+ = \mathrm{cl}(X_+), S_- = \mathrm{cl}(X_-)$. Next we show that X_+ is convex. Take $p, q \in X_+$. There is a point $y \in X_-$, close enough to the center of I, such that both segments $(p, y), (q, y)$ intersect I, meaning that y sees neither p nor q via X, and therefore $[p, q] \subset X$, hence $[p, q] \subset X_+$. Similarly, X_- is convex.

Next we show that $M_i \subset L$. Note that $S \cap L_+ \subset \mathrm{cl}X_+$, and therefore $L_+ \cap \mathrm{int}S = \mathrm{int}(S \cap L_+) \subset \mathrm{int}\,\mathrm{cl}X_+ = \mathrm{int}X_+ \subset X$. Therefore $L_+ \cap M_i = \phi$. Similarly, $L_- \cap M_i = \phi$, hence $M_i \subset L$.

Define: $B_+ = S_+ \cap L$, $B_- = S_- \cap L$. B_+ is an edge of S_+ (think of it as the base of S_+). If a point u lies in $\mathrm{rel\,int}B_+$, then u sees every point of $S \cap L_+$ via $\mathrm{int}S_+$, hence via X. Thus, a point $u \in X \cap B_+$ may fail to see some point of X_+ via X only if u is an endpoint of B_+. Similarly for B_- and X_-.

If X contains a point y that is in $L \setminus (B_- \cup B_+)$, then S is not 2-dimensional at y, and therefore $\gamma(X) = 2$, by Theorem B. Assume therefore that $X \cap L \subset B_- \cup B_+$. Note that the segment $I(\subset M_i)$ lies in $B_- \cap B_+$.
Next we show that $\gamma(X) = 2$, unless $X \cap L \subset B_- \cap B_+$. Assume $X \cap L \not\subset$

$B_- \cap B_+$. Pick a point $y \in X \cap L \setminus (B_- \cap B_+)$. Think of L as a horizontal line, and suppose, w.l.o.g., that $y \notin B_-$, and that y is to the right of B_-.

Denote by L_2 the component of y in $X \cap L$. L_1 is the other component of $X \cap L$, if $X \cap L$ is not convex. If $X \cap L$ is convex, then $L_1 = \phi$.

Clearly, y does not see any point of L_1 via X. Since $y \notin B_-$, y doesn't see any point of X_- via X. Since $\alpha(X) = 2$, every point of L_1 sees every point of X_- via X, hence via X_-. In other words, $L_1 \cup X_-$ is convex. But, as we shall see immediately, $L_2 \cup X_+$ is convex as well. Indeed, consider a point $x \in X_+$ and a point $y' \in L_2$, to the right of y ($y' = y$ included). x doesn't see via X some point $z \in X_-$, that lies beyond I. $y' \notin B_-$ and therefore doesn't see via X any point in X_-. It follows that y' sees x via X. Now consider a point $y'' \in L_2$, strictly to the left of y. Since $y'' \in L_2$ lies to the right of I, and $I \subset B_+$, we conclude that $y'' \in \operatorname{rel\,int} B_+$, and therefore sees via X_+ every point of X_+. Now we can represent X as the union of two convex sets: $X = (L_1 \cup X_-) \cup (L_2 \cup X_+)$.

Assume from now on that $X \cap L \subset B_- \cap B_+$. Let us first dispose of the case where $M_b = \phi$ or $M_b = \operatorname{bd} S$.

$M_b = \phi$: If $c \in X \cap L (\subset B_- \cap B_+)$ then c sees every point of X_+ via $S \cap L_+$, which is a subset of X_+. Same for X_- and L_-. Denote by L_1, L_2 the components of $X \cap L$. ($L_1 = \phi$ if $X \cap L$ is convex). Then X is the union of the two convex sets $L_1 \cup X_+$, $L_2 \cup X_-$.

$M_b = \operatorname{bd} S$: If $x \in X \cap L$ then $x \in \operatorname{rel\,int} B_+$. (The endpoints of B_+ are boundary points of S and therefore not in X.) Similarly, $x \in \operatorname{rel\,int} B_-$. Define L_1, L_2 as above. Then X is again the union of the two convex sets $L_1 \cup X_+$ and $L_2 \cup X_-$.

Now we return to the general case: $|M_i| > 1$ and M_b unrestricted, and try to show that $\gamma(X) \leq 3$.

If $X \cap L$ is convex then X is the union of three convex sets and we are done. Assume $X \cap L$ is not convex, so it is the union of two non-empty components L_1, L_2, where L_1 is to the left of L_2. If L_1 has no left endpoint, then $L_1 \subset \operatorname{rel\,int} B_+$, and therefore X is the union of the three convex sets $L_1 \cup X_+, X_-, L_2$. The same argument works when L_1 has a left endpoint c_1, but c_1 is not the left endpoint of B_+. We can repeat this argument with

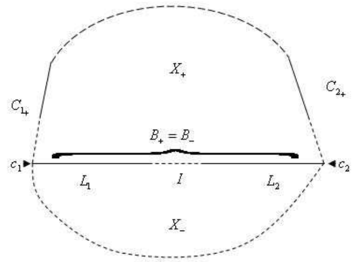

Figure 1

B_-, X_- instead of B_+, X_+, and also with L_2 instead of L_1.

Assume, therefore, that L_1 has a left endpoint c_1, L_2 has a right endpoint c_2, and $B_+ = B_- = [c_1, c_2]$. The point c_1 still sees every point of $S \cap L_+$ via int$S_+(\subset X)$ unless S_+ has an edge C_{1_+} with endpoint c_1, other than B_+. Assume, therefore that S_+ has such an edge C_{1_+}, and, by the same token, that S_+ has has an edge C_{2_+} with endpoint c_2, other than B_+ (see Figure 1). If $X \cap C_{1_+}$ is convex then c_1 still sees every point of X_+ via X_+, and thus $L_1 \cup X_+$ is again convex, as before.

Assume therefore that $X \cap C_{1_+}$is not convex. It is the union of $\{c_1\}(= C_{1_+} \cap L)$ and the convex set $C_{1_+} \cap X_+$. By the same token, assume that $X \cap C_{2_+}$ is not convex. It follows that $X \cap C_{1_+} \cap C_{2_+} = \phi$ since a point $z \in X \cap C_{1_+} \cap C_{2_+}$ would form a 3-clique of invisibility with c_1 and c_2.

We could play the same game with X_-, but this is not necessary, since X is the union of the three convex sets X_-, $(X_+ \setminus C_{1_+}) \cup L_1$, $(X_+ \setminus C_{2_+}) \cup L_2$.

Examples 1,2 show that the number three is best possible: We describe two sets $X_1, X_2 \subset \mathbb{R}^2$ with $|M_i| > 1$ and show that α of each set is 2 and that γ of each set is three. Notice that $X_1 \cap L$ is not convex, while $X_2 \cap L$ is convex.

Example 1:

Let P be a regular hexagon with center O and vertices $p_0, p_1, ..., p_5$. Take $[a, b]$ to be a short segment lying on $[p_5, p_2]$ with O in its center. We define

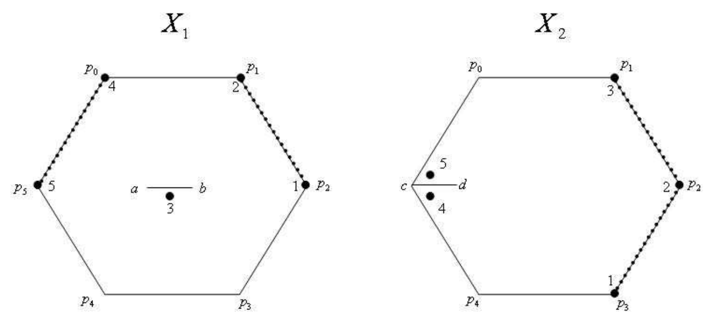

Figure 2

$X_1 = P \setminus ((p_5, p_0) \cup (p_1, p_2) \cup [a, b])$ (see Figure 2).

$\alpha(X_1) = 2$: The set $X_1 \setminus \{p_0\}$ is the union of two convex sets. The same holds for $X_1 \setminus \{p_1\}$. Therefore, if there is a 3-circuit of invisibility in X_1, it must contain both p_0 and p_1. But these two points see each other via X_1.

$\gamma(X_1) \geq 3$ since, as shown in Figure 2, there is a 5-circuit of invisibility.

Example 2:

Let P be as above and take $[c, d]$ to be a short segment lying on $[p_5, p_2]$ with $c = p_5$. Define $X_2 = P \setminus ((p_1, p_2) \cup (p_2, p_3) \cup [c, d])$ (see Figure 2).

$\alpha(X_2) = 2$: The set $X_2 \setminus \{p_2\}$ is the union of two convex sets. Therefore, if there is a 3-clique of invisibility in X_2, it must contain p_2. But there are only two points that p_2 doesn't see via X_2: p_1 and p_3, and fortunately $[p_1, p_3] \subset X_2$.

$\gamma(X_2) \geq 3$ since, as shown in Figure 2, there is a 5-circuit of invisibility.

7 Proof of Theorem D

Lemma 7.1. *$M_i \subset kerS$*

Proof. Assume $x \in M_i$ and suppose there is a point $y \in S$ such that $[x, y] \not\subseteq S$. In other words, there is a point $z \in (x, y)$ such that $z \notin S$. As S is closed, there is a neighborhood U of z, disjoint from S. $y \in S = \text{cl}X$, so there is a point $y' \in X$, close to y, satisfying $[x, y'] \cap U \neq \phi$. $x \in \text{int}S$, so there is an open circular neighborhood V of x, $V \subset \text{int}S$, such that no point in V sees y' via S. Thus no point of $V \cap X$ sees y' via X. It follows that every two points of $V \cap X$ see each other via $V \cap X$ (since $\alpha(X) = 2$), or, in other words, that $V \cap X$ is convex. Since V is open and $V \cap X$ is dense in V, it follows that $V \cap X = V$, i.e. $V \subset X$. This contradicts our assumption that $x \in V \cap M_i \subset V \setminus X$. $\square$

Before we start to prove theorem D we quote a result of Breen and Kay [1976]: Let $S \subset \mathbb{R}^2$ be a closed set with finite $\alpha(S)$. If S is starshaped with respect to a point that lies on a line that supports S, then $\gamma(S) = \alpha(S)$.

We now return to the proof of theorem D.

<u>If $M_b = \phi$</u>: Assume $M_i = \{(0,0)\}$, thus $X = S \setminus \{(0,0)\}$. Define $S_+ = S \cap \{(x,y) \in \mathbb{R}^2 | y \geq 0\}$ and $S_- = S \cap \{(x,y) \in \mathbb{R}^2 | y \leq 0\}$. $S = S_+ \cup S_-$. Since S_+ is the intersection of S with a convex set, $\alpha(S_+) \leq 2$. By Lemma 7.1, S_+ is starshaped with respect to $(0,0)$ and therefore, due to the result quoted above, $\gamma(S_+) = \alpha(S_+) \leq 2$, so $S_+ = A \cup B$ for some convex sets A, B. Similarly, $\gamma(S_-) \leq 2$, so $S_- = C \cup D$ for some convex sets C, D, hence $S = A \cup B \cup C \cup D$.

Define $T_+ = \{(x,y) \in \mathbb{R}^2 | y > 0 \vee (y = 0 \wedge x > 0)\}$ and $T_- = \{(x,y)\mathbb{R}^2 | y < 0 \vee (y = 0 \wedge x < 0)\}$. T_+ and T_- are both convex, and $T_+ \cup T_- = \mathbb{R}^2 \setminus \{(0,0)\}$.

We wish to show that in this case, when $M_b = \phi$, X is the following union of four convex sets: $X = (A \cap T_+) \cup (B \cap T_+) \cup (C \cap T_-) \cup (D \cap T_-)$. It is clear that the union of these four sets is included in X. We shall prove the opposite inclusion: Suppose $p = (x,y) \in X$. Note that $p \neq (0,0)$.
If $y > 0$, or if $y = 0$ and $x > 0$, then $p \in S_+ \cap T_+ = (A \cap T_+) \cup (B \cap T_+)$.
If $y < 0$, or if $y = 0$ and $x < 0$, then $p \in S_- \cap T_- = (C \cap T_-) \cup (D \cap T_-)$.

There is an alternative proof of Theorem D which avoids the result of Breen and Kay quoted above. Instead, it uses the necessary condition for $\gamma(S) = 3$ in Valentine's Theorem. We shall use this alternative approach in the proof of the slightly more complicated case $M_b = \text{bd}S$.

<u>If $M_b = \text{bd}S$</u>: Assume $M_i = \{p\}$, $p \in \text{int}S$. Then $X = S \setminus \text{bd}S \setminus \{p\} = \text{int}S \setminus \{p\}$. Assume $\gamma(S) = k$, $1 \leq k \leq 3$. Then S is the union of k

closed convex sets C_i $(i = 1, ..., k)$. Replacing C_i by $\text{conv}(\text{ker}S \cup C_i)$, if necessary, we may (and shall) assume that $\text{ker}S \subset C_i$ $(i = 1, ..., k)$, and clearly, $\cap_{i=1}^{k} C_i = \text{ker}S$. Now, consider the following cases:

1) $k = 1$, i.e., S is convex. So is $\text{int}S$, and $\text{int}S \setminus \{p\}$ is the union of two convex sets.

2) $k = 2$ and $\dim \text{ker}S = 2$. From $\text{int}C_1 \cap \text{int}C_2 \neq \phi$ it follows that $\text{int}S = \text{int}C_1 \cup \text{int}C_2$. (Clearly, $\text{int}C_1 \cup \text{int}C_2 \subset \text{int}S$. Conversely, if $x \in \text{int}S$, and $z \in \text{int}C_1 \cap \text{int}C_2$, then, for some sufficiently small $\epsilon > 0$, $x' = (1+\epsilon)x - \epsilon z \in S = C_1 \cup C_2$. Suppose, say, that $x' \in C_1$, then $x \in (x', z] \subset \text{int}C_1$.)

Therefore, $X = \text{int}S \setminus \{p\} = (\text{int}C_1 \setminus \{p\}) \cup (\text{int}C_2 \setminus \{p\})$ is the union of at most 4 convex sets. Example 3 shows that sometimes X is not the union of fewer than 4 convex sets.

3) $k = 2$ and $\dim \text{ker}S = 1$. Put $K = \text{ker}S$, $L = \text{aff}K$, and let L_+, L_- be the two closed half-planes bounded by L. Then K is a closed line segment (or a ray) within L.

Since S has no lnc points outside L, it follows that the sets $S_+ = S \cap L_+$, $S_- = S \cap L_-$ are convex, $S = S_+ \cup S_-$ and $S_+ \cap S_- = K$. It follows easily that $\text{int}S = \text{int}S_+ \cup \text{int}S_- \cup \text{rel int}K$.

Now $M_i = \{p\}$, where $p \in \text{int}S \cap \text{ker}S$. Thus $p \in \text{rel int}K$. The point p divides $\text{rel int}K$ into two (one-dimensional) convex sets $K_1, K_2 \subset L$, and $X = \text{int}S \setminus \{p\}$ is the union of two convex sets: $X = (K_1 \cup \text{int}S_-) \cup (K_2 \cup \text{int}S_+)$.

4) $k = 2$ and $\dim \text{ker}S \leq 0$. Thus $S = C_1 \cup C_2$, where C_1 and C_2 are closed convex sets and $|\text{ker}S| = |C_1 \cap C_2| \leq 1$. If $C_1 \cap C_2 = \phi$ then there is no room for $p \in M_i \subset \text{ker}S$. If $|C_1 \cap C_2| = 1$ then $\text{ker}S = C_1 \cap C_2 = \{p\}$, but $p \notin \text{int}S$, hence $p \notin M_i$.

5) $k = 3$. From $\alpha(S) = 2$ and $\gamma(S) = 3$ it follows (due to Valentine[57]) that S is the union of an odd-sided convex polygon $P = \text{conv}Q$ (where $Q = \text{lnc}S = \{q_1, ..., q_m\}$, $m \geq 3$, m odd) and m "leaves" $W_1, ..., W_m$. Each leaf W_i is a closed convex set that includes the edge $[q_i, q_{i+1}]$ of Q (where $q_{m+1} = q_1$), and lies beyond that edge and beneath all other edges of P. Note the subset $P \cup \cup_{i=1}^{m-1} W_i$ of S is the union of two convex sets: $P \cup \{W_i : 1 \leq i < m, i \text{ odd}\}$ and $P \cup \{W_i : i \text{ even}\}$.

The missing point $p \in \text{int}S \cap \text{ker}S$ may lie in P or outside P. It is

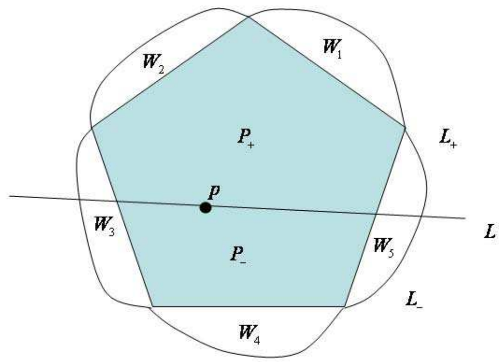

Figure 3

certainly not a vertex of P. Pass a line L through p that passes through int$P$ but misses all vertices of P. Denote by L_+, L_- the two closed half-planes determined by L, and define:

$S_+ = S \cap L_+, \quad S_- = S \cap L_-,$

$P_+ = P \cap L_+, \quad P_- = P \cap L_-,$

$W_{i_+} = W_i \cap L_+, \quad W_{i_-} = W_i \cap L_- \ (i = 1, 2, ..., m),$

P_+ is a convex polygon, even-sided or odd-sided. For $i = 1, ..., m$, W_{i_+} is either empty, or a closed, convex leaf that sits on an edge of P_+ and lies beneath all other edges of P_+. But there is no leaf sitting on the edge $P \cap L$ of P_+ (see Figure 3). It follows that S_+ is the union of two closed convex sets: $S_+ = C_{1_+} \cup C_{2_+}$. These two convex sets can be extended to include the convex kernel of S_+. We shall therefore assume that $\{p\} \cup P_+ \subset C_{i_+}$ for $i = 1, 2$. The same argument, with $+$ replaced by $-$, applies to S_-.

Denote by K_1, K_2 the two components of the set $S \cap L \setminus \{p\}$. Then $\text{int}(S \setminus \{p\}) = \text{int}S_+ \cup \text{int}S_- \cup \text{relint}K_1 \cup \text{relint}K_2$.

Since $K_1 \subset S_+ = C_{1_+} \cup C_{2_+}$, and $p \in C_{1_+} \cap C_{2_+}$, one of the sets C_{1_+}, C_{2_+}, say C_{1_+}, must include K_1. By the same argument, applied to K_2 and S_-, one of the sets C_{1_-}, C_{2_-}, say C_{2_-}, must include K_2. Thus $X = \text{int}S \setminus \{p\}$ is the union of the four convex sets: $\text{int}C_{1_+} \cup \text{relint}K_1$, $\text{int}C_{2_+}$, $\text{int}C_{1_-}$, $\text{int}C_{2_-} \cup \text{relint}K_2$.

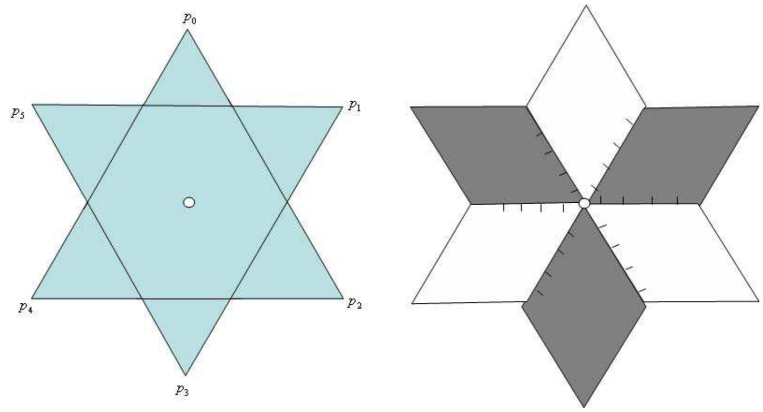

Figure 4

A well known example shows that the number four is best possible.

Example 3:

We describe a set $X \subset \mathbb{R}^2$ with $|M_i| = 1$ and $M_b = \phi$, and show that $\alpha(X) = 2$ and $\gamma(X) \geq 4$. Let X be a closed Star of David with its center O removed. Denote by $p_0, p_1, ..., p_5$ the outer vertices of the Star of David (see the left side of Figure 4). $\alpha(X) = 2$: The right side of Figure 4 shows a representation of X as the union of two seeing subset, hence $\beta(X) = 2$. Therefore $\alpha(X) = 2$.

$\gamma(X) \geq 4$: Let C be a convex subset of X. Define $A = \{p_0, p_2, p_4\}$ and $B = \{p_1, p_3, p_5\}$. No point of A sees any point of B, therefore C cannot contain points of both sets. $A \nsubseteq C$ and $B \nsubseteq C$, since $O \notin C$. Therefore, C contains at most two points of A or two points of B. Hence, we need at least two convex subsets of X to cover A, and two other convex subsets of X to cover B. It follows that $\gamma(X) \geq 4$. One can easily represent X as a disjoint union of 4 convex sets.

For an example with $|M_i| = 1$ and $M_b = \text{bd}S$, take the same set X and remove its boundary.

8 Proof of Theorem E, Preliminary Considerations

Let us recall our assumptions: $X \subset \mathbb{R}^2$, $\alpha(X) = 2$, $S = \mathrm{cl}X$ (hence $\alpha(S) \leq 2$ and therefore $\gamma(S) \leq 3$, by Valentine[57]), $M_i = \phi$, which means just that $\mathrm{int}S \subset X \subset S$. We wish to show that $\gamma(X) \leq 3$. Put $K = \mathrm{ker}S$. Our arguments will depend on the dimension of K. We shall deal with the cases where $\mathrm{dim}K < 2$ in this section, and treat the main case, $\mathrm{dim}K = 2$, in the subsequent sections. We first need the following lemmata:

Lemma 8.1. *S has no triangular holes.*

Proof. If S is convex, then of course, there are no holes in S.

If S is not connected, Then S is the union of two disjoint, closed convex sets, and again there are no holes in S.

Otherwise, if S is connected but not convex, then by Tietze's Theorem, S contains an lnc point q. According to Valentine [1957], $q \in \mathrm{ker}S$. In other words, S is starshaped with respect to q. But a starshaped set has no holes. □

Lemma 8.2. *If $M_i = \phi$, then $\beta(X) = \gamma(X)$.*

Proof. It suffices to show that if A is a seeing subset of X then $\mathrm{conv}A \subset X$. Every point in $\mathrm{conv}A$ is a convex combination of at most three points of A. If x is a convex combination of two points of A, then $x \in X$. Assume x is a convex combination of three affinely independent points $a, b, c \in A$. The edges of the triangle $\Delta = [a, b, c]$ lie in X. By Lemma 8.1, S has no triangular holes, therefore $\Delta \subset S$. This implies $x \in \mathrm{int}\Delta \subset \mathrm{int}S \subset X$. □

In view of Lemma 8.2, we only have to find how many seeing subsets of X are needed in order to cover X.

Case 1: $K = \phi$. If the kernel is empty then S is not connected and so is X, so by Theorem A, X is the union of two convex sets.

Case 2: dim$K = 0$. We will show that in this case X is the union of two convex sets. If $|K| = 1$, then according to the the proof in Valentine [1957], S is the union of two convex sets. We can assume that S is the union of two closed, convex sets A, B, with $A \cap B = K = \{q\}$. We can also assume that both A, B are of full dimension, otherwise we are back to Theorem B. If $q \notin X$ then X is not connected, so assume $q \in X$.

We claim that $X \cap (A \setminus \{q\})$ is a seeing subset of X. Indeed, suppose $x, y \in X \cap (A \setminus \{q\})$. Note that if $b \in B \cap X$, then x cannot see b via X (even via S) unless $q \in [x, b]$. Similarly for y. Chooses a point $b \in B$ that is not collinear with x, q, nor with y, q (dim$B = 2$). Then b sees neither x nor y via X, and therefore $[x, y] \subset X$. By the same token, $X \cap (B \setminus \{q\})$ is also a seeing subset of X.

We still have to take care of the point q. We would like to add q to either $X \cap (A \setminus \{q\})$ or to $X \cap (B \setminus \{q\})$ and obtain a seeing subset of X. This is always possible, unless q fails to see via X some point $a \in X \cap (A \setminus \{q\})$ and some other point $b \in X \cap (B \setminus \{q\})$. But then a fails to see b via X. (If $[a, b] \subset X$ then $q \in [a, b]$, as we explained above.) This contradicts our assumption that $\alpha(X) = 2$.

Case 3: dim$K = 1$. We will show that in this case X is the union of at most three convex subsets. As in Case 2, S is the union of two closed convex sets A, B of full dimension, such that $A \cap B = K$. As K is convex, K is either a segment,a ray or a line. If K were a line, then both A, B would be strips or half-planes, and their union would be convex, which is impossible. So assume K is a segment or a ray.

Suppose K is a ray. W.l.o.g., K lies on the x-axis and has a rightmost point $(w, 0)$. If A $(\subset \{(x, y)|y \geq 0\})$ has a supporting line L at $(w, 0)$ that is not horizontal, say $y = m(x - w)$ $(m \neq 0)$ or $x = w$, then every point $b \in B$ that lies below the x-axis and to the left of L sees via S every point a of A. (The segment $[b, a]$ crosses the x-axis within K.) The point b sees, of course, every other point of B via S. Thus $b \in \ker S = K$, contrary to our assumption that K is part of the x-axis. By the same token, the set B $(\subset \{(x, y)|y \leq 0\})$ does not have a supporting line L through $(w, 0)$ that is not horizontal.

At most one of the sets A, B contains points on the x-axis to the right of $(w, 0)$. Assume B does not. We claim that X is the union of two convex subsets: $B \cap X$ and $(A \setminus K) \cap X$. We shall first show that $B \cap X$ is a convex subset of X. Suppose $x, y \in B \cap X$.

If both x, y do not belong to the x-axis, take a non-horizontal line L passing through $(w, 0)$ with the points x, y to its right. As noted before, L does not support A, hence there is a point $a \in A$ to the right of L (see Figure 5). The segment $[x, a]$ meets the x-axis to the right of $(w, 0)$. Therefore, a does not see x via S. (Otherwise, $[x, a]$ would be the union of two disjoint non-empty closed sets $[x, a] \cap A$ and $[x, a] \cap B$.) By the same token, a does not see y via S, and therefore $[x, y] \subset X$. $[x, y] \subset B$ as well, since B is convex.

If both x, y do belong to the x-axis, then $(x, y) \subset \mathrm{rel\,int} K \subset \mathrm{int} S \subset X$, and $(x, y) \subset B$ as well. If, say, $x \in \mathrm{rel\,int} K$ and $y \in B \setminus K$ then $(x, y) \subset \mathrm{rel\,int} B \subset \mathrm{int} S \subset X$, hence $(x, y) \subset B \cap X$.

The last case is when, say, x is the endpoint $(w, 0)$ of K, and $y \in B \setminus K$. Since K is the only edge of B through $(w, 0)$, $(w, 0)$ sees every point $y \in B \setminus K$ via $\mathrm{int} B$, hence via X.

We now show that $(A \setminus K) \cap X$ is a convex subset of X. Assume $x, y \in (A \setminus K) \cap X$. Take a non-horizontal line L passing through $(w, 0)$ with the points x, y to its right. L does not support B, hence there is a point $b \in B$ to the right of L. According to the considerations brought in the first case above, b sees via X neither x nor y, hence $[x, y] \subset X$, and $[x, y] \subset A \setminus K$, so $(A \setminus K) \cap X$ is a convex subset of X as well.

Now suppose K is a segment $[u, v]$. If A has a non-horizontal supporting line at u and a non-horizontal supporting line at v, then there are points in $B \setminus K$ that are in $\ker S$, contrary to our assumption. Therefore, in at least one of the endpoints of K, the only supporting line of A is horizontal. By the same token, in at least one of the endpoints of K, the only supporting line of B is horizontal.

By considerations similar to those brought in the case where $\ker S$ is a ray, the sets $(A \setminus \{(x, y) | y = 0\}) \cap X$, $(B \setminus \{(x, y) | y = 0\}) \cap X$ are convex subsets

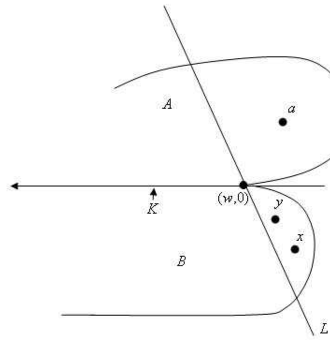

Figure 5

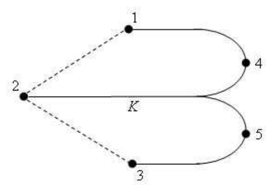

Figure 6

of X. Now, if $D = \{(x,y)|y=0\} \cap X$ is connected, then we are done as X is the union of three convex subsets. Otherwise, D is the union of two convex sets: D_1, the component that includes $\mathrm{rel\,int}K$, and D_2. Assume, w.l.o.g., that D_2 is included in A and is disjoint from B. Considerations similar to those brought above show that in this case X is the union of two convex subsets: $[(B \setminus \{(x,y)|y=0\}) \cap X] \cup D_1$ and $[(A \setminus \{(x,y)|y=0\}) \cap X] \cup D_2$. Example 4 shows that in the case where $M_i = \phi$ and $\dim K = 1$, the number three is best possible:

Example 4:

Figure 6 describes the set X. It is easy to verify that $\alpha(X) = 2$. $\gamma(X) = 3$, as there is a 5-circuit of invisibility.

9 Proof of Theorem E, Reduction to the Polygonal Case

Now we have reached the main case.
Case 4: $\dim K = 2$. This is the most complicated case of the four, which Breen [1974] relates to lengthily. She claims that in this case, X is the union of four convex sets. We will show that X is the union of three convex sets. This result can be viewed as the focal point of the whole paper, since it trivially implies theorem F.

Stage 1: Reduction to the polygonal case:

In this section we intend to show why it is possible to assume that S is a compact, polygonal set. Our approach depends heavily on the following important result of Lawrence, Hare and Kenelly [1972]:

Let T be a subset of a real vector space. Assume that every finite subset $F \subseteq T$ has a k-partition, $\{F_1, ..., F_k\}$, with $\text{conv} F_i \subseteq T$ for $i = 1, ..., k$. Then $\gamma(T) \leq k$, i.e., T is a union of k or fewer convex sets.

Let F be a finite subset of X. We wish to show that F has a 3-partition, $\{F_1, F_2, F_3\}$, with $\text{conv} F_i \subseteq X$ for $i = 1, 2, 3$. We intend to construct a set H such that: $\alpha(H) \leq 2$, $\text{cl} H$ is polygonal, $F \subset H \subset X$, $\text{cl} H \setminus H \subset \text{bd}\, \text{cl} H$ and $\dim \ker \text{cl} H = 2$. A representation of H as a union of three convex sets will imply, in particular, that F has a partition as required. Thus, in order to complete our proof, it will suffice to deal with sets X for which $S = \text{cl}(X)$ is polygonal. In Theorem 9.1 below we construct a closed polygonal set P. We then define $H = P \cap X$ and show that H satisfies the conditions above. Before embarking on the construction of P, we pause to discuss the important notions of relative convexity and relative convex hull.

Let S be a subset of $\mathbb{R}^d$, or, for that matter, of any real vector space. (Here we do not necessarily assume that S is closed.) We say that a subset

C of S is convex relative to S if, for any two points $x, y \in C$, $[x, y] \subset S$, implies $[x, y] \subset C$. The set S itself, or the intersection of S with any convex set, are examples of relatively convex subsets of S.

The intersection of any family of relatively convex subsets of S is again convex relative to S. If F is any (finite or infinite) subset of S, then the intersection of all relatively convex subsets of S that include F is clearly the smallest relatively convex subset of S that includes F. It is called the relative convex hull of F (relative to S). This relative convex hull can also be defined constructively, as follows: Put $F_0 = F$. Define inductively, for $n \in \mathbb{N}$, F_n to be the union of F_{n-1} and all closed line segments $[x, y]$, where $x, y \in F_{n-1}$ and $[x, y] \subset S$. Then the relative convex hull of F (relative to S) is the union $\cup_{n=0}^{\infty} F_n$. In many important cases this construction ends after a finite number of steps, i.e., $F_{n+1} = F_n$ for some finite n.

Finally, let us note that if C is a relatively convex subset of S, then $\alpha(C) \leq \alpha(S)$, $\beta(C) \leq \beta(S)$ and $\gamma(C) \leq \gamma(S)$. Reasons:

α: If $F \subset C$ is a visually independent subset of C, then F is visually independent in S as well.

β: The invisibility graph of C is a (spanned)subgraph of the invisibility graph of S.

γ: If A is a convex subset of S, then $A \cap C$ is a convex subset of C.

Now we are ready to formulate Theorem 9.1:

Theorem 9.1. *Suppose S is a closed subset of $\mathbb{R}^2$, $\alpha(S) \leq 2$, $\underline{0} \in intK$ ($K = kerS$), and F is a finite subset of S. Then there exists a set P such that:*

1) $F \subset P \subset S$;

2) P is convex relative to S (hence $\alpha(P) \leq 2$);

3) $\underline{0} \in int\ kerP$;

4) P is polygonal, i.e., P consists of a simple closed polygonal line bdP and its interior.

Proof. We construct the set P in several steps:

Step 1: Add to F the origin $\underline{0}$, and, if necessary, a few more points of S (never more than three), so as to make the origin $\underline{0}$ an interior point of the

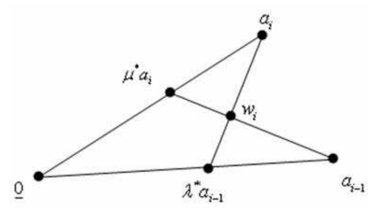

Figure 7

convex hull of the resulting set. Call the resulting set F_1.

Step 2: Define $S_1 = S \cap \text{conv}F_1$. S_1 satisfies all our assumptions on S, and is, in addition, compact. Proceed with S replaced by S_1.

Step 3: Replace each point $a \in F_1 \setminus \{\underline{0}\}$ by the intersection of S_1 with the closed ray $\{\lambda a : \lambda \geq 0\}$. Denote the resulting "sun" (union of segments emanating from $\underline{0}$) by G. The polygonal set P promised in the theorem will be the convex hull of the "sun" G relative to S_1 (or to S, doesn't matter).

Step 4: Now we start to construct the convex hull of G relative to S. Assume $G = \cup_{i=0}^{n-1}[\underline{0}, a_i]$, where the points a_i are arranged in order of increasing argument. Define $a_n = a_0$ and denote by Δ_i ($i = 1, 2, ..., n$) the triangle $[\underline{0}, a_{i-1}, a_i]$. For each i, $1 \leq i \leq n$, we define a subset P_i of Δ_i as follows:

Define:

$\lambda^* = \max\{\lambda : 0 \leq \lambda \leq 1 \wedge [\lambda a_{i-1}, a_i] \subset S\}$

$\mu^* = \max\{\mu : 0 \leq \mu \leq 1 \wedge [a_{i-1}, \mu a_i] \subset S\}$

The maxima do exist, since S is closed.

Define $P_i = [\underline{0}, \lambda^* a_{i-1}, a_i] \cup [\underline{0}, a_{i-1}, \mu^* a_i] \subset \Delta_i$. If $[a_{i-1}, a_i] \subset S$, then $\lambda^* = \mu^* = 1$ and $P_i = \Delta_i$. If not, then $0 < \lambda^* < 1$ and $0 < \mu^* < 1$. (λ^* and μ^* are strictly positive, since an initial subinterval of $[\underline{0}, a_{i-1}]$ (and of $[\underline{0}, a_i]$) lies in ker$S$) In this case , the intervals $[\lambda^* a_{i-1}, a_i]$ and $[a_{i-1}, \mu^* a_i]$ cross at some point $w_i \in \text{int}\Delta_i$, and we obtain: $P_i = [\underline{0}, a_{i-1}, w_i] \cup [\underline{0}, a_i, w_i]$ (see Figure 7).

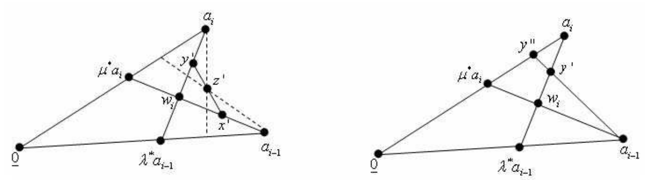

Figure 8

Claim 9.2. *The set P_i is convex relative to S.*

Proof. This is obvious when $P_i = \Delta_i$. Assume, therefore, that $P_i \neq \Delta_i$, i.e., $P_i = [\underline{0}, a_{i-1}, w_i] \cup [\underline{0}, a_i, w_i] = \Delta_i \setminus (\mathrm{int}[a_{i-1}, a_i, w_i] \cup (a_{i-1}, a_i))$. Suppose, on the contrary, that some two points $x, y \in P_i$ see each other via S, but not via P_i. It follows that, say, $x \in [\underline{0}, a_{i-1}, w_i]$, $y \in [\underline{0}, a_i, w_i]$, and the segment $[x, y]$ passes through $\Delta_i \setminus P_i$ $(= \mathrm{int}[w_i, a_{i-1}, a_i] \cup (a_{i-1}, a_i))$.

The segment $[x, y]$ cannot meet (a_{i-1}, a_i), unless $x = a_{i-1}$ and $y = a_i$, in which case $P_i = \Delta_i$, contrary to our assumption. It follows that the segment $[x, y]$ crosses $[w_i, a_{i-1}]$ at some point $x' \neq w_i$ and $[w_i, a_i]$ at some point $y' \neq w_i$. If $x = x' = a_{i-1}$ and $y' \neq a_i$, extend the segment $[x, y']$ beyond y' into P_i, until it hits $[\underline{0}, a_i]$ at some point y'' (see the right side of Figure 8). We find that $y'' = \mu a_i$ for some $\mu^* < \mu < 1$, but a_{i-1} does see y'' via S, contrary to our definition of μ^*. We obtain the same type of contradiction when $y = y' = a_i$ but $x' \neq a_{i-1}$.

Now suppose $y' \neq a_i$, $x' \neq a_{i-1}$. In this case $x' \in (w_i, a_{i-1})$, $y' \in (w_i, a_i)$. Put $z = 1/2(x' + y')$. If a_{i-1} sees z via S, then it sees via S some point beyond $\mu^* a_i$ on $[\underline{0}, a_i]$, which is impossible. (Note that $S \cap \Delta_i$ is starshaped with respect to $\underline{0}$)

We conclude that $[a_{i-1}, z] \nsubseteq S$. By the same token, $[a_i, z] \nsubseteq S$. But $[a_{i-1}, a_i] \nsubseteq S_1$, as well, since $P_i \neq \Delta_i$. This contradicts our assumption that $\alpha(S) \leq 2$. (See the left side of Figure 8) $\square$

Step 5: Define $P = \cup_{i=1}^n P_i$. Let us check that P satisfies the requirements of Theorem 9.1.

By our construction, $F \subset F_1 \subset G \subset P \subset S_1 \subset S$.

To prove that P is convex relative to S, we take two points $x, y \in P$ that see each other via S, and show that $[x, y] \subset P$.

If x and y belong to the same part P_i, then $[x, y] \subset P_i \subset P$, by Claim 9.2.

If $\underset{\sim}{0} \in [x, y]$, then $[x, y] \subset P$, since P is starshaped with respect to $\underset{\sim}{0}$.

Assume, therefore, that $x \in P_i$ and $y \in P_j$, $i < j$, and that the line through x, y does not pass through the origin. Note that both x and y lie in S_1 ($= S \cap \text{conv} F_1$) and therefore $[x, y] \subset S$ implies $[x, y] \subset S_1$.

For $\nu = 0, 1, ..., n$, denote by R_ν the ray emanating from $\underset{\sim}{0}$ through a_ν ($R_\nu = \{\lambda a_\nu : \lambda \geq 0\}$). The segment $[x, y]$ crosses the rays $R_i, R_{i+1}, ..., R_{j-1}$ (or $R_j, R_{j+1}, ..., R_n, R_1, ..., R_{i-1}$) in this order. Assume, for the sake of simpler notation, that it crosses $R_i, R_{i+1}, ..., R_{j-1}$.

Assume that $[x, y]$ meets R_ν at the point $b_\nu = \lambda_\nu a_\nu$, where $\lambda_\nu > 0$. If $\lambda_\nu > 1$ then $b_\nu \notin S_1$, since a_ν is the last point of S_1 on R_ν. It follows that $0 < \lambda_\nu \leq 1$, and therefore $b_\nu \in S_1$, hence $b_\nu \in P_\nu \cap P_{\nu+1}$ for $\nu = i, i+1, ..., j-1$. Thus $[x, y] = [x, b_i] \cup [b_i, b_{i+1}] \cup \cdots \cup [b_{j-2}, b_{j-1}] \cup [b_{j-1}, y]$. By Claim 9.2, $[x, b_i] \subset P_i$, $[b_{j-1}, y] \subset P_j$ and $[b_{\nu-1}, b_\nu] \subset P_\nu$ for $i < \nu < j$, hence $[x, y] \subset P$.

To show that $\underset{\sim}{0} \in \text{int ker} P$, note that $\underset{\sim}{0} \in \text{int ker} S$ and $\underset{\sim}{0} \in \text{int} P$. Let U be a neighborhood of $\underset{\sim}{0}$ that lies in $P \cap \ker S$. Every point $u \in U$ sees every point $p \in P$ ($\subset S$) via S, and therefore via P, since P is convex relative to S.

Finally, note that the number of edges of the boundary of P never exceeds $2|F_1|$. $\square$

We can now define the set H as follows: $H = P \cap X$. Let us show that H satisfies our requirements: (Recall that we need H such that: $F \subset H \subset X$, $\alpha(H) \leq 2$, cl$H$ is polygonal, $\text{cl}H \setminus H \subset \text{bd}\,\text{cl}H$ and $\dim \ker \text{cl}H = 2$.)

According to our construction, $F \subset H \subset X$.

Let us show that H is convex relative to X: Take two points $a, b \in H$ such that $[a, b] \subset X$. $a, b \in P$, $[a, b] \subset S$, so, since P is convex relative to S,

$[a,b] \subset P$. Hence, $[a,b] \subset X \cap P = H$. Therefore, $\alpha(H) \leq 2$.

$\mathrm{int}P \subset \mathrm{int}S \subset X$, so $\mathrm{int}P \subset H = P \cap X$. Since $P = \mathrm{cl}\,\mathrm{int}P$, we find that $\mathrm{cl}H = P$ is polygonal, and $\mathrm{cl}H \setminus H \subset P \setminus \mathrm{int}P = \mathrm{bd}P = \mathrm{bd}\,\mathrm{cl}H$.

Finally, $\underset{\sim}{0} \in \mathrm{int}\ \mathrm{ker}P$, so $\dim \ker \mathrm{cl}H = 2$. This concludes the reduction to the polygonal case. Therefore, we may assume that $S = \mathrm{cl}X$ is polygonal.

10 Proof of Theorem E, The Polygonal Case. Perliminaries

S is a compact polygonal set. Denote by Q the set of lnc points of S. Let $q_1, ..., q_n$ be the points of Q ordered in clockwise direction along bd(conv$Q$). We assume, for the moment, that $n \geq 3$. The simpler cases $n = 0, 1, 2$ will be considered afterwards. conv$Q$ is a polygon with vertices $q_1, ..., q_n$ and edges $e_i = [q_i, q_{i+1}]$, $i = 1, ..., n$ (where $q_{n+1} = q_1$). By e_{i_+} we denote the closed half-plane determined by $\mathrm{aff}\, e_i$ that misses $\mathrm{int}\,\mathrm{conv}Q$. According to Valentine's proof [1957], S is the union of conv$Q$ and n 'bumps' $W_1, ..., W_n$, where $W_i = S \cap e_{i_+}$. We shall refer to $W_1, ..., W_n$ as the **leaves** of S. Each W_i is a convex polygon and so is the union $W_i \cup \mathrm{conv}Q$, for $i = 1, ..., n$. Actually, the union of conv$Q$ with any set of leaves not containing two adjacent leaves, is a convex polygon.

If we orient the boundary of S clockwise, the boundary of each leaf W_i (excluding the base edge e_i) becomes a directed polygonal path, with a first edge starting at q_i and a last edge ending at q_{i+1}. Take l_i to be the line spanned by the last edge of W_{i-1}, and m_i to be the line spanned by the first edge of W_{i+1}. Notice that if α, β, γ are the angles subtended by W_{i-1}, conv$Q$ and W_i at q_i, as in Figure 9, then the following holds:
$\alpha + \beta \leq 180°$, $\beta + \gamma \leq 180°$ and $\alpha + \beta + \gamma > 180°$. Therefore, l_i passes either through intW_i or through the basis e_i. The same holds for m_i. (See Figure 10.)

Denote by l_{i_+} the closed half-plane determined by l_i, that misses $\mathrm{int}\,\mathrm{conv}Q$ and by m_{i_+} the closed half-plane determined by m_i, that misses $\mathrm{int}\,\mathrm{conv}Q$.

Done with the description of S, we move on to describe X: Define for

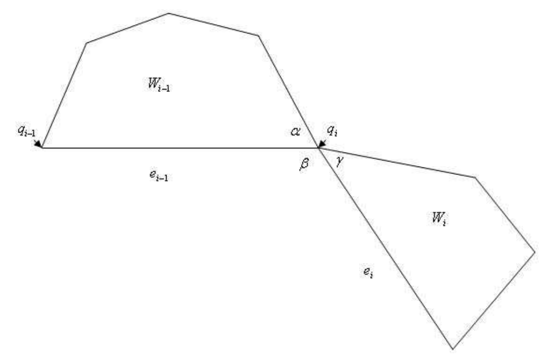

Figure 9

$i = 1, ..., n$, $A_i = (W_i \cap X) \setminus Q$. These are the 'leaves' of X. (Note that A_i includes the relative interior of e_i, but not its endpoints q_i, q_{i+1}.) Now, since $\mathrm{int} S \subset X$, X can be represented as the following disjoint union:
$X = \mathrm{int}\,\mathrm{conv}Q \cup (\cup_{i=1}^{n} A_i) \cup (Q \cap X)$.

Before entering into more technicalities, we would like to give the reader an idea of how we properly color X with three colors.

In the original proof of Valentine's Theorem (for S), each leaf W_i (or, more precisely, $W_i \setminus e_i$), is colored uniformly, and two adjacent leaves get different colors. The central part, $\mathrm{conv}Q$, is part of $\ker S$, and need not be colored at all. Thus, two colors suffice locally, and the third color is only needed to close the circuit when n is odd.

Passing to X, the set A_i ($=X \cap W_i \setminus Q$) may miss some boundary points of W_i, and fail to be convex. This necessitates more than one color for A_i. We pass through each leaf A_i (of X) the line l_i, that divides A_i into an upper left part NE ($C_i \cup D_i$ in Figure 10) and a lower right part SW ($F_i \cup E_i$ in Figure 10). The precise definition of this partition (i.e., which part includes $A_i \cap l_i$) will be given below.

The NE part is convex, and consists precisely of all points $x \in A_i$ that fail to see via X some points in A_{i-1}. The SW part is also convex, except (possibly) for some local invisibilities on the boundary.

We color each of these two parts (NE and SW) uniformly with different colors. We also have to keep in mind that the color assigned to NE should be different from the colors assigned to the adjacent leaf A_{i-1}.

Such a coloring will also take care of at least part of the invisibilities along the boundary of A_i. If there is some invisibility left within the SW part,(this can happen only if the lines l_i, m_i do not cross within A_i), then we fix the coloring along the boundary using the third color (see the set G_i below).

We can play the same trick with the line m_i, coming from the right, instead of l_i, but we shall not use this option. The following point, however, is important: Whenever the line l_i, or m_i, happens to be "horizontal", i.e., coincides with affe_i, the leaf A_i will be convex, and we may color it uniformly.

After having colored the A_i's, we finish the job by coloring Q. Each point $q_i \in Q$ belongs to ker$S$, and sees via int$S$ (hence via X) almost all of X. q_i may fail to see via X only points that lie on the last edge of A_{i-1} or A_{i-2}, or on the first edge of A_i or A_{i+1} (actually, on at most two of these edges simultaneously). Here the term "first (last) edge of A_i" should be understood as "the intersection of A_i with the first (last) edge of W_i". We shall see to it that the points that q_i does not see use at most two colors, so there is a third color left for q_i.

At this point we start the precise, technical description of the promised 3-coloring of X. First, we define two partitions of A_i into two parts:

$$l_{i(+)} = \begin{cases} A_i \cap \text{int}(l_{i_+}) & \text{if } X \cap l_i \text{ is convex,} \\ A_i \cap l_{i_+} & \text{otherwise.} \end{cases} \quad , \qquad l_{i_{(-)}} = A_i \setminus l_{i_{(+)}}$$

$$m_{i_{(+)}} = \begin{cases} A_i \cap \text{int}(m_{i_+}) & \text{if } X \cap m_i \text{ is convex,} \\ A_i \cap m_{i_+} & \text{otherwise.} \end{cases} \quad , \qquad m_{i_{(-)}} = A_i \setminus m_{i_{(+)}}$$

We shall now see that $l_{i_{(+)}}$ is convex:

If $X \cap l_i$ is not convex then $l_{i_{(+)}}$ includes $l_i \cap A_i$ and there is a point in the last edge of W_{i-1} which is in X and does not see any point in $l_{i_{(+)}}$, so due to $\alpha(X) = 2$, every two points in $l_{i_{(+)}}$ see each other via $l_{i_{(+)}}$. Otherwise, if $X \cap l_i$ is convex, then $l_{i_{(+)}}$ does not include $l_i \cap A_i$, so for any two points a, b in $l_{i_{(+)}}$ there is a point in A_{i-1} (close enough to the last edge of W_{i-1}) which

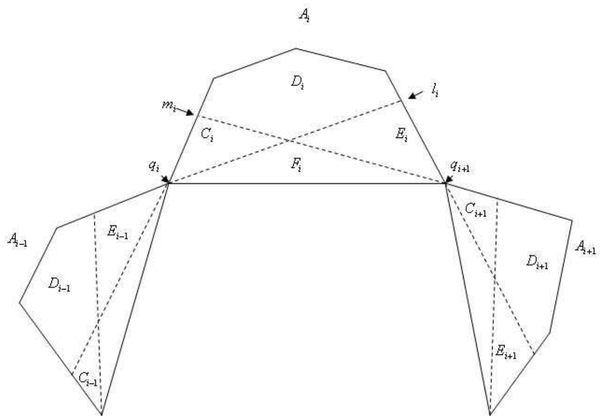

Figure 10

sees neither a nor b, hence $[a,b] \subset l_{i_{(+)}}$. Similarly, $m_{i_{(+)}}$ is convex as well.

It is easy to see that any point in $l_{i_{(-)}}$ sees all points in A_{i-1} via X. (The union $U_i = W_{i-1} \cup \mathrm{conv}Q \cup (l_{i_-} \cap W_i)$ is locally convex, and therefore a convex polygon, by Tietze's Theorem. Since $\mathrm{int}S \subset X$, the only possible invisibilities in $X \cap U_i$ are along boundary edges of U_i. The edge determined by l_i is taken care of by the exact definition of $l_{i_{(-)}}$. In case $n = 3$ there may be another boundary edge of U_i that reaches from A_i to A_{i-1}, namely, the edge determined by the line $\mathrm{aff}(q_{i+1}, q_{i+2})$ ($=\mathrm{aff}(q_{i+1}, q_{i-1})$). If this edge contains a point $x \in A_i$ and a point $y \in A_{i-1}$ then both x and y fail to see via X any point $z \in \mathrm{int}W_{i+1}$, and therefore, x sees y via X.) Similarly, any point in $m_{i_{(-)}}$ sees all points in A_{i+1} via X.

Now define:

$D_i = l_{i_{(+)}} \cap m_{i_{(+)}}$
$C_i = l_{i_{(+)}} \cap m_{i_{(-)}}$
$E_i = l_{i_{(-)}} \cap m_{i_{(+)}}$
$F_i = l_{i_{(-)}} \cap m_{i_{(-)}} \cap \ker X$
$G_i = l_{i_{(-)}} \cap m_{i_{(-)}} \setminus \ker X$ (see Figure 10).

Notice that Figure 10 describes the case where l_i, m_i meet in $\mathrm{int}A_i$ (then $D_i \neq \phi$ and $G_i = \phi$). G_i may be non-empty when l_i, m_i do not meet within A_i (as in Figure 11), or even when they meet on the boundary of A_i.

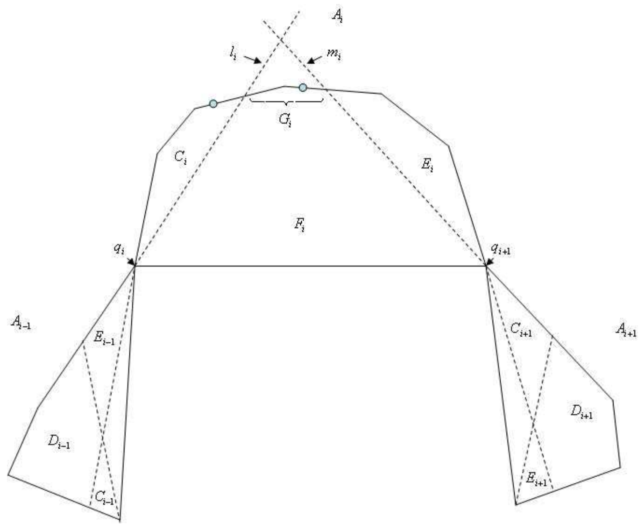

Figure 11

11 Proof of Theorem E, The Polygonal Case. Cont.

Stage 3: The requirements: As we are going to color each of $C_i \cup D_i, E_i$ uniformly, we will first see why each of them is convex:

Recall that $D_i = l_{i_{(+)}} \cap m_{i_{(+)}}$, $C_i = l_{i_{(+)}} \cap m_{i_{(-)}}$. So $C_i \cup D_i = l_{i_{(+)}}$, which is convex. Now, $E_i = l_{i_{(-)}} \cap m_{i_{(+)}}$ so either $E_i = l_{i_-} \cap m_{i_{(+)}}$ or $E_i = \text{int}(l_{i_-}) \cap m_{i_{(+)}}$. In any case, E_i is convex, as the intersection of two convex sets.

Next, we need to check what are other requirements there are for a coloring $c : X \to \{0, 1, 2\}$:

Within each leaf A_i :

$F_i \subset \ker(X)$, hence can be given any color. It is left to check the requirements for its complement in A_i. Since W_i is convex, and $\text{int}W_i \subset A_i$, invisibility within A_i can occur only along edges of W_i. Indeed, two points $a, b \in A_i$ do not see each other via X iff:

(i) Both a and b belong to an edge e of W_i (not the base edge $[q_i, q_{i+1}]$, of course), and

(ii) the intersection $e \cap A_i$ is not convex, and
(iii) a, b belong to different components of $e \cap A_i$.

If l_i and m_i cross in intW_i, then invisibility within A_i can be along at most one edge e of W_i, that goes all the way from C_i to E_i, with one component of $e \cap A_i$ in C_i, and the other in E_i. Our coloring will take care of this invisibility if we require:

Requirement 1: $c(C_i \cup D_i) \neq c(E_i)$

If l_i and m_i do not cross in intW_i, then invisibility within A_i can occur within edges of W_i that are not entirely confined to C_i or to E_i, i.e., edges that cross from C_i to $l_{i(-)} \cap m_{i(-)}$ or lie entirely in $l_{i(-)} \cap m_{i(-)}$, or cross from $l_{i(-)} \cap m_{i(-)}$ to E_i (or, possibly, a single edge that reaches from C_i through $l_{i(-)} \cap m_{i(-)}$ all the way to E_i). G_i consists of all points $a \in A_i$ that belong to $l_{i(-)} \cap m_{i(-)}$ and fail to see some other points $b \in A_i$.

A detailed recipe for a 3-coloring that takes care of all these invisibilities is given in Stage 4.

Between two adjacent leaves: Two points in adjacent leaves, $a \in A_i, b \in A_{i+1}$, may not see each other. This can happen only if $a \in m_{i_{(+)}} = D_i \cup E_i$ and $b \in l_{i+1_{(+)}} = C_{i+1} \cup D_{i+1}$. Therefore we require:

Requirement 2: For each i, $c(E_i) \neq c(C_{i+1} \cup D_{i+1})$ *and* $c(C_i \cup D_i) \neq c(C_{i+1} \cup D_{i+1})$, Where the addition of the indices is modulo n, i.e., $n + 1 = 1$.

Involvement of an lnc point q_i:

q_i may fail to see a point that is in one of the following locations: a point in the last edge of A_{i-1} ($\subset D_{i-1} \cup E_{i-1}$), a point in the first edge of A_i ($\subset C_i \cup D_i$), a point in the last edge of A_{i-2} ($\subset D_{i-2} \cup E_{i-2}$) (this can happen only if $l_{i-1} = \text{aff}(q_{i-1}, q_i)$, in which case A_{i-1} is convex), or a point in the first edge of A_{i+1} ($\subset C_{i+1} \cup D_{i+1}$) (this can happen only if $m_i = \text{aff}(q_i, q_{i+1})$, in which case A_i is convex). Now assume that q_i does not see two points $a, b \in X$ at two different locations. Since $\alpha(X) = 2$, there are 3 cases that cannot occur: $a \in A_{i-1}, b \in A_i$, $a \in A_{i-2}, b \in A_{i-1}$ and $a \in A_i, b \in A_{i+1}$. This leaves three possible cases:

1. $a \in A_{i-2}, b \in A_{i+1}$
2. $a \in A_{i-2}, b \in A_i$
3. $a \in A_{i-1}, b \in A_{i+1}$.

Thus it is impossible that q_i won't see points at three different locations. In stage 4 we shall produce a coloring for q_i that copes with all three possible cases.

Between two leaves that are not adjacent: If a point $x \in A_{i-1}$ does not see a point $y \in A_{i+1}$, then x is necessarily on the last edge of W_{i-1} and y is on the first edge of W_{i+1}, and these edges lie on the same line, i.e., $m_i = l_i = \text{aff}(q_i, q_{i+1})$. In this case, both x and y do not see any point of A_i. This leads to a contradiction to $\alpha(X) = 2$. Therefore, invisibility is impossible among two leaves of X which are two edges apart.

In any other case, for any two points $a, b \in X$ such that $a \in A_i, b \in A_j$ and $i+2 < j < i+n-2$, the segment (a, b) is in int$S$ (according to Valentine [1957]), and therefore is in X.

12 Proof of Theorem E, The Polygonal Case. Coloring.

Stage 4: A recipe for a coloring $c : X \to \{0, 1, 2\}$: Since $\text{int}(\text{conv}Q) \subset \ker X$, we only need to color $(\cup_{i=1}^{n}(A_i \setminus F_i)) \cup (Q \cap X)$. We start by coloring $A_i \setminus F_i$:

The coloring of $C_i \cup D_i, E_i$:

General rule: $c(C_i \cup D_i) \equiv i (\text{mod } 3)$ and $c(E_i) \equiv i + 2 \pmod 3$ for all $1 \leq i \leq n$.

Exceptions:

•Case 0, $n \equiv 0 (\text{mod } 3)$: No exceptions.

•Case 1, $n \equiv 1 (\text{mod } 3)$: $c(C_n \cup D_n) = 2$ and $c(E_{n-1}) = 1$. See Figure 12 for an example with $n = 4$.

•Case 2, $n \equiv 2 (\text{mod } 3)$: $c(E_n) = 0$. See Figure 13 for an example with $n = 5$.

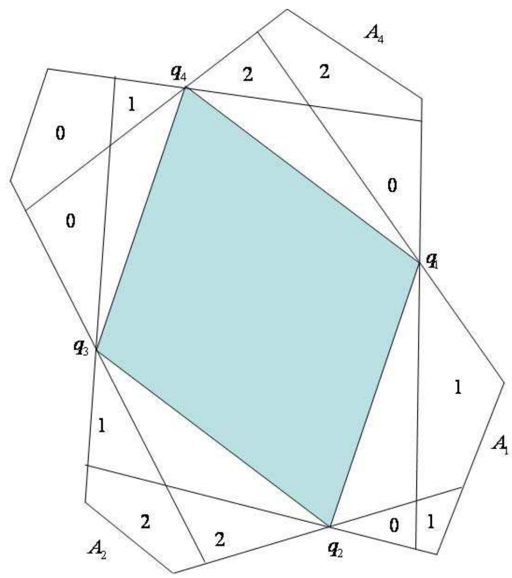

Figure 12

Let us now check that the proposed coloring does satisfy Requirements 1 and 2, namely, that

a) $c(C_i \cup D_i) \neq c(E_i)$

b) $c(C_i \cup D_i) \neq c(C_{i+1} \cup D_{i+1})$

c) $c(E_i) \neq c(C_{i+1} \cup D_{i+1})$ for $i = 1, 2, ..., n$, where the addition of the indices is modulo n.

These conditions clearly hold in Case 0 (i.e., when $3/n$). In Cases 1 and 2 we only have to check conditions a), b), c) in the instances where the definition of $c(C_i \cup D_i)$, or of $c(C_{i+1} \cup D_{i+1})$, or of $c(E_i)$, is exceptional, and conditions b), c) for $i = n$, because of the "seam" modulo 3 between $i = n$ and $i + 1 = 1$. Let us do this:

<u>Case 1</u>

a) $c(C_{n-1} \cup D_{n-1}) = 0 \neq 1 = c(E_{n-1})$, $c(C_n \cup D_n) = 2 \neq 0 = c(E_n)$

b) $c(C_{n-1} \cup D_{n-1}) = 0 \neq 2 = c(C_n \cup D_n)$, $c(C_n \cup D_n) = 2 \neq 1 = c(C_1 \cup D_1)$

c) $c(E_{n-1}) = 1 \neq 2 = c(C_n \cup D_n)$, $c(E_n) = 0 \neq 1 = c(C_1 \cup D_1)$

<u>Case 2</u>

a) $c(C_n \cup D_n) = 2 \neq 0 = c(E_n)$

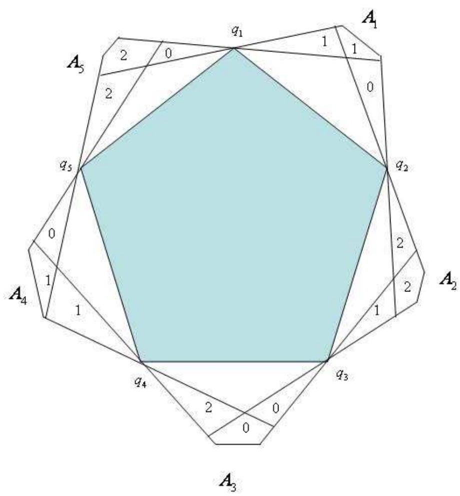

Figure 13

b) $c(C_n \cup D_n) = 2 \neq 1 = c(C_1 \cup D_1)$
c) $c(E_n) = 0 \neq 1 = c(C_1 \cup D_1)$
(As a matter of fact, the check can be performed by a close look at Figures 12 and 13.)

The coloring of $q_i \in X$**:**

As mentioned in stage 3, there are three possible 'maximal' cases where q_i does not see via X points a, b at two different locations:

Case 1: $a \in A_{i-2},\ b \in A_{i+1}$: In this case, $a \in D_{i-2} \cup E_{i-2}$ and $b \in C_{i+1} \cup D_{i+1}$. In addition, $a \in \text{aff}(q_{i-1}, q_i)$, $b \in \text{aff}(q_i, q_{i+1})$. It follows that Case 1 cannot occur for two adjacent lnc points q_i, q_{i+1}. Otherwise, if q_i does not see via X two points $a \in A_{i-2},\ b \in A_{i+1}$ and q_{i+1} does not see via X two points $a' \in A_{i-1},\ b' \in A_{i+2}$, then $a' \in \text{aff}(q_i, q_{i+1})$, $b \in \text{aff}(q_i, q_{i+1})$. Hence, the points a', q_i, b do not see each other via X, a contradiction to $\alpha(X) = 2$. Now we specify the color of q_i in Case 1, dealing with the three congruence classes modulo 3 separately.

If $n = 0 \pmod 3$: According to the coloring c, in this case, for every $1 \le i \le n$, $c(C_{i+1} \cup D_{i+1}) = i + 1 (\text{mod } 3)$, $c(C_{i-2} \cup D_{i-2}) = i - 2 (\text{mod } 3)$, therefore $c(C_{i+1} \cup D_{i+1}) = c(C_{i-2} \cup D_{i-2})$ so those two sets are colored by the same color. The set E_{i-2} is colored by a second color. As these sets are the only constraints for q_i, we will color q_i by the third color that is left 'free'.

If $n = 1 \pmod 3$:

For $3 \le i \le n-2$, the same considerations as in the case $n = 0 \pmod 3$ hold. ($i = n-2$ is not exceptional, since we make no use of E_{i+1})

For $i = n$, $c(C_{i+1} \cup D_{i+1}) = c(C_1 \cup D_1) = 1$, $c(C_{i-2} \cup D_{i-2}) = c(C_{n-2} \cup D_{n-2}) = n - 2 (mod\ \ 3) = 2$, $c(E_{i-2}) = c(E_{n-2}) = n \pmod 3 = 1$, so we define $c(q_n) = 0$.

For $i = 2$, $c(C_{i+1} \cup D_{i+1}) = c(C_3 \cup D_3) = 0$, $c(C_{i-2} \cup D_{i-2}) = c(C_n \cup D_n) = 2$, $c(E_{i-2}) = c(E_n) = n + 2 \pmod 3 = 0$, so we define $c(q_2) = 1$.

For $i = 1$ or $i = n - 1$, there is no color left to assign to q_i, since all three colors are used by $C_{i+1} \cup D_{i+1}$,$C_{i-2} \cup D_{i-2}$ and E_{i-2}. We cope with this situation by renumbering the lnc points in a way that Case 1 occurs neither in q_1, nor in q_{n-1}. This is certainly possible if Case 1 does not occur at all.

If it does occur, mark one lnc point where it occurs by q_n, and recall that Case 1 cannot occur in two adjacent lnc points.

If $n = 2 \pmod 3$:

For $3 \leq i \leq n-1$, the same considerations as in the case $n = 0 \pmod 3$ hold.

For $i = 2$, $c(C_{i+1} \cup D_{i+1}) = c(C_3 \cup D_3) = 0$, $c(C_{i-2} \cup D_{i-2}) = c(C_n \cup D_n) = n \pmod 3 = 2$, $c(E_{i-2}) = c(E_n) = 0$, so we define $c(q_2) = 1$.

For $i = 1$ or $i = n$, there is no color left to assign to q_i, since all three colors are used by $C_{i+1} \cup D_{i+1}$, $C_{i-2} \cup D_{i-2}$ and E_{i-2}. If Case 1 does not occur in some two adjacent lnc points, then we renumber the lnc points in such a way that Case 1 occurs neither in q_1, nor in q_n. Notice that if n is odd, then this must happen, since Case 1 cannot occur in two adjacent lnc points. If n is even, then it may happen that Case 1 occurs in every second lnc point, but then all the sets A_i are convex. In this special case we color X differently: we color all A_i's alternately by two colors and the lnc points by the third color. (See Figure 14)

Case 2: $a \in A_{i-2},\ b \in A_i$: Color q_i by $c(C_{i-1} \cup D_{i-1})$. Notice that $b \in C_i \cup D_i$. Now, $c(C_{i-1} \cup D_{i-1}) \neq c(C_i \cup D_i)$, so q_i and b are colored differently. Similarly, $c(C_{i-1} \cup D_{i-1})$ differs from both $c(C_{i-2} \cup D_{i-2})$ and (E_{i-2}), so q_i and a are colored differently.

Case 3: $a \in A_{i-1},\ b \in A_{i+1}$: Color q_i by $c(C_i \cup D_i)$. Considerations similar to those in Case 2 show that q_i is colored differently from both a, b .

The coloring proposed in Case 2 or in Case 3 clearly works also when q_i fails to see points in only one (or none) of the leaves $A_{i-2}, A_{i-1}, A_i, A_{i+1}$.

The coloring of G_i: G_i is the disjoint union of finitely many connected components b_j. Each component is either a single point, or a line segment, or the union of two line segments with a common endpoint. (This common endpoint must, of course, be a vertex of W_i) Each edge of W_i meets at most two components of G_i. If $G_i \neq \phi$, then $D_i = \phi$, and all the components b_j lie in the gap between C_i and E_i (see Figure 11). Number the components $b_1, ..., b_t$ in the order they appear on the boundary of A_i, with b_1 closest to C_i and b_t closest to E_i.

Points of b_1 may fail to see points of C_i, and points of b_t may fail to see

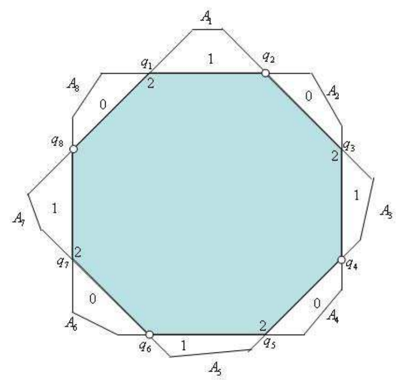

Figure 14

points of E_i. Beyond that, points of G_i may fail to see each other via X only if they belong to adjacent components b_j, b_{j+1}.

Assume $\{0, 1, 2\} = \{p, q, r\}$, where $p = c(E_i)$ and $q = c(C_i)$. Color the components b_j as follows:

$$c(b_j) = \begin{cases} p & \text{if } j \text{ is odd and } j \neq t, \\ q & \text{if } j \text{ is even,} \\ r & \text{if } j \text{ is odd and } j = t. \end{cases}$$

Note that $c(b_1) \neq c(C_i)$, $c(b_t) \neq c(E_i)$ and $c(b_j) \neq c(b_{j+1})$ for $j = 1, 2, ..., t-1$. This finishes the description of a 3-coloring of X.

We still have to deal with the cases $n = 0, 1, 2$.

13 Proof of Theorem E, The Polygonal Case, Some Residual Cases

For $n = 0$: Let us show that in this case, when $M_i = \phi$ and S is convex, $\gamma(X) \leq 3$. Take $a, b \in X$. If $[a, b] \cap \text{int}S \neq \phi$ then $(a, b) \subset \text{int}S$ and since $\text{int}S \subset X$, $[a, b] \subset X$. Hence, $\text{int}S \subset \ker X$. Therefore, we only need to show a coloring of $X \cap \text{bd}S$ with three colors.

S is a polygon with, say, m vertices $p_1, ..., p_m$ and m edges $[p_{i-1}, p_i]$ ($i = 1, ..., m$), where $p_0 = p_m$ ($3 \leq m \leq \infty$). Let $c : X \cap \text{vert}S \to \{0, 1, 2\}$ be a coloring, such that if $p, q \in X$ are adjacent vertices of S, then $c(p) \neq c(q)$. Now extend the coloring c to $X \cap (p_{i-1}, p_i)$ according to the following rules:

1) If $[p_{i-1}, p_i] \subset X$, then $\forall a \in (p_{i-1}, p_i)$, $c(a) = c(p_i)$.

2) If $\{p_{i-1}, p_i\} \subset X$ but $[p_{i-1}, p_i] \not\subseteq X$, then $X \cap [p_{i-1}, p_i]$ has two components. Color the component that contains p_{i-1} by $c(p_{i-1})$, and the one that contains p_i by $c(p_i)$.

3) If $p_i \in X$ but $p_{i-1} \notin X$, and $[p_{i-1}, p_i] \cap X$ is connected, color it by $c(p_i)$.

4) If $p_i \in X$ but $p_{i-1} \notin X$, and $[p_{i-1}, p_i] \cap X$ is not connected, color the component that contains p_i by $c(p_i)$, and the other component by a different number.

5,6) If $p_i \notin X$ but $p_{i-1} \in X$, act as in cases 3,4, with the roles of p_{i-1} and p_i interchanged.

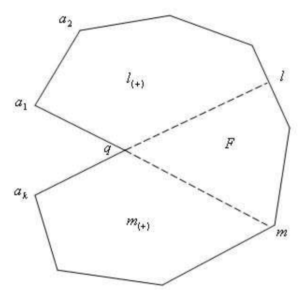

Figure 15

7) If $p_{i-1}, p_i \notin X$, and $[p_{i-1}, p_i] \cap X$ is connected, color it by 1.
8) If $p_{i-1}, p_i \notin X$, and $[p_{i-1}, p_i] \cap X$ is not connected, color one component by 1 and the other component by 2.

Let us mention that in this case (where $\text{cl}X = S$ is convex and $M_i = \phi$), $\gamma(X) = 3$ iff S is an odd-sided convex polygon (not a triangle), X contains all vertices of S and misses at least one point in each edge of S.

For $n = 1$: In this case $Q = \{q\}$ and $\text{bd}S$ is a polygon $< q, a_1, a_2, ..., a_k, q >$. (see Figure 15). Define l to be the line spanned by $[a_k, q]$ and m to be the line spanned by $[q, a_1]$. l_+ is the closed half-plane determined by l that contains a_1, and m_+ is the closed half-plane determined by m that contains a_k. Note that $\ker S = S \cap l_- \cap m_-$. Define:

$$l_{(+)} = \begin{cases} X \cap \text{int}(l_+) & \text{if } X \cap l \text{ is convex} \\ (X \setminus [a_k, q]) \cap l_+ & \text{otherwise} \end{cases}$$

and similarly define:

$$m_{(+)} = \begin{cases} X \cap \text{int}(m_+) & \text{if } X \cap m \text{ is convex} \\ (X \setminus [a_1, q]) \cap m_+ & \text{otherwise} \end{cases}$$

By considerations identical to those appearing in the case $n \geq 3$ (regarding $l_{i_{(+)}}$ and $m_{i_{(+)}}$), both $l_{(+)}$ and $m_{(+)}$ are convex.

Define $F = X \setminus \{q\} \setminus l_{(+)} \setminus m_{(+)}$. A glance at Figure 15 shows that $\mathrm{int}F \subset \mathrm{ker}X$. A close look at the definition of $l_{(+)}$ and $m_{(+)}$ shows that every point $x \in F \cap \mathrm{int}S$ belongs to $\mathrm{ker}X$. There may be some invisibilities in X along edges of S that meet F. Define $G = F \setminus \mathrm{ker}X$. G is composed of connected components along $F \cap \mathrm{bd}S$.

We now define the coloring c of X: $\mathrm{ker}X$ need not be colored at all, $c(l_{(+)}) = 1$, $c(m_{(+)}) = 2$. G will be colored in the same manner as G_i is colored in the case $n \geq 3$ (here the third color might be needed). We still have to color q (if $q \in X$). Note that q sees via X every point of F. If q sees via X every point of $l_{(+)}$, define $c(q) = 1$. Otherwise define $c(q) = 2$. Since $\alpha(X) = 2$, q cannot fail to see via X both a point of $l_{(+)}$ and a point of $m_{(+)}$.

For $n = 2$: Here we shall produce a coloring that is similar to the one we gave in the case $n \geq 3, n \equiv 2 \pmod 3$.

Let $Q = \{q_1, q_2\}$ be the set of lnc points of S. Define $e = [q_1, q_2]$, and assume that $\mathrm{aff}\, e$ is the x-axis. Denote by e_+ (e_-) the upper (lower) closed halfplane determined by $\mathrm{aff}\, e$, and define: $W_1 = S \cap e_+$ and $W_2 = S \cap e_-$. W_1, W_2 are two convex polygons with $W_1 \cap W_2 = e$. Note that since $e = \mathrm{conv}Q \subset \mathrm{ker}S$, q_1 and q_2 cannot be two adjacent vertices of the polygon $\mathrm{bd}S$. Order the edges of W_1, excluding e, clockwise from q_1 to q_2, and denote them by $e_{11}, ..., e_{1n_1}$. Note that the first edge e_{11} and the last edge e_{1n_1} may be horizontal, i.e., collinear with e. If this happens, then they are, strictly speaking, not edges, but parts of the horizontal base edge of W_1. Do the same for W_2, to obtain a sequence $e_{21}, ..., e_{2n_2}$ of edges, stretching from q_2 to q_1. Denote by l_1, m_1 the lines spanned by the last edge e_{2n_2} and by the first edge e_{21} of $\mathrm{bd}W_2$, respectively. (See Figure 16.) Similarly, define $l_2 = \mathrm{aff}\, e_{1n_1}$, $m_2 = \mathrm{aff}\, e_{11}$. Denote by l_{1+} (respectively m_{1+}) the closed half-plane determined by l_1 (respectively by m_1) that misses $\mathrm{int}W_2$. The half-planes l_{2+}, m_{2+} are defined in the same way, with the roles of the indices 1,2 interchanged.

The event that some of the edges $e_{11}, e_{1n_1}, e_{21}, e_{2n_2}$ are horizontal will require some special scrutiny. Therefore we list all possible combinations of such events:

0) None is horizontal.
1) Exactly one is horizontal. Since we can reflect Figure 16 in the horizontal and in the vertical axis, we may assume, w.l.o.g., that e_{2n_2} is horizontal.
2) Exactly two are horizontal: either e_{11} and e_{1n_1}, or e_{21} and e_{2n_2}. Again, we may assume, w.l.o.g., that e_{21} and e_{2n_2} are horizontal.

If e_{11} and e_{21} (or e_{1n_1} and e_{2n_2}) are horizontal, then $\dim \ker S = 1$, contrary to our assumption that $\dim \ker S = 2$.

Done with the description of S, we move on to describe X. For $i = 1, 2$, let $A_i = (W_i \cap X) \setminus e$. Note that here the sets A_i do <u>not</u> include the base e (which is common to $X \cap W_1$ and $X \cap W_2$), as opposed to the definition of the sets A_i in the main case $n \geq 3$.

Now we can represent X as a disjoint union $X = A_1 \cup A_2 \cup (q_1, q_2) \cup (Q \cap X)$. Each A_i has the following partition:

$$l_{i(+)} = \begin{cases} A_i \cap \mathrm{int}(l_{i_+}) & \text{if } X \cap l_i \text{ is convex,} \\ A_i \cap l_{i_+} & \text{otherwise} \end{cases} \quad , \qquad l_{i_{(-)}} = A_i \setminus l_{i_{(+)}}$$

$$m_{i_{(+)}} = \begin{cases} A_i \cap \mathrm{int}(m_{i_+}) & \text{if } X \cap m_i \text{ is convex,} \\ A_i \cap m_{i_+} & \text{otherwise} \end{cases} \quad , \qquad m_{i_{(-)}} = A_i \setminus m_{i_{(+)}}$$

We claim that $l_{1_{(+)}}$ is convex. This is clear when l_1 is not horizontal, as in the case $n \geq 3$. When l_1 is horizontal (i.e., when e_{2n_2} is horizontal), then neither e_{11} nor e_{1n_1} is horizontal, as we noted above, so A_1 lies entirely within the open upper half-plane $\mathrm{int}\, e_+$. Therefore $l_{1_{(+)}} = A_1$ (whether $X \cap l_1$ is convex or not), and A_1 is convex, since every two points of A_1 fail to see a common point of A_2 slightly below e_{2n_2}. By the same token, $l_{2_{(+)}}$, $m_{1_{(+)}}$ and $m_{2_{(+)}}$ are convex as well.

Now, define:
$D_i = l_{i_{(+)}} \cap m_{i_{(+)}}$
$C_i = l_{i_{(+)}} \cap m_{i_{(-)}}$
$E_i = l_{i_{(-)}} \cap m_{i_{(+)}}$
$F_i = l_{i_{(-)}} \cap m_{i_{(-)}} \cap \ker X$

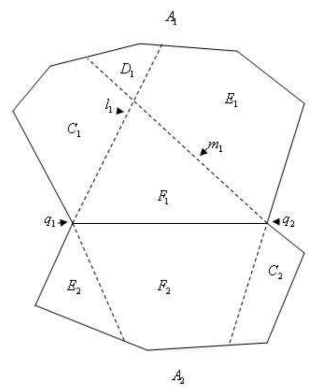

Figure 16

$G_i = l_{i_{(-)}} \cap m_{i_{(-)}} \setminus \ker X$. (See Figure 16.)

The convexity of $l_{i_{(+)}}$ and $m_{i_{(+)}}$ entails the convexity of $C_i \cup D_i$ and E_i for $i = 1, 2$. All points in $l_{1_{(-)}}$ see all points in $l_{2_{(-)}}$. In addition, all points in $m_{1_{(-)}}$ see all points in $m_{2_{(-)}}$. Therefore, C_1, C_2 see each other and E_1, E_2 see each other. All the above enables the same coloring as the one described above in the case $n \geq 3$ and $n \equiv 2 \mod 3$, which leads to the following: $c(C_1 \cup D_1) = 1, c(E_1) = 3, c(C_2 \cup D_2) = 2, c(E_2) = 3$. The sets G_1, G_2 will also be colored as in the case $n \geq 3$.

We still have to color $\operatorname{rel\,int} e = (q_1, q_2)$ and $Q \cap X$. Let us start with $\operatorname{rel\,int} e$. If none of the edges $e_{11}, e_{1n_1}, e_{21}, e_{2n_2}$ is horizontal, then $\operatorname{rel\,int} e \subset \operatorname{int} \ker S \subset \ker X$, so $\operatorname{rel\,int} e$ need not be colored at all. If one or two of these edges is horizontal, then, by our conventions, e_{2n_2} (and maybe also e_{21}) is horizontal. In this case $C_1 \cup D_1 = l_{1_{(+)}} = A_1$ is convex and lies in the open upper half-plane, as we noted earlier. Thus every point of $\operatorname{rel\,int} e$ sees every point of A_1 via $\operatorname{int} A_1$, so $A_1 \cup \operatorname{rel\,int} e$ is convex. Therefore we define $c(\operatorname{rel\,int} e) = c(C_1 \cup D_1) = 1$ in this case.

Finally, we color $Q \cap X$. Assume first that none of the four "problematic" edges is horizontal. If $q_1 \in X$, then q_1 may fail to see via X points on e_{11} or on e_{2n_2} (but not on both). $e_{11} \subset C_1 \cup D_1$ is colored 1, which leaves us the choice of 2 or 3 for $c(q_1)$. $e_{2n_2} \subset m_{2_{(+)}} = E_2 \cup D_2$ is colored 2 and/or 3, which leaves color 1 for q_1. Similarly for q_2.

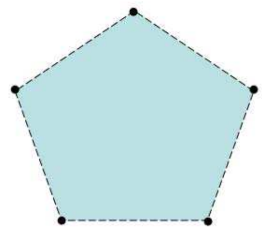

Figure 17

Next assume that e_{2n_2} is horizontal, but e_{21} is not. This leaves for q_1 the same choices as in the previous case. q_2, however, may fail to see points on $e_{21}(\subset C_2 \cup D_2$, colored 2), or on $e_{1n_1}(\subset E_1 \cup D_1$, colored 2,3), or on $e_{2n_2}(\subset E_2$, colored 3). The only possible combination of two of these is e_{21} and e_{2n_2}, which still leaves color 1 for q_2.

If both e_{2n_2} and e_{21} are horizontal, then each of the points q_1, q_2 may fail to see points on three edges, like q_2 in the previous case. But now no combination of two edges is possible. If, e.g., q_2 fails to see points on both e_{21} and e_{2n_2}, then X contains three visually independent points on the x-axis. This completes the proof that if $M_i = \phi$ and $\dim K = 2$, then $\gamma(X) \leq 3$, and with it the proof of theorem E.

Example 5 (due to Breen [1974]) shows that the number three is best possible.

Example 5:

Let P be a regular pentagon. Define $X = (P \setminus \mathrm{bd}P) \cup \mathrm{vert}P$ (see Figure 17).

$\alpha(X) = 2$: The only points in X that are not in $\ker X$ are the vertices of P. The only points that a vertex does not see via X are the two adjacent vertices, but these two see each other via X.

There is a 5-circuit of invisibility, therefore $\gamma(X) \geq 3$.

14 Proof of Theorem F

Assume $M_i = \{(0,0)\}$. Define $A = \{(x,y) \in \mathbb{R}^2 : y > 0 \vee (y = 0 \wedge x > 0)\}$, $B = \{(x,y) \in \mathbb{R}^2 : y < 0 \vee (y = 0 \wedge x < 0)\}$. $A \cup B = \mathbb{R}^2 \setminus \{(0,0)\}$ and $(0,0) \notin X$, therefore $X = (X \cap A) \cup (X \cap B)$. A, B are both convex, so each of the sets $X \cap A, X \cap B$ satisfies the conditions of theorem E, and therefore each is a union of three convex sets. Hence, X is the union of six convex sets.

Example 6 shows that the number six is best possible.

Example 6:

We describe a set $X \subset \mathbb{R}^2$ and show that $\alpha(X) = 2$ and $\gamma(X) > 5$. Let P be a regular 48-gon with center O, vertices $p_0, p_1, ..., p_{48}$ ($p_0 = p_{48}$) and edges $e_i = [p_{i-1}, p_i]$ ($i = 1, 2, ..., 48$). Above each edge e_i erect a triangular dome $T_i = [p_{i-1}, p_i, , t_i]$. The interior angles of each T_i are as follows:

At the odd-numbered base vertex: $7.5°$ ($= 360°/48$).

At the even-numbered base vertex: $6°$.

At the tip t_i: $166.5°$.

Define $S = P \cup (\cup_{i=1}^{48} T_i)$. X is obtained from S by removing the odd-numbered vertices p_{2k-1} ($k = 1, 2, ...24$) and the center O.

Note that each odd-numbered vertex p_{2k-1} is the crossing point of the segments $[p_{2k-2}, t_{2k}]$ and $[t_{2k-1}, p_{2k}]$. Moreover, the sum of the interior angles of T_{2k}, P and T_{2k+1} at the even-numbered vertex p_{2k} is $184.5°(> 180°)$. Therefore, t_{2k} and t_{2k+1} do not see each other via X. Thus we see that, for each k, the points $< p_{2k}, t_{2k-1}, t_{2k}, t_{2k+1}, t_{2k+2}, p_{2k} >$ form a 5-circuit of invisibility in X. Note also that t_i and t_{i+2} always see each other via int$X$. (See Figure 18)

Next, we show that $\alpha(X) = 2$. Note that $S = \text{cl}X$ is the union of the (closed) 48-gon P and 48 triangular (closed) domes, with ker$S = P$. The union of P and any collection of non-adjacent (closed) domes is convex. Thus two points of S fail to see each other via S only if they belong to adjacent domes. A close look at Figure 18 shows that if $z, w \in X$, and the open segment (z, w) passes through one of the removed vertices of P, then z and w must belong to two adjacent domes.

Assume that three points a, b, c form a 3-circuit of invisibility in X.

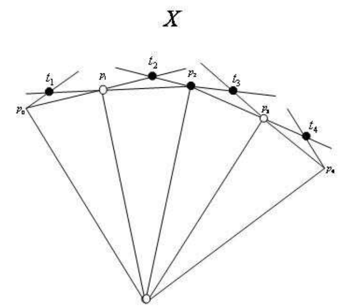

Figure 18

a) If $a \in \text{int}P$ then all points of X that a does not see via X, in particular b and c, lie on the ray $R_{-a} = \{(1+\lambda)O - \lambda a : \lambda > 0\}$. The intersection of this ray with X is convex, and thus $[b,c] \subset X$, contrary to our assumption. Thus we may assume that each of a, b and c belongs to one of the 48 closed triangular domes $T_1, ..., T_{48}$.
b) If a is an even-numbered vertex of P, say p_2 (see Figure 18), then the only points of $X \setminus \text{int}P$ that do not see a via X are: the opposite vertex (p_{26}), the points of the segment $[t_1, p_1)$ and the points of the segment $(p_3, t_4]$. But all these points see each other via X, so again, $[b,c] \subset X$.
c) Assume, therefore, that $a, b, c, \in X \setminus (\text{int}P \cup \text{vert}P)$. It follows that each of these three points belongs to a unique dome (the only points of X that are common to two domes are the even-numbered vertices of P). Denote by T_a, T_b, T_c the domes that contain a, b and c respectively. If $[a,b] \not\subseteq X$, then T_a and T_b must be adjacent or (if $O \in (a,b)$) opposite domes. Same for T_a and T_c, T_b and T_c. But there are no three different domes such that each two are either adjacent or opposite. Thus $\alpha(X) = 2$.

It is left to convince the reader that $\gamma(X) > 5$. The 12 rays $\overrightarrow{OP_{4k}}$ $(k = 0, ..., 11)$ divide X into twelve congruent sectors. Each sector includes four consecutive edges of P, and the corresponding domes, and contains a 5-circuit of invisibility. Figure 18 represents one sector. (The central angles are, of course, exaggerated.)

It follows that each sector is not a union of two convex sets, and therefore, in any covering of X by convex subsets, each sector will meet at least three of the covering subsets. Now, assume to the contrary that X is the union of 5 convex sets. Let's try to evaluate the number of incidences between the five convex sets and the 12 sectors. On the one hand, as every sector meets at least three convex sets, this number is no less than $3 \cdot 12 = 36$. On the other hand, as none of the convex sets includes the center O, each convex set lies on one side of a line through O, and therefore meets at the most 7 sectors. Therefore, the number of incidences is not more than $7 \cdot 5 = 35$, a contradiction.

15 Proof of Theorem G

In the proof of Theorem E, in the case where $n = 0$ (Section 13), we stated the following characterization: When S is convex and $M_i = \phi$, $\gamma(X) = 3$ iff S is an odd-sided convex polygon (not a triangle), X contains all vertices of S and misses at least one point in each edge of S. (Otherwise $\gamma(X) \leq 2$.)

Assume $M_i = \{(0,0)\}$. Define A and B as in the proof of Theorem F (Section 14), to obtain $X = (X \cap A) \cup (X \cap B)$. A, B and $\mathrm{cl}X$ are all convex.

Let us show that the set $\mathrm{cl}(X \cap A)$ is convex. Since $(0,0) \in \mathrm{int}S$, we can choose a point $z = (x_0, y_0) \in \mathrm{int}S$ with $y_0 > 0$, i.e., $z \in \mathrm{int}S \subset X$. Now assume that $p, q \in \mathrm{cl}(X \cap A)(\subset \mathrm{cl}X \cap \mathrm{cl}A)$, and $r = (1-\lambda)p + \lambda q$ for some $0 < \lambda < 1$. We must show that $r \in \mathrm{cl}(X \cap A)$ as well. Define: $p_n = (1 - \frac{1}{n})p + \frac{1}{n}z$, $q_n = (1 - \frac{1}{n})q + \frac{1}{n}z$, $r_n = (1 - \frac{1}{n})r + \frac{1}{n}z$. Then $p_n, q_n \in \mathrm{int}A \cap \mathrm{int}S$ for all n. The sets $\mathrm{int}A$, $\mathrm{int}S$ are convex. Therefore, for all n, $r_n = (1-\lambda)p_n + \lambda q_n \in \mathrm{int}A \cap \mathrm{int}S \subset A \cap X$, and $r = lim_{n\to\infty} r_n \in \mathrm{cl}(A \cap X)$. By the same token, $\mathrm{cl}(X \cap B)$ is convex as well. Therefore, each of $X \cap A$, $X \cap B$ satisfies the conditions of the characterization brought above. Each of these sets has an edge with a missing vertex. Therefore, according to that characterization, each of $X \cap A$, $X \cap B$ is the union of at most two convex sets, hence $\gamma(X) \leq 4$.

Example 7 shows that the number four is best possible.

Example 7:

We describe a convex set $X \subset \mathbb{R}^2$ and show that $\alpha(X) = 2$ and $\gamma(X) \geq 4$. Let P be a regular 7-gon with center O. We define $X = P \setminus (\{O\} \cup \text{bd}P) \cup \text{vert}P$. ($X$ is obtained from P by removing the center O and the relative interiors of all edges.) Let C be a convex subset of X. Since $O \notin X$, C is included in a closed half-plane H with $O \in \text{bd}H$. H intersects vert$P$ in a stretch of three or four consecutive vertices. But two adjacent vertices of X do not see each other via X. Therefore C contains at most two vertices of P. It follows that $\gamma(X) \geq 4$.

We show now that $\alpha(X) = 2$: $X \setminus \text{vert}P$ is the union of two convex sets. Therefore, if there is a 3-circuit of invisibility in X, then it must contain a vertex of P. For each vertex v of P, the points in X that it does not see via X are the two vertices adjacent to v in P and the opposite radius. Notice that all these points see each other via X. Therefore, a vertex of P cannot participate in a 3-circuit of invisibility, hence $\alpha(X) = 2$.

It is left to show that if $M_b = \phi$ or $M_b = \text{bd}S$, then $\gamma(X) = 2$. Indeed, if $M_b = \phi$ then $X = S\setminus\{O\}) = (S \cap A) \cup (S \cap B)$, where A, B are the convex sets defined above. If $M_b = \text{bd}S$, then $X = \text{int}S \setminus \{O\} = (\text{int}S \cap A) \cup (\text{int}S \cap B)$.

16 Proof of Main Theorem 2

If $\beta(X) = 2$ then $\alpha(X) = 2$. If $M_i = \phi$ then according to Theorem E, $\gamma(X) \leq 3$. If $|M_i| > 1$ then according to Theorem C, $\gamma(X) \leq 3$. It is left to handle the case $|M_i| = 1$:

Assume $M_i = \{p\}$. We claim that for every point $x \in X$, $(p, x] \subset X$. Recall that $p \in \text{int}S$ and that $\text{int}S \setminus \{p\} \subset X$. If $(p, x] \not\subseteq X$ for some $x \in X$, then, for some sufficiently small $\varepsilon > 0$, the three points x, $p + \varepsilon(x - p)$, $p - \varepsilon(x - p)$ are in X and fail to see each other via X, and thus $\alpha(X) > 2$.

Define $X^* = X \cup \{p\}$. The argument in the preceding paragraph shows that $p \in \ker X^*$. It follows that a 2-coloring of the invisibility graph of X can be extended to X^* by assigning to p either color. Therefore $\beta(X^*) \leq 2$. Moreover, $\text{cl}(X^*) = \text{cl}X = S$, and thus $M_i(X^*) = \phi$. Hence, by Lemma 8.2, $\gamma(X^*) = \beta(X^*) \leq 2$. If X^* is the union of two convex sets A, B, then

$X = X^* \setminus \{p\} = (A \setminus \{p\}) \cup (B \setminus \{p\})$. Each of the sets $A \setminus \{p\}$, $B \setminus \{p\}$ is the union of at most two convex sets, and therefore $\gamma(X) \leq 4$.

Example 3 in Section 7 (a punctured Star of David) satisfies $\beta(X) = 2$ and $\gamma(X) = 4$. A proper 2-coloring is shown in the right part of Figure 4.

17 Example 8

We describe a set $X \subset \mathbb{R}^2$ and show that $\alpha(X) = 2$ and $\beta(X) = 4$. Start with a square $ABCD$ topped by the upper half of a regular 16-gon with vertices $p_{-4}, p_{-3}, ..., p_0, ..., p_3, p_4$ ($p_{-4} = D$, $p_4 = C$) and edges $e_i = [p_{i-1}, p_i]$, $e_{-i} = [p_{-i}, p_{-i+1}]$ ($i = 1, 2, 3, 4$). Above each edge e_i (e_{-i}) erect a triangular dome $T_i = [p_{i-1}, p_i, t_i]$ $\quad$ ($T_{-i} = [p_{-i}, p_{-i+1}, t_{-i}]$), $i = 1, 2, 3, 4$. The interior angles of each T_i and T_{-i} are as follows:
At the odd-numbered base vertex: 22.5°.
At the even-numbered base vertex: 18°.
At the tip t_i (t_{-i}): 139.5°.

Define $S = \text{conv}\{A, B, p_{-4}, ..., p_0, ..., p_4\} \cup \cup_{i=1}^4 (T_i \cup T_{-i})$. X is obtained from S by removing the odd-numbered vertices p_{-3}, p_{-1}, p_1, p_3, the center O of the square $ABCD$ and the midpoint q of the base $[A, B]$ (See Figure 19).
$\alpha(X) = 2$: Assume, on the contrary, that there is a 3-circuit of invisibility $\Delta \subset X$, i.e., a set of three points of X that fail to see each other via X.

If $a \in X \setminus ([A, B] \cup \cup_{i=1}^4 (T_i \cup T_{-i}))$, then $\text{inv}(a, X)$, i.e., the set of points of X that fail to see a via X, is a convex set, namely, a line segment with endpoint O on the line $\text{aff}(a, O)$, and therefore $a \notin \Delta$. If $a \in X \cap [A, B]$, assume, w.l.o.g., that $a \in [A, q)$. Now $\text{inv}(a, X)$ is the union of $(q, B]$ and a segment with left endpoint O on $\text{aff}(a, O)$. Since the convex hull of these two segments lies in X, we conclude again that $a \notin \Delta$. If $a = p_0$, then $\text{inv}(a, X) = (p_1, t_2] \cup [t_{-2}, p_{-1}) \cup (O, q)$. The points of (O, q) are already out of the game (i.e., cannot participate in a 3-circuit of invisibility) and it follows that $p_0 \notin \Delta$. The same type of argument works for p_{-2} and p_2 as well.

So far we have shown that Δ is included in the union of the eight triangular domes, $\Delta \subset \cup_{i=1}^4 (T_i \cup T_{-i})$, and does not contain any of their intersection

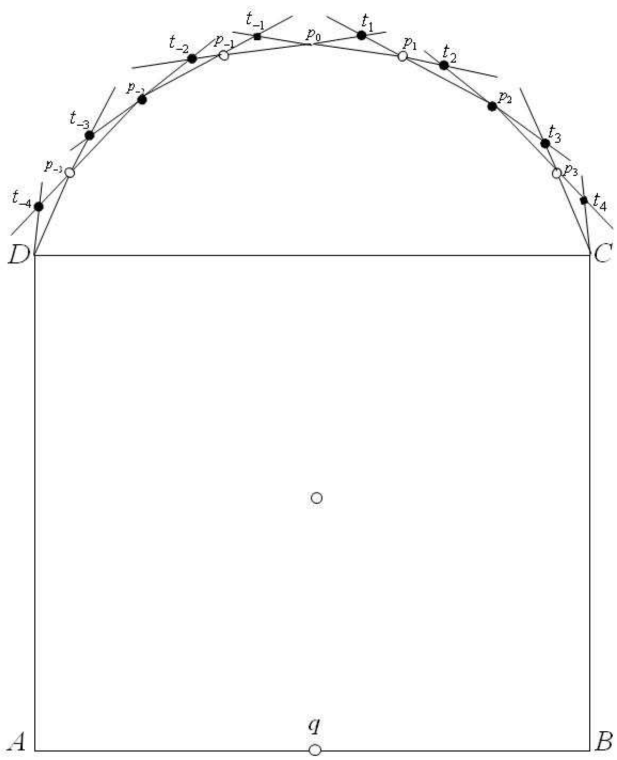

Figure 19

points $p_{-3}, p_{-2}, p_{-1}, p_0, p_1, p_2, p_3$. Thus each point of Δ belongs to a unique triangular dome. Two points of Δ do not see each other only if they belong to adjacent domes. But there is no triple of mutually adjacent domes, so Δ cannot exist at all.

$\beta(X) \geq 4$: Assume, on the contrary, that there exists a 3-coloring $c : X \rightarrow \{1, 2, 3\}$, such that any two points colored alike see each other via X. If $x \in [A, q)$ and $y \in (q, B]$ then $[x, y] \nsubseteq X$, and therefore $c(x) \neq c(y)$. It follows that one of the half-bases $[A, q)$, $(q, B]$ (say $[A, q)$) is colored uniformly. But this leaves only two colors for the 5-circuit of invisibility $< t_1, p_2, t_4, t_3, t_2 >$, in view of the missing center O.

$\beta(X) \leq 4$: Use color 1 for the point p_{-2} and for the segments $[A, D)$ and $[A, q)$, color 2 for the point p_2 and for the segments $[B, C)$ and $[B, q)$, color 3 for the intersection of $X \setminus \{p_{-2}, p_0, p_2\}$ with T_{-4}, T_{-2}, T_1 and T_3, and color 4 for the intersection of $X \setminus \{p_{-2}, p_0, p_2\}$ with T_{-3}, T_{-1}, T_2 and T_4. Now extend the coloring radially from O. Finally, use color 1 for the segments $(O, p_{-3}), (O, p_{-1})$ and $(O, p_0]$, and color 2 for the segments $(O, p_1), (O, p_3)$ and (O, q).

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