Partitions, measurable partitions and disintegration

Partitions and $\sigma$-algebras

Let $(X,F)$ be a standard Borel space. A partition $\xi$ of $X$ always means a partition into measurable sets. $\xi(x) \in \xi$ is the element containing $x$. A set $A \in B$ is $\xi$-saturated if $x \in A$ implies $\xi(x) \subseteq A$. Thus $A = \bigcup_{x \in A} \xi(x)$.

A partition $\xi$ defines the $\sigma$-algebra of measurable $\xi$-saturated sets,

$$F_\xi = \{ A \in F : A \text{ is } \xi\text{-saturated} \}$$

Example 1. If $\xi$ is the partition into points then $F_\xi$ is the full $\sigma$-algebra. If $X = [0,1]^2$ and $\xi$ is the partition into vertical lines then $F_\xi = \{ A \times [0,1] : A \subseteq \text{Borel}([0,1]) \}$.

One way to define a partition is as the atoms of a countably generated sub-$\sigma$-algebra: if $A = \sigma(A_1,A_2,\ldots) \subseteq F$ write $A^0 = A$ and $A^1 = X \setminus A$, and for $i \in \{0,1\}^\mathbb{N}$ write

$$A^i = \bigcap_{k=1}^{\infty} A^{ik}$$

Then define

$$\xi_A = \{ A^i : i \in \{0,1\}^\mathbb{N} \}$$

$$= \{ A \in F : \forall i \in \mathbb{N} \in F, A \subseteq C_i \text{ or } A \subseteq X \setminus C_i \}$$

$$= \{ A \in F : \forall B \in F, A \subseteq B \text{ or } A \subseteq X \setminus B \}$$

This construction is related to the previous one by

$$F_{\xi_A} = A$$

and if $\xi$ is a partition and $F_\xi$ is countably generated then also $\xi = \xi_{F_\xi}$. (These statements are exercises).

Example 2. If $X = \{0,1\}^\mathbb{N}$ and $A$ is the $\sigma$-algebra determined by coordinates $2,3,4,\ldots$, then $\xi_A$ is the partition whose sets are pairs of points differing in their first coordinate.

Example 3. In general $F_\xi$ is not countably generated. For example if $T : X \to X$ is a measurable automorphism then the partition $\xi$ into orbits consists of countable, hence measurable, sets, and $F_\xi$ is the $\sigma$-algebra of $T$-invariant sets. If $\mu$ is a non-atomic ergodic invariant measure for $\xi$ then every orbit has measure 0 and every invariant set has measure 0 or 1, and to $\xi_{F_\xi}$ contains an atom of measure 1, which cannot be an orbit.

Example 4. Let $A \in SL_d(\mathbb{Z})$ be a hyperbolic matrix, $V \leq \mathbb{R}^d$ the span of the expanding eigendirections, and $\xi$ the partition of $\mathbb{T}^d$ such that $x \sim y$ if and only if
x = y + v \text{ mod } 1 \text{ for some } v \in V. \text{ This is called the } \textit{partition into unstable leaves.} \\
One can characterize it dynamically by the property that \( x \sim y \) if and only if \\
d(\T_A^{-n}x, \T_A^{-n}y) \to 0. \text{ In general, } \mathcal{B}_\xi \text{ is not countably generated. For example for } \\
\text{hyperbolic } A \in SL_2(\mathbb{Z}), \text{ the expanding eigenspace is a line of irrational slope,} \\
\text{and } \xi \text{ is the orbit relation on the torus of the flow in this direction, which is} \\
\text{ergodic. Therefore by the previous example, it is not countably generated.}

\textbf{Definition 5.} \text{ A partition } \xi \subseteq \mathcal{B} \text{ is } \textit{measurable} \text{ if } \mathcal{B}_\xi \text{ is countably generated.}

\textbf{Adding a measure to the picture}

\text{Now let } \mu \text{ be a probability measure on } (X, \mathcal{F}). \text{ We identify partitions and } \\
\sigma\text{-algebras that differ on a zero-measure set.} \\
\text{Specifically, we say that } \xi = \xi' \mod \mu \text{ if there is a set } X_0 \subseteq X \text{ of full measure} \\
such that } \xi|_{X_0} = \xi'|_{X_0}. \text{ This implies that } \mathcal{F}_\xi = \mathcal{F}_{\xi'} \mod \mu.

\text{We say that } \sigma\text{-algebras } \mathcal{A}, \mathcal{A}' \subseteq \mathcal{F} \text{ satisfy } \mathcal{A} \subseteq \mathcal{A}' \mod \mu \text{ if for every } A \in \mathcal{A} \\
\text{there exists } A' \in \mathcal{A}' \text{ with } \mu(A \Delta A') = 0. \text{ If both } \mathcal{A} \subseteq \mathcal{A}' \mod \mu \text{ and } \mathcal{A}' \subseteq \mathcal{A} \mod \mu \text{ then } \mathcal{A} = \mathcal{A}' \mod \mu.

\text{When } \mathcal{A}, \mathcal{A}' \text{ are countably generated, if } \mathcal{A} = \mathcal{A}' \mod \mu \text{ then } \xi_A = \xi_{A'} \mod \mu \\
(\text{Find generating sequences } \{C_i\} \text{ and } \{C'_i\} \text{ for } \mathcal{A}, \mathcal{A}' \text{ respectively with } \mu(C_i \Delta C'_i) = 0 \text{ and take } X_0 = X \setminus \bigcup (C_i \Delta C'_i)).

\textbf{Lemma 6.} \text{ For every } \sigma\text{-algebra } \mathcal{A} \subseteq \mathcal{F} \text{ there is a countably generated sub-} \sigma\text{-}

\text{algebra } \mathcal{A}' \subseteq \mathcal{A} \text{ such that } \mathcal{A} = \mathcal{A}' \mod \mu.

\textit{Proof.} \text{ Let } d(A, B) = \mu(A \Delta B). \text{ This is a separable metric on } \mathcal{F} \text{ (this uses the fact that } (X, \mathcal{F}) \text{ is standard Borel), so } \mathcal{A} \subseteq \mathcal{F} \text{ is separable as well; choose a countable dense sequence } A_1, A_2, \ldots \subseteq \mathcal{A} \text{ and let } \mathcal{A}' = \sigma(\{A_i\}). \text{ It is an exercise to show that } \mathcal{A} = \mathcal{A}' \mod \mu. \quad \square

\textbf{Remark 7.} \text{ This shows that the property of being countably generated is not preserved under equality mod } \mu! \text{ But of course, the property of being countably generated mod } \mu, \text{ is.}

\textbf{Measure-valued integration}

\text{Given a measurable space } (X, \mathcal{B}), \text{ a family } \{\nu_x\}_{x \in X} \text{ of probability measures on} \\
(Y, \mathcal{C}) \text{ is measurable if for every } E \in \mathcal{C} \text{ the map } x \mapsto \nu_x(E) \text{ is measurable (with respect to } \mathcal{B}). \text{ Equivalently, for every bounded measurable function } f : Y \to \mathbb{R}, \\
\text{the map } x \mapsto \int f(y) \, d\nu_x(y) \text{ is measurable.}

\text{Given a measure } \mu \in \mathcal{P}(X) \text{ we can define the probability measure } \nu = \int \nu_x \, d\mu(x) \text{ on } X \text{ by} \\
\nu(E) = \int \nu_x(E) \, d\mu(x)

\text{For bounded measurable } f : Y \to \mathbb{R} \text{ this gives} \\
\int f \, d\nu = \int (\int f \, d\nu_x) \, d\mu(x)
and the same holds for \( f \in L^1(\nu) \) by approximation (although \( f \) is defined only
on a set \( E \) of full \( \nu \)-measure, we have \( \nu_x(E) = 1 \) for \( \mu \)-a.e. \( x \), so the inner
integral is well defined \( \mu \)-a.e.).

Example 8. Let \( X \) be finite and \( \mathcal{B} = 2^X \). Then

\[
\int \nu_x d\mu(x) = \sum_{x \in X} \mu(x) \cdot \nu_x
\]

Any convex combination of measures on \( Y \) can be represented this way, so the
definition above generalizes convex combinations.

Example 9. Any measure \( \mu \) on \((X, \mathcal{B})\) the family \( \{\delta_x\}_{x \in X} \) is measurable since
\( \delta_x(E) = 1_E(x) \), and \( \mu = \int \delta_x d\mu(x) \) because

\[
\mu(X) = \int 1_E(x) d\mu(x) = \int \nu_x(E) d\mu(x)
\]

In this case the parameter space was the same as the target space.

In particular, this representation shows that Lebesgue measure on \([0, 1]\) is
an integral of ergodic measures for the identity map.

Example 10. \( X = [0, 1] \) and \( Y = [0, 1]^2 \). For \( x \in [0, 1] \) let \( \nu_x \) be Lebesgue
measure on the fiber \( \{x\} \times [0, 1] \). Measurability is verified using the definition
of the product \( \sigma \)-algebra, and by Fubini’s theorem

\[
\nu(E) = \int \nu_x(E) d\mu(x) = \int_0^1 \int_0^1 1_E(x, y) dy dx = \int \int_E 1 dxdy
\]

so \( \nu \) is just Lebesgue measure on \([0, 1]^2\).

One could also represent \( \nu \) as \( \int \nu_{x,y} d\nu(x, y) \) where \( \nu_{x,y} = \nu_x \). Written this
way each fiber measure appears many times.

Disintegration

We now reverse the procedure above and study how a measure may be decom-
posed as an integral of other measures. Specifically, we will study the decom-
position of a measure with respect to a partition.

Example 11. Let \((X, \mathcal{B}, \mu)\) be a probability space and let \( \xi = \{P_1, \ldots, P_n\} \) be
finite or countable partition of it. For simplicity assume also that \( \mu(P_i) > 0 \).
let \( \mu_x^\xi \) denote the conditional measure on \( \xi(X) \), i.e. \( \mu_x = \frac{1}{\mu(P_i)} \mu|_{P_i} \). Then
it is easy to check that \( \mu = \int \mu_x d\mu(x) \).

Our goal is to give a similar decomposition of a measure with respect to
an infinite (usually uncountable) partition of \( X \). Then the partition elements
\( E \in \mathcal{E} \) typically have measure 0, and the formula \( \frac{1}{\mu(E)} \mu|_E \) no longer makes
sense. As in probability theory one can define the conditional probability of an
event \( E \) given that \( x \in E \) as the conditional expectation \( \mathbb{E}(1_E|\mathcal{P}) \) evaluated at
Theorem 12. Let \( \mathcal{E} \) be a standard Borel space, \( \xi \) a measurable partition and \( \mathcal{E} = \mathcal{B}_\xi \subseteq \mathcal{B} \) the corresponding countably generated sub-\( \sigma \)-algebra and \( \xi \). Then there is an \( \mathcal{E} \)-measurable family \( \{ \mu^\xi_y \}_{y \in X} \subseteq \mathcal{P}(X) \) such that \( \mu^\xi_y \) is supported on \( \mathcal{E}(y) \) and
\[
\mu = \int \mu^\xi_y \, d\mu(y)
\]
i.e. for every \( f \in L^2(\mu) \), we have \( f \in L^1(\mu^\xi_y) \) for \( \mu \)-a.e. \( x \) and
\[
\int fd\mu = \int \left( \int fd\mu^\xi_y \right) d\mu(x)
\]
Furthermore if \( \{ \mu'_y \}_{y \in X} \) is another such system then \( \mu^\xi_y = \mu'_y \) a.e.

Note that \( \mathcal{E} \)-measurability has the following consequence: For \( \mu \)-a.e. \( y \), for every \( y' \in \xi(y) \) we have \( \mu^\xi_{y'} = \mu^\xi_y \) (and, since since \( \mu^\xi(y) = 1 \), it follows that \( \mu^\xi_{y'} = \mu^\xi_y \) for \( \mu^\xi \)-a.e \( y' \)).

Definition 13. The representation \( \mu = \int \mu^\xi_y \, d\mu(y) \) in the proof is often called the disintegration of \( \mu \) over \( \mathcal{E} \) (or \( \xi \)).

The Martingale theorem

Assume that \( (X, \mathcal{F}) \) is endowed with a countable family \( \{ f_n \} \) of bounded measurable test functions generating \( \mathcal{F} \) (e.g. when \( X \) is compact, a dense set of continuous functions). We define convergence of measures \( \mu_n \to \mu \) on \( X \) by
\[
\int f_k \, d\mu_n \to \int f_k \, d\mu \text{ for all } k.
\]
Let \( \xi_1 \leq \xi_2 \leq \xi_3 \ldots \) be measurable partitions and \( \xi_\infty = \bigvee \xi_n \) the coarsest common refinement, given by \( \xi_\infty(x) = \bigcap_{n=1}^\infty \xi_n(x) \). Then \( \xi_\infty \) is measurable.

Theorem 14 ("Forward" Martingale theorem). If \( \xi_1 \leq \xi_2 \leq \xi_3 \ldots \) are measurable partitions, and \( \xi_\infty = \bigvee \xi_n \), then
\[
\mu^\xi_x \to \mu^\xi_\infty \quad \mu \text{-a.e. } x
\]
Let \( \xi_1 \geq \xi_2 \geq \xi_3 \geq \ldots \) be measurable partitions, let \( B_\infty = \bigcap B_\xi_n \) mod \( \mu \) be a countably generated \( \sigma \)-algebra and \( \xi_\infty = \xi_{B_\infty} \); we denote \( \xi_\infty = \bigwedge \xi_n \), and note that it is measurable, but defined mod \( \mu \) (because \( B_\infty \) is only defined mod \( \mu \)).

Theorem 15 ("Backward" Martingale theorem). Let \( \xi_1 \geq \xi_2 \geq \xi_3 \geq \ldots \) be measurable partitions, let \( B_\infty = \bigcap B_\xi_n \) mod \( \mu \) be a countably generated \( \sigma \)-algebra and \( \xi_\infty = \xi_{B_\infty} \). Then
\[
\mu^\xi_x \to \mu^\xi_\infty \quad \mu \text{-a.e. } x
\]
Remark 16. Note that in general, $\xi_\infty(x) \neq \bigcup \xi_n(x)$. For example let $X = \{0,1\}^\mathbb{N}$ with the product measure $\mu$ with marginal $(\frac{1}{2}, \frac{1}{2})$. Let $\xi_n$ denote the partition according to coordinates $n, n+1, n+2, \ldots$. Then $B_\infty = \bigcap B_{\xi_n}$ is trivial mod $\mu$, by the Kolmogorov 0,1-law, and $\xi_\infty$ is trivial (consists of a set of full measure and a nullset). On the other hand, for every $x$, $\bigcup \xi_n(x)$ is a countable set consisting of all which eventually agree with $x$. 