Polynomials with Disconnected Julia Sets and Green Maps[1]

by

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Let T be a polynomial of degree $d \geq 2$, and let T^n be the n-th iterate of T. Recall the following definitions [F], [B], [DH]:

$A(\infty) = \{z : T^n(z) \to \infty,\ n \to \infty\}$ is the basin of attraction to infinity,

$J = \partial A(\infty)$ is the Julia set of T,

$u(z)$ is the Green function of the domain $A(\infty)$ with the pole at infinity.

Define $u = 0$ outside $A(\infty)$, then

$$u(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |T^n(z)|, z \in \mathbb{C},$$

and

$$u(T(z)) = d \cdot u(z).$$

Let $B(z)$ be the Bottcher function of T,

i.e.

$$B(z) = \lim_{n\to\infty} [T^n(z)]^{1/d^n},$$

$$B(z) \sim z, z \to \infty.$$

so $u(z) = \log |B(z)|$.

Denote by C^* the set of all critical points of T, contained in $A(\infty)$. If $C^* = \emptyset$ then the Julia set J is connected and the function $B(z)$ gives the conformal map of the simply connected domain $A(\infty)$ onto the exterior of the unit disk. If additionally the set J is locally connected, then, by the Caratheodory theorem, $B(z)$ extends to the map of boundaries, namely J and $\mathbb{T}$. This map semi-conjugates $T|_J$ and $t \to d \cdot t(mod 2\pi)$ on $\mathbb{T}$. If J is connected, but not locally connected, $B(z)$ can be extended to some points of $J = \partial A(\infty)$, for example, to the points of repulsive cycles of T, and this fact is also useful [D], [DH] [EL1].

The main purpose of this paper is to consider similar (and other) questions in the situation when $C^* \neq \emptyset$ and, consequently, the domain $A(\infty)$ is infinitely connected.

The paper consists of two parts.

In the first part (Sects. 1-3) we investigate the Green map [AJ] of the domain $A^*(\infty)$ obtained from $A(\infty)$ by cutting along the Green lines going from critical points of the function $u(z)$ to the Julia set. This Green map is the extension of the Bottcher function, and the image of $A^*(\infty)$ under this map is the domain with a hedgehog-like boundary. For polynomials with real Julia set such a map was considered in [SY] (see also [BGM]).

In the second part (Sects. 4-5) we apply the map onto hedgehog in order to generalize two known results. The first of them estimates multipliers of repulsive fixed points of polynomials with connected Julia sets [P], [Y], [L], [EL1]. The second one concerns the sizes of the parts of the Mandelbrot set, so-called “limbs”. The concept of the limb has been introduced by Douady, Hubbard and Branner (see, for example, [GM]). Yoccoz [Y] estimated diameters of the limbs using the first result (about multipliers). We give analogous estimates for rational visible periodic points of quadratic polynomials with disconnected Julia sets (see also [EL2]), and for diameters of neighborhoods of the limbs (we call them by “shades” of the limbs).

1. The Bottcher Function and the Green Map

Introduce the following definitions:

$C^* = \{c_1, \cdots, c_s\}, \ 1 \le s \le d-1,$

$a = \max\{u(c_j) : 1 \le j \le s\},$

$K(r) = \{z : u(z) \le r\},$

$\Gamma(r) = \partial K(r) = \{z : u(z) = r\},$

$G(r) = \mathbb{C} \setminus K(r).$

If $r \ge a$, the domain $G(r)$ is simply connected. In this domain the Bottcher function $B(z)$ is well defined and gives a conformal map of $G(r)$ onto $\{w : |w| > r\}$. The function $B(z)$ satisfies the functional equation

$$B(T(z)) = [B(z)]^d. \tag{1.1}$$

This equation yields an analytic continuation of the function $B(z)$ on the whole domain $A(\infty)$. The continued function has branch points in the points of the set

$$C(\infty) = \bigcup_{m=0}^{\infty} T^{-m} C^*,$$

i.e. in the critical points of the Green function $u(z)$, but the function $|B(z)|$ is single-valued .

To make the function $B(z)$ single-valued we use the technique similar to [AJ]. Let us generalize some definitions from [AJ] and [DH].

Fix positive direction on the level lines $\Gamma(r)$,so that $K(r)$ stays on the left.If a curve γ crosses $\Gamma(r)$ at a point z,define a positive direction on γ in a neighbourhood of z in such a way that the γ passes $\Gamma(r)$ from $G(r)$ to $K(r)$.

Fix an angle (slope) $\tau \in (0, \pi)$. The Green line of the slope τ is a C^1-curve γ ,which crosses any level line at the angle τ. The function $u(z)$ is monotone along an arbitrary Green line. The Green line γ is maximal if γ is not contained properly in any other Green line. Thus if $z \in A(\infty) \setminus C(\infty)$ then exists a unique maximal Green line $\gamma^\tau(z)$, passing through this point. So the origin of every maximal Green line is either ∞ or some point from $C(\infty)$. In the former case the maximal Green line is *the external radius of the slope* τ, in the later case it is *the cut*(of the same slope).

Let $A^*_\tau(\infty)$ be a subset of $A(\infty)$ formed by points which belong to the external radii, i.e. $A^*_\tau(\infty)$ is $A(\infty)$ with cuts deleted. Particularly, $G(a) \subset A^*_\tau(\infty)$.

There exists a unique analytic continuation of the function $B(z)$,$z \in G(a)$, along arbitrary external radius R^τ of the slope τ.This defines the continued function B^τ in the whole domain $A^*_\tau(\infty)$.

The image $B^\tau(R^\tau)$ of an external radius R^τ is a C^1-curve in $\mathbb{D}^* = \{x : 1 < |x| < \infty\}$,which crosses every circle $|x| = r$ of large enough radius at the angle τ.If we consider the extention L^τ of$B^\tau(R^\tau)$ in $\mathbb{D}^*$,which preserves this property,it is ended at a point with an argument $t \in [0, 1)$ of the unit circle.We will call the t the external argument arg_{B^τ} of the radius R^τ (and its points) in the dynamical plane or the argument arg_τ of the curve L^τ (and its points) in $\mathbb{D}^*$:

$$t = arg_{B^\tau}(R^\tau) = arg_\tau(L^\tau)$$

. In particular,we have defined the $arg_\tau(x)$ of every point $x \in \mathbb{D}^*$ (of course,the arg_τ coincides with the usual argument in the complex plane iff $\tau = \pi/2$).

The continued function maps conformally the domain $A^*_\tau(\infty)$ onto some domain

$$U_\tau \subset \mathbb{D}^*.$$

We set $S_\tau = \partial U_\tau$.

Let $x \in \mathbb{C},\ |x| > 1,\ I_x = \{\zeta : 1 \le |\zeta| \le |x|, arg_\tau(\zeta) = arg_\tau(x)\}$. We call this curve *the needle*, the point $\exp(it)$,where $t = arg_\tau(x)$, is *the base* of this needle.

Consider the map

$$\sigma : t \to d.t(mod 1). \tag{1.2}$$

If $arg_{B^\tau}(z) = t$, then $arg_{B^\tau}(T(z)) = \sigma(t)$.

Every point $c \in C^*$ is the common end of a finite set of external radii. Denote by $\Lambda_\tau(c)$ the set of the external arguments t of the corresponding radii R^τ_t with the end at c. Note that, for every $t \in \Lambda_\tau(c)$,

$$\sigma(t) = arg_{B^\tau}(T(c)),$$

and there exists

$$\lim_{\substack{z\to c\\ z\in R_t^\tau}} B^\tau(z)=\tilde{B}^\tau(c,t)$$

(this equality is just a notation).

Proposition 1.1. *The following statements are true*

1. $T: A_\tau^*(\infty)\to A_\tau^*(\infty)$.
2. *The continued function* $B^\tau(z)$ *satisfies the functional equation* (1.1), *in other words, the function*

$$T_0=B^\tau\circ T\circ(B^\tau)^{-1}: x\mapsto x^d$$

maps U_τ *into* U_τ.

3.

$$S_\tau=\mathbb{T}\bigcup_{c\in C^*}\bigcup_{t\in\Lambda_\tau(c)}\bigcup_{n=0}^{\infty}\bigcup_{T_0^n(x)=\tilde{B}^\tau(c,t)} I_x\ .$$

Proof. The map T takes every external radius to an external radius. This implies p.1 and, then, p.2.

To prove p.3, let us introduce the set of critical arguments:

$$\Lambda_\tau^*=\bigcup_{c\in C^*}\bigcup_{n=0}^{\infty}\sigma^{-n}(\Lambda_\tau(c)).$$

If t is not such a critical value, then the corresponding external radius R_t^τ extends up to the Julia set. Hence $e^{2\pi i\,t}\in S$.

On the other hand, if $\sigma^n(t)\in\Lambda_\tau(c)$ for some $n\geq 0$ and $c\in C^*$, the radius R_t^τ joins ∞ and a critical point q of Green map u, so that

$$u(q)=\frac{u(c)}{d^n}.$$

It completes the proof.

We will call the set S_τ the *hedgehog* (of the equal slope τ). *The skin* of this hedgehog is the circle $\mathbb{T}$ minus bases of hedgehog's needles. Let us note that the hedgehog S_τ is uniquely determined by the slope τ and the set of limit values $\tilde{B}^\tau(c,t)$, for all $c\in C^*, t\in\Lambda_\tau(c)$.

Define now the external rays as follows(cf.[GM]).It is just an external radius $R=R^\tau$,if R extends up to the Julia set(i.e. R is not ended at a point of $C(\infty)$).Let the end point of

R be a point of $C(\infty)$.Then $B^\tau(R)$ lands at the top of some needle N_x.The function B^τ extends to two continuous functions B_+ and B_- on two banks of N_x.It allows us to define the two external rays (of the slope τ) corresponding to the external radius R:

$$R_t^{\pm} = B_{\pm}^{-1}(\{\zeta : 1 < |\zeta| < \infty, arg_\tau(\zeta) = t\}),$$

where $t = arg_\tau(x)$.

Let $R(t) = \{z \in A^*(\infty) : arg_B z = t\}$ be some external radius. Such radius can walk approaching to the Julia set $J \subset \partial A^*(\infty)$, but not ending to a definite point. Besides, the Julia set can contain inaccessible from $A^*(\infty)$ points. But, as it was proved in [DH, p. 70], any external radius with a rational argument is sure to finish either in some point from $C(\infty)$, or in some pre-image of the point from a repulsive or an indifferent rational cycle of T. Moreover, if at least one rational external radius lands at a point of such cycle, then there are finitely many external radii landing at this cycle, all are rational and the same period (see [GM]). We will denote by $N = N(\alpha)$ the number of such external radii, which land at point of the cycle (α).

2. The Hyperbolic Case

In this Sect. we assume that the polynomial T is hyperbolic, i.e. there exists a compact neighborhood V of J and numbers $L > 1, \beta > 0$ such that

$$|(T^{-n})'(z)| \leq \beta L^{-n}$$

for any $z \in V$ and for any branch T^{-n}.

We give without proofs two propositions concerning this case. The proofs of them are quite similar to the proofs of corresponding propositions from [DH].

Proposition 2.1. *Let T be the hyperbolic polynomial. Then all maximal Green lines (i.e. the external radii and the cuts) have their ends on the Julia set.*

Define the set $J^* = \partial A^*(\infty)$. By the definition, J^* is a continuum (i.e. connected compact). In the hyperbolic case this set consists of the Julia set J and of the cuts, which connect the points of $C(\infty)$ and J.

Proposition 2.2. *In the hyperbolic case J^* is locally connected.*

So in this case $B(z)$ extends to the map of J^* onto S which semi-conjugates $T|_J$ and T_o on the skin of the hedgehog S.

Remark. These propositions remain valid in some other cases, for example, when the polynomial T is semi-hyperbolic [DH], i.e. when all critical points, lying on J, are preperiodic, and other critical points tend to attractive cycles.

3. The Dependence on Coefficients of the Polynomial T.

Let $T(z)$ be a monic centered polynomial, i.e.

$$T(z) = z^d + b_{d-2}z^{d-2} + \cdots + b_1 z + b_0.$$

Parametrize such polynomials by points $\underline{b} \in \mathbb{C}^{d-2}$. Let $\mathbb{H} \subset \mathbb{C}^{d-2}$ be the closure of the set $\underline{b} = (b_0, \cdots, d_{d-2})$ of such polynomials with disconnected Julia set. We will use the notations $A_{\underline{b}}(\infty), A^*_{\underline{b}}(\infty), u_{\underline{b}}(z)$ and so on.

Theorem 3.1. *Let $(\underline{b}_n) \subset int(\mathbb{H}), \underline{b}_n \to \underline{b}$. Then the domains $A^*_{\underline{b}_n}(\infty)$ are converging to $A_{\underline{b}}(\infty)$, if $\underline{b} \in \partial\mathbb{H}$ and to $A^*_{\underline{b}}(\infty)$, if $\underline{b} \in int(\mathbb{H})$. Here we mean the convergence as to the kernel in the sense of Caratheodory.*

Proof. We consider here only the case $\underline{b} \in \partial\mathbb{H}$. The case $\underline{b} \in int(\mathbb{H})$ is quite similar. We use that the function $u_{\underline{b}}(z)$ is the continuous function on $(z, \underline{b}) \in \mathbb{C} \times \mathbb{C}^{d-2}$ [DH].

We divide the proof into two steps.

Step 1. Let $K \subset A_{\underline{b}}(\infty)$ be an arbitrary compact. We show that

$$K \subset A^*_{\underline{b}_n}(\infty), \; n \geq n_0.$$

Choose such $\varepsilon > 0$ that $u_{\underline{b}}(z) > \varepsilon, \; z \in K$.

Then

$$u_{\underline{b}_n}(z) \geq \varepsilon/2, \quad z \in K, \quad n \geq n_0. \tag{3.1}$$

If $c_1, \cdots, c_{d-1}$ are the critical points of the polynomial $T_{\underline{b}}$, then $u_{\underline{b}}(c_j) = 0, \; 1 \leq j \leq d-1$, and, therefore, (3.2) $$u_{\underline{b}_n}(c_{j,n}) < \varepsilon/2,\ n \geq n_0,$$

where $c_{j,n}$ are the critical points of $T_{\underline{b}_n}$. Let

$$a_n = \max\{u_{\underline{b}_n}(c_{j,n}) : 1 \leq j \leq d-1\}.$$

It follows from (3.1) and (3.2) that

$$K \subset G_{\underline{b}_n}(a_n) \subset A^*_{\underline{b}_n}(\infty),\ n \geq n_0.$$

Step 2. Show that $A_{\underline{b}}(\infty)$ is the largest domain with the property mentioned in Step 1.

Assume that this is not true, and there exists a subsequence, which we also denote by $\underline{b}_n$, and a domain $D \underset{\neq}{\supset} A_{\underline{b}}(\infty)$ such, that for any compact $K \subset D$ it holds

$$K \subset A^*_{\underline{b}_n}(\infty),\ \ n \geq n_0.$$

Then $J_{\underline{b}} \cap D \neq \emptyset$. Choose such a neighborhood V of $z \in J_{\underline{b}} \cap D$ that $\bar{V} \subset D$, and take $K = \bar{V}$. Then $V \subset A^*_{\underline{b}_n}(\infty),\ n \geq n_0$, so

(3.3) $$V \cap J_{\underline{b}_n} = \emptyset,\ n \geq n_0.$$

On the other hand

(3.4) $$J_{\underline{b}} \subset \mathbb{D}_R = \{z : |z| < R\},$$

for some $R < \infty$. There exists a number $N \in \mathbb{N}$ such that $T^N_{\underline{b}}(V) \supset \bar{\mathbb{D}}_{2R}$. In view of $T^N_{\underline{b}_n} \to T^N_{\underline{b}},\ n \to \infty$, we get

(3.5) $$T^N_{\underline{b}_n}(V) \supset \bar{D}_{3R/2},\ n \geq n_0.$$

By (3.4) and (3.5) it holds $J_{\underline{b}} \subset T^N_{\underline{b}_n}(V),\ n \geq n_0$, and we can find such points $y \in J_{\underline{b}_n}, x \in V$, that $T^N_{\underline{b}_n}(x) = y$. Hence

(3.6) $$x \in J_{\underline{b}_N} \cap V,\ n \geq n_0.$$

The relations (3.3) and (3.6) give the contradiction, and Theorem 3.1 is proved.

Let

$$\Phi = B^{-1} : U \to A^*(\infty).$$

Corollary 3.2. *Under the conditions of the previous theorem we have* $\Phi_{\underline{b}_n} \to \Phi_{\underline{b}}$ *uniformly on every compact in* $\mathbb{C} \setminus \bar{\mathbb{D}}$ *in the case* $\underline{b} \in \partial\mathbb{H}$, *and uniformally on every compact in* $\bar{U}_{\underline{b}}$, *when* $\underline{b} \in int(\mathbb{H})$.

Proof. We use the theorem 3.1 and the Caratheodory theorem.

4. One Property of the Hedgehog

Consider exterior $\mathbb{D}^* = \{\omega : 1 < |\omega| < \infty\}$ of the unit disk and its universal covering $\mathbb{C}_+ = \{\zeta : Im\zeta > 0\}$ with a covering projection $p : \mathbb{C}_+ \to \mathbb{D}^*$,

$$p : \omega \mapsto \exp(-2\pi i\omega).$$

The pre-image $H^* = p^{-1}(U)$ of the hedgehog's exterior $U = \mathbb{D}^* \setminus S$ is a universal covering of U. Moreover,

$$H^* = \mathbb{C}_+ \setminus Q,$$

where $Q = p^{-1}(S)$ is a one-periodic comb. The ground of this comb is the real axis, and the ends of teeth of Q are the points with coordinates

$$\omega_{c,\tau}(n,k) = -\frac{\tau + k}{d^n} + \frac{i}{2\pi}\frac{u(c)}{d^n},$$

where $c \in C^*, \tau \in \Lambda(c), n \geq 0$ and $k \in \mathbb{Z}$.

A lifting $\tau^* : \omega \mapsto d\omega$ of the map $z \mapsto z^d, z \in U$, acts in H^* as well as in the ground $\mathbb{R}$ of Q.

Let us fix $t^* \in \mathbb{R}$. For every $c \in C^*$, denote by $d_c^{(r)}(t^*)$ the distance between t^* and the point of the set $\Lambda^*(c) = \{\tau + k : \tau \in \Lambda(c), k \in \mathbb{Z}\}$, which is nearest to the point t^* and has a smaller value. Similarly, the number $d_c^{(\ell)}(t^*)$ is the distance between t^* and the nearest point of $\Lambda^*(c)$ bigger than t^*. Set

$$D_c^{(r)}(t^*) = \inf_{n \geq 0} d_c^{(r)}(d^n t^*),$$
$$D_c^{(\ell)}(t^*) = \inf_{n \geq 0} d_c^{(r)}(d^n t^*).$$

The following two numbers $\gamma^{(r)}$ and $\gamma^{(\ell)}$ are uniquely defined by the conditions:

$$tg\gamma^{(r)} = \max_{c \in C^*} \frac{1}{2\pi}\frac{u(c)}{D_c^{(r)}(t^*)}, \gamma^{(r)} \in (0, \frac{\pi}{2}],$$
$$tg\gamma^{(\ell)} = \max_{c \in C^*} \frac{1}{2\pi}\frac{u(c)}{D_c^{(\ell)}(t^*)}, \gamma^{(\ell)} \in (0, \frac{\pi}{2}].$$

Proposition 4.1. *The angles*

$$W^{(r)}(t^*) = -t^* + \{\omega \in \mathbb{C}_+ : \gamma^{(r)} < arg\omega < \frac{\pi}{2}\},$$
$$W^{(\ell)}(t^*) = -t^* + \{\omega \in \mathbb{C}_+ : \frac{\pi}{2} < arg\omega < \pi - \gamma^{(\ell)}\}$$

belong to H^*, and they are the maximal open angles with this property.

Proof. If an angle

$$-t^* + \{\omega \in \mathbb{C}_+ : \gamma < arg\omega < \frac{\pi}{2}\}$$

is the maximal such an angle, which lies in H^*, then

$$tg\gamma = \sup \frac{Im\omega_{c,\tau}(n,k)}{Re\omega_{c,\tau}(n,k) - (-t^*)} =$$
$$\sup \frac{u(c)/2\pi}{d^n t^* - \tau - k},$$

where supremum is taken over all points $c \in C^*, \tau \in \Lambda(c)$, and over such values n, k, that $n \geq 0, k \in \mathbb{Z}$, and the denominator is positive. Hence, $\gamma = \gamma^{(r)}$. The angle $\gamma^{(\ell)}$ is found similarly.

Remind, that $\tau t = dt (mod 1)$. Let $t = \{t^*\}$ be the fractional part of t^*.

Corollary 4.1. *The following conditions are equivalent*:

(a) *none point of the union* $\underset{c\in C^*}{\cup} \Lambda(c)$ *is a point of the closure of the orbit* $\{\tau^n t\}_{n=0}^{\infty}$,

(b) *there is some nonsingular open angle*

$$W(t^*) = -t^* + \{\omega \in \mathbb{C}_+ : \gamma_1 < arg\omega < \pi - \gamma_2\}$$
$$0 < \gamma_1, \gamma_2 < \frac{\pi}{2},$$

which lies within H^*.

Let E be the set of the points $t^* \in \mathbb{R}$, for which such an angle exists. Note that $E + 1 = E$.

Introduce the Poincare metric ρ^* in H^* and the standard Poincare metric ρ in $\mathbb{C}_+$ ($d\rho = \frac{|d\omega|}{Im\omega}$). We will call two Riemannian metrics ρ' and ρ'' in a domain $V \subset \bar{\mathbb{C}}$ are equivalent ($\rho' \sim \rho''$) within an open subset $V' \subset V$, iff, for some constants $C', C'', 0 < C' < C'' < \infty$,

$$C' d\rho' < d\rho'' < C'' d\rho' \qquad \text{in} V'.$$

Let now $t^* \in E$, so that the nonsingular open angle $W(t^*)$ lies in H^*.

Proposition 4.2 *The metrics $\rho^*(\omega)$ and $\rho(\omega), \omega \in H^*$, are equivalent whenever ω belongs to any smaller angle $W_1(t^*)$, which is in $W(t^*)$ together with its sides.*

Proof. On the one hand, $d\rho < d\rho^*$ in H^*, since $H^* \subset \mathbb{C}_+$. On the other hand, if $\rho_1(\omega)$ is the Poincare metric of the domain $W(t^*) \subset H^*$, then $d\rho^* < d\rho_1$ and $\rho_1 \sim \rho$ in $W_1(t^*)$.

Corollary 4.2 *Let $R_t \subset A^*(\infty)$ be an external radius at angle t, such that $t \in E$. If $z_0 \in R_t$ is fixed, and a point z goes to the Julia set along R_t, then the length of the arc of R_t joining z_0 and z, measured in the Poincare metric of the domain $A^*(\infty) \subset \mathbb{C}$, is equivalent to the function $\log u(z_0)/u(z)$.*

5. Estimates for Multipliers of Repulsive Cycles of Quadratic Polynomials

In this Sect. we consider the case, when $T_c(z) = z^2 + c$, and the parameter c lies in the exterior of so-called Mandelbrot set $M = \{c \in \mathbb{C} : J_{T_c} \text{ is connected}\}$.

Let $\alpha = (z_1, \cdots, z_q)$ be a repulsive cycle of the polynomial T_c for some $c \in \mathbb{C} \setminus M$. We suppose that there is an external radius with a rational argument landing at a point z of the cycle (α). Remind that in this case only finitely many, $N = N(\alpha)$, external radii finish in z, and the map T^q permutes them.

A number $\lambda = (T^q)'(z)$ does not depend on a point $z \in (\alpha)$ and is called the multiplier of the cycle (α).

Theorem 5.1

$$N \log|\lambda| \le \frac{|\log \lambda^N|^2}{\log|\lambda^N|} \le \frac{2\pi q \log 2}{arcctg[(2^{Nq}-1)a/\pi]}, \tag{5.1}$$

where $a = u(0)$ and $u = u_c$ is the corresponding Green function.

Remark. The following inequality (5.2) is related to (5.1). If $z \in J$, then the value

$$\chi(z) = \limsup_{n\to\infty} \frac{1}{n} \log|(T^n)'(z)|$$

is the characteristic exponent of the polynomial T at the point z. In [EL2] the estimate

$$\chi(z) \le \frac{\pi \log d}{arcctg \frac{ad}{\pi}} \tag{5.2}$$

was proved for an arbitrary point $z \in J$ and for an arbitrary hyperbolic polynomial $T, \deg T = d$. Our method in the present paper is different.

Proof of theorem. Set $p = Nq$. Instead of T we will consider the polynomial T^p, which we again denote by T. This polynomial has the same hedgehog with tops of needles

$$x(n,k) = \exp\{\frac{2a + 2\pi i(t_c + k)}{2^n}\},\ n \in \mathbb{N},$$

where $a = \log|B(0)| = u(0),\ t_c = \arg B(c)$. So z is a repulsive fixpoint of $T, \deg T = 2^p,\ T'(z) = \lambda^N$, and let $\zeta_1, \cdots, \zeta_N$ be the different fixpoints of $T_0 : \zeta \to \zeta^{2^p}$ such that $\Phi(\zeta_i) = z$.

We need two changes of variables.

1. In the hedgehog's plane we set $w = \frac{i}{2\pi}\log\zeta$. After this the hedgehog S turns in to the 1-periodic comb Q with the ends of teeth (see Sec. 4)

$$w(n,k) = -\frac{t_c + k}{2^n} + i\frac{2a}{2\pi \cdot 2^n} + \mathbb{Z},\ n \in \mathbb{N}. \tag{5.3}$$

The map $\tau : t \to 2t(mod 1)$ acts in the ground of this comb and in $H^* = \mathbb{C}_+ \setminus Q$.

2. Linearizing the function T in a neighborhood of the point z, we replace T by $f : z \to \lambda^N z$, and the point z by the point $z = 0$.

Put $u_j = \frac{i}{2\pi}\log\zeta_j$. Let V_j be an upper half-neighborhood of the point u_j without points of the comb Q. Then there exists a holomorphic univalent function h in V_j such that

$$h(\tau^p(u)) = \lambda^N h(u).$$

Let us assume that there exists such a sector

$$W_j = \{w \in \mathbb{C}_+ : |w| < \varepsilon, \gamma_j^{(r)} < arg\ w < \gamma_j^{(\ell)}\},$$

that $W_j + u_j \subset V_j$. Then after the obvious normalization we get the equation

$$h(2^p u) = \lambda^N h(u)$$

in this sector. Now we can apply the theorem from [L]. This theorem states that the previous equation yields the inequality

$$\frac{|\log\lambda^N|^2}{\log|\lambda^n|} \le \frac{2\Delta_j \pi \log 2^p}{\gamma_j}, \tag{5.4}$$

where

$$\Delta_j = \liminf_{r\to 0}\frac{1}{\log r}\iint_{h(W_j)\cap\mathbb{D}_r}\frac{dxdy}{|z|^2}\ , z = x+iy,$$

is the lower logarithmic density of the domain $h(W_j)$ and

$$\gamma_i = \pi - \gamma_j^{(r)} - \gamma_j^{(\ell)}$$

is the angle of the sector W_j.

Now we *claim* that

$$\gamma_i \geq arcctg\frac{(2^p-1)a}{\pi} \tag{5.5}$$

If this claim is correct, then the inequalities (5.4) yield

$$\frac{|\log\lambda^N|^2}{\log|\lambda^N|} \leq \frac{2\pi q\log 2}{arcctg\frac{(2^p-1)a}{\pi}}\ ,$$

since $\Delta_1+\Delta_2+\cdots+\Delta_N\leq 1$, and the required inequality (5.1) is proved.

Proof of Claim. Fix $j\in\{1,\cdots,N\}$ and set $t_0=-u_j$. Let $\delta_c^{(r)}$ be the distance between $t_c = arg_B c$ and the set

$$P=\{t_0,\tau(t_0),\cdots,\tau^{p-1}(t_0)\},$$

measured clockwise on the circumference of the unit length. By proposition 4.1,

$$tg\gamma_j^{(\gamma)} = \max_{n,k}\frac{2a}{2\pi(2^nt_j-t_c-k)} = \frac{a}{\pi\delta_c^{(r)}}.$$

Similarly,

$$tg\gamma_j^{(\ell)} = \frac{a}{\pi\delta_c^{(\ell)}},$$

where $\delta_c^{(\ell)}$ is the distance between t_c and P measured counterclockwise on the circumference of the unit length. Because of $\delta_c=\delta_c^{(\ell)}+\delta_c^{(r)}\geq 1/(2^P-1)$, we have

$$\gamma_j^{(r)}+\gamma_j^{(\ell)} = arctg\frac{a}{\pi\delta_c^{(r)}} + arctg\frac{a}{\pi\delta_c^{(\ell)}} \leq$$
$$\leq \frac{\pi}{2}+arctg\frac{a}{\pi\delta_c}\leq\frac{\pi}{2}+arctg\frac{(2^P-1)a}{\pi},$$
$$\text{and }\ \gamma_j=\pi-\gamma_j^{(r)}-\gamma_j^{(\ell)}\geq arcctg\frac{(2^P-1)a}{\pi}.$$

The claim and the theorem are proved.

6. Sizes of "shades".

If J is connected (i.e. $a = 0$), the Pommerenke-Yoccoz inequality [P], [Y], [L] states

$$\frac{|\log \lambda^N|^2}{\log|\lambda^N|} \leq 2q\, logd, \tag{6.1}$$

for every repulsive cycle of every polynomial $T, degT = d$. Our inequality (4.1), for $a = 0$ and $d = 2$, gives two times worse estimate.

Yoccoz [Y] derived from (6.1) a bound for diameters of the limbs in the set M. Following Yoccoz's method, we obtain a generalization of his result.

Let us give definitions of a "*limb*" and a "*wake*". Much more information can be found in [GM].

Fix an integer $N \geq 2$ and consider a set $W(N)$ of all points c such that the corresponding map T_c has a repulsive fixed point and there exist precisely N external radii, which land at this point (so they are permuted by T_c).$W(N)$ is an open set and it consists of a finitely many components $(W_i(N))$. Each component $W_i(N)$ is bounded by two curves R_i^- and R_i^+, which are two external radii in the plane of the parameter c. The external radii in the parameter plane can be described as the orthogonal trajectories with respect to the family of "level curves" $\{c : u_c(0) = a\}, a > 0$. It is important, that, for every component $W_i(N)$, the radii R_i^- and R_i^+ unite in a common point $c = c_i(N) \in \partial M$, namely, for which T_c has a neutral fixed point with a multiplier λ such that $\lambda^N = 1$. The closure of every component $W_i(N)$ contains exactly one such point $c_i(N)$. It splits the set M into two connected part. The first part is a central core $M_0 = \{c : T_c$ has a repulsive or neutral fixed point $\}$. The second part of M is called *the limb* $M_i(N)$. It is in $\overline{W_i(N)}$. The complement $W_i(N) \setminus M_i(N)$ is a *wake* of this limb.

Now we want to define "*shades*" of the limbs. Fix a number $a > 0$ and a component $W_i(N)$ (so we fix also a limb $M_i(N)$). A set $M_i(a, N) = W_i(N) \cap \{c : u_c(0) < a\}$ is said to be a-shade of this limb. Notice that

$$M_i(a_1, N) \subset M_i(a_2, N), \text{ if } 0 < a_1 < a_2,$$

and

$$\cap_{a>0} \overline{M_i(a, N)} = M_i(N).$$

Yoccoz [Y] *proved that there is* $C_0 > 0$ *such that the diameter of the limb* $M_i(N)$ *less than* C_0/N, *for all* i *and* $N \geq 2$.

Theorem 4.1 allows us to generalize slightly this result.

Theorem 5.1 *There exists* $C > 0$ *such that*

$$\text{diam}\, M_i(a, N) < \frac{c}{N}, \tag{6.2}$$

whenever

$$(2^N - 1)a < \pi arcctg \log 2. \tag{6.3}$$

Proof. Let $c \in \partial M_i(a, N)$ and let λ be a multiplier of a nonrepulsive fixed point of T_c. Then

$$c = \frac{\lambda}{2} - (\frac{\lambda}{2})^2 \tag{6.4}$$

and (5.1) implies

$$\log \lambda^N \in \{z : (x - \frac{D}{2})^2 + y^2 \leq (\frac{D}{2})^2\}, \tag{6.5}$$

where

$$D = 2\pi \log 2 / arcctg \frac{(2^N - 1)a}{\pi}.$$

The condition (6.3) provides $D < 2\pi$, and, together with (6.5), we obtain that $\log \lambda$ belongs to the one and only one among N mutually disjointed connected sets; each of them has a diameter less than $2\pi/N$ and contains a point λ such that $\lambda^N = 1$. Then the inequality (6.2) follows from the explicit expression (6.4).

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