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ESTIMATING THE CHARACTERISTIC EXPONENTS OF POLYNOMIALS

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1. We consider a polynomial f of degree $d \geq 2$, and we denote its n-th iteration by f^n. The results of the theory of iterations that are used in the present article may be found in [1, 2].

A root of the equation $f^n z = z$ is called a periodic point (with period n). The quantity $\chi(z) = \frac{1}{n}\log|(f^n)'(z)|$ is the characteristic exponent of this point. When the Julia set of the polynomial f is connected, we have

$$\chi(z) \leq 2\log d, \tag{1.1}$$

for any periodic point z, and this bound is sharp only when the Julia set is a line segment with z for its end [3]. In the present paper we obtain an upper bound for $\chi(z)$ for arbitrary polynomials, as well as a lower bound for $\chi(z)$ for the case in which the Julia set is totally disconnected.

We set

$$u_f(z) = \lim_{n\to\infty} \frac{1}{d^n} log^+|f^n(z)|. \tag{1.2}$$

This limit exists and is a subharmonic function in C ([4] is a standard reference for the theory of subharmonic functions). The function u_f is nonnegative and continuous on C. It is harmonic and positive in the domain $D = \{z: f^n z \to \infty,\ n \to \infty\}$, and $u_f(z) = 0$ in $C \setminus D = K$. We have the functional equation

$$u_f \circ f = du_f. \tag{1.3}$$

The Riesz measure μ_f of the function u_f is concentrated in the Julia set $J = \partial D = \partial K$. This is the only probability measure in C that has the following property: For any Borel set $E \subset C$ on which the function f is univalent, we have

$$d\mu_f(E) = \mu_f(fE). \tag{1.4}$$

The measure μ_f is called the equilibrium measure or the measure of maximum entropy.

Let $c_1, c_2, \ldots, c_{d-1}$ be all of the critical points (with zero derivative) of the polynomial f. We set

$$a = \max\{u(c_j)\colon 1 \leq j \leq d-1\}, \tag{1.5}$$

$$b = \min\{u(c_j)\colon 1 \leq j \leq d-1\}. \tag{1.6}$$

The numbers a and b are natural parameters characterizing the degree of disconnection of the Julia set: $a = 0$ if and only if J is connected; on the other hand, J is a Cantor set (totally disconnected) if $b > 0$. We should also note the connection between the number a and mean of the characteristic exponent

$$\chi_f = \int \log|f'|\, d\mu_f.$$

We have

$$\chi_f = \log d + \sum_{j=1}^{d-1} u_j(c_j),$$

so $a \leq \chi_f - \log d \leq (d-1)a$. In particular, $\chi_f = \log d$ if and only if $a = 0$, i.e., J is connected.

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