GAME OPTIONS

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Abstract. I introduce and study new derivative securities which I call game options (or Israeli options to put them in line with American, European, Asian, Russian etc. ones). These are contracts which enable both their buyer and seller to stop them at any time and then the buyer can exercise the right to buy (call option) or to sell (put option) a specified security for certain agreed price. If the contract is terminated by the seller he must pay certain penalty to the buyer. A more general case of game contingent claims is considered. The analysis is based on the theory of optimal stopping games (Dynkin's games). Game options can be sold cheaper than usual American options and their introduction could diversify financial markets.

1. Introduction

A standard (B, S)-securities market consists of a nonrandom (riskless) component B_t, which is described as a savings account (or price of a bond) at time t with an interest r, and of a random (risky) component S_t, which can be described as the price of a stock at time t. Both discrete time $t \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and continuous time $t \in \mathbb{R}_+ = \{t \geq 0\}$ models are considered. A standard American option is a

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contract which enables its buyer to exercise it, i.e. to sell (put option) or to buy (call option) the stock for a specific price K, at any time t which amounts to the gain $(K-S_t)^+$ in the put and $(S_t-K)^+$ in the call option cases. The problem of fair pricing of American options leads to the optimal stopping of certain stochastic processes (see [Be], [Ka1,2], [My], [SKKM1,2] and references there). In this paper I introduce game options in which the seller of an option can cancel the contract at any time t. In this case the buyer's gain is the sum $(K-S_t)^+ + \delta_t$ in the put and $(S_t-K)^+ + \delta_t$ in the call option case where $\delta_t \geq 0$ is certain penalty paid by the seller. The pricing of these options leads to a game version of the optimal stopping problem considered first in the discrete time case by Dynkin [Dy] (a continuous time version was first treated in [Ki]) but for financial applications it is more appropriate to employ another more general set up studied in [Ne], [El], and [Oh1] in the discrete time case and developed in the continuous time case in [Kr], [Fr], [BF1], [Bi] (Markov case) and in [LM] (general case).

The formal set up consists of a probability space $(\Omega, \mathcal{F}, P)$ together with a stochastic process $S_t \geq 0$, $t \in \mathbb{Z}_+$, or $t \in \mathbb{R}_+$ describing the price of a unit of stock, of a family of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$ such that $\mathcal{F}_t$ is generated by all S_u, $0 \leq u \leq t$, and of two right continuous with left limits stochastic payoff processes $X_t \geq Y_t \geq 0$ adapted to the filtration $\{\mathcal{F}_t, t \in \mathbb{Z}_+ \text{ or } t \in \mathbb{R}_+\}$.

A game contingent claim (GCC) is a contract between a seller A and a buyer B which enable A to cancel (terminate) it and B to exercise it at any time t up to a maturity date (horizon) T when the contract is terminated anyway. If B exercises the contract at time t then he gets from A the payment Y_t but if A cancels before B exercises then A should pay to B the sum X_t. If A cancels and B exercises at the same time t then A pays to B the sum Y_t. It turns out (see Remarks 2.2 and 3.2) that if, instead A pays to B in the latter case the amount X_t all results remain the same provided there is no penalty at maturity date. Assuming that clairvoyance is not possible A and B have to use only stopping times with respect to the filtration $\{\mathcal{F}_t\}$ as their cancellation and exercise times. The difference $\delta_t = X_t - Y_t \geq 0$ is interpreted as a penalty which A pays to B for cancellation of the contract.

What is the fair price V^* that B should pay to A for such contract? In accordance with the modern ideology of option pricing based on hedging it is natural to require that V^* should be the minimal capital which enables A to invest it into a skillfully managed self-financing portfolio which will cover his liability to pay to B up to a cancellation stopping time σ no matter what exercise time B chooses. Namely, for any initial capital $Z_0 > V^*$ the seller A should be able to choose a stopping time σ and to manage a self-financing portfolio having a wealth Z_t at time t and being redistributed in discrete times or continuously between the savings account and the stock shares so that Z_t is sufficient for payment to B provided he exercises GCC at the time $t \in [0, \sigma]$. Thus hedging in GCC consists in a choice of both a hedging investment policy and of a cancellation time of the contract. I shall show that this leads to the zero sum optimal stopping game of two players with the payoffs $e^{-rt}X_t$ and $e^{-rt}Y_t$.

If A is not allowed to terminate the contract before the maturity time T then we arrive at an American Contingent Claim. The same can be achieved in the framework of my model if the penalty is chosen large enough, for instance, if $X_t = Y_t + \delta$ and $\delta > \sup\limits_{0 \leq \tau \leq T} EY_\tau$. On the other hand, I could modify the above model so that B is not allowed to terminate the contract until the maturity date T in the spirit of European (game) options. This also can be considered in the framework of my model if I take $Y_t = 0$ for $t < T$ and $Y_t = Y_T > 0$ for $t = T$. Observe, that if the penalty δ_0 is zero then either A or B should terminate the contract at once and the price V^* equals Y_0. It follows from Theorems 2.1 and 3.1 that the price V^* is a continuous increasing function of penalty which varies, thus, from Y_0 to $\sup_{0 \leq \tau \leq T} Y_\tau$.

In the next section I consider the discrete time case where the stock evolution is described by the popular binomial CRR-model introduced in [CRR]. In Section 3 I deal with the continuous time situation where the stock evolution is described by the geometric Brownian motion. The payoff functions X_t and Y_t are supposed to be right continuous and having left limits. In particular, one can take $Y_t = (K - S_t)^+$ or $Y_t = (S_t - K)^+$ and $X_t = Y_t + \delta_t$. These cases are naturally to call put or call game options, respectively, with a penalty process δ_t, $t \geq 0$. Other payoff functions leading to exotic game options can be considered, as well. In Section 4 I discuss the case when Y_t and X_t have the form $\beta^t Y(S_t)$ and $\beta^t X(S_t)$, $\beta \leq 1$.

In this paper I consider only basic problems concerning extension of the option pricing theory to game options and many problems still remain to deal with. First one can consider a multidimensional case of several stocks which can be treated in the same way. Next, the model may include transaction costs and uncertainty (random environments) which are important in real stock exchange trading but, of course, complicate the study. Furthermore, it is important for applications to find convenient formulas and algorithms for computation of prices of game options. On the other hand, one can consider more general options based on nonzero sum stopping games (see [BF2] and [Oh2]) and on stopping games with more than two players (see [YNK]).

Game options are safer for an investment company which issues them, and so it can sell them cheaper than usual American options. In addition, such options contain some elements of games of chance which may be attractive for some investors and could help to diversify financial markets. As a market name for such contracts I suggest to call them Israeli contingent claims (Israeli options) to put them in line with American, Asian, European, Russian etc. ones. All commercial rights on game contingent claims and game options described in this paper are reserved with the author.

2. Discrete time

Let $\Omega = \{1, -1\}^N$ be the space of finite sequences $\omega = (\omega_1, \omega_2, \dots, \omega_N)$; $\omega_i = 1$ or $= -1$ with the product probability $P = \{p, q\}^N$, $q = 1-p$ so that $p(\omega) = p^k q^{N-k}$ where $k = \frac{1}{2}\left(N + \sum_{i=1}^{N} \omega_i\right)$. In this section I consider the CRR-model of financial market which functions at times $n = 0, 1, \dots, N < \infty$ and consists of a savings account B_n with an interest rate r, so that

$$B_n = (1+r)^n B_0, \quad B_0 > 0, \quad r > 0, \tag{2.1}$$

and of a stock whose price at time n equals

$$S_n = S_0 \prod_{k=1}^{n} (1 + \rho_k), \quad S_0 > 0, \tag{2.2}$$

where $\rho_k(\omega) = \frac{1}{2}(a + b + \omega_k(b-a))$, $\omega = (\omega_1, \omega_2, \dots, \omega_N)$.

Thus the "random growth rates" ρ_k, $k = 1, \dots, N$ form a sequence of independent identically distributed random variables on the probability space (Ω, P) taking values a and b with probabilities q and p, respectively. As usual, I assume

$$-1 < a < r < b, \quad 0 < p < 1. \tag{2.3}$$

Introduce also the (finite) σ-algebras $\mathcal{F}_n$, $n = 0, 1, \ldots, N$ where $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n$, $n = 1, 2, \ldots, N$ is generated by the random variables $\{\rho_k, k = 1, \ldots, n\}$.

Recall, (see, for instance, [SKKM1]) that a portfolio strategy π with an initial capital $Z_0^\pi = z > 0$ and a horizon N is a sequence $\pi = (\pi_1, \ldots, \pi_N)$ of pairs $\pi_n = (\beta_n, \gamma_n)$ where β_n, γ_n are $\mathcal{F}_{n-1}$-measurable random variables representing the number of units on the savings account and of the stock, respectively, at time n so that the price of the portfolio at time n is given by the formula

$$Z_n^\pi = \beta_n B_n + \gamma_n S_n. \tag{2.4}$$

A portfolio strategy π is called self-financing if all changes in the portfolio value are due to capital gains or losses but not to withdrawal or infusion of funds. This means that (see [SKKM1]),

$$B_{n-1}(\beta_n - \beta_{n-1}) + S_{n-1}(\gamma_n - \gamma_{n-1}) = 0. \tag{2.5}$$

Denote by $\mathcal{J}_{nN}$ the finite set of stopping times ξ with respect to the filtration $\{\mathcal{F}_n\}_{0 \le n \le N}$ (i.e. $\{\omega : \xi(\omega) \le k\} \in \mathcal{F}_k, k = n, \ldots, N$) with values in $\{n, n+1, n+2, \ldots, N\}$.

A Game Contingent Claim (GCC) is a contract between investors A and B consisting of a maturity date $N < \infty$, of selection of a cancellation time $\sigma \in \mathcal{J}_{0N}$ by A, of selection of an exercise time $\tau \in \mathcal{J}_{0N}$ by B and of $\mathcal{F}_n$-adapted payoff processes $\infty > X_n \ge Y_n \ge 0$, so that A pledges to pay to B at time $\sigma \wedge \tau = \min(\sigma, \tau)$ the sum

$$R(\sigma, \tau) \overset{\text{def}}{=} X_\sigma \mathbb{I}_{\sigma < \tau} + Y_\tau \mathbb{I}_{\tau \le \sigma} \tag{2.6}$$

where $\mathbb{I}_Q = 1$ if an event Q occurs and $= 0$ if not. It turns out (see Remark 2.2 below) that if I replace in (2.6) $\mathbb{I}_{\sigma<\tau}$ by $\mathbb{I}_{\sigma\le\tau}$ and $\mathbb{I}_{\tau\le\sigma}$ by $\mathbb{I}_{\tau<\sigma}$ then the results below remain the same provided $X_N = Y_N$.

A hedge against a GCC with a maturity date N is a pair (σ, π) of a stopping time $\sigma \in \mathcal{J}_{0N}$ and a self-financing portfolio strategy π such that $Z^\pi_{\sigma\wedge n} \ge R(\sigma, n)$ for all $n = 0, 1, \ldots, N$.

The fair price V^* of a GCC is the infimum of $V \ge 0$ such that there exists a hedge (σ, π) against this GCC with $Z_0^\pi = V$.

2.1 Theorem. *Let $P^* = \{p^*, 1-p^*\}^N$ be the probability on the space Ω with $p^* = \frac{r-a}{b-a}$, $N < \infty$ and E^* denotes the corresponding expectation. Then the fair price V^* of the above GCC equals V^*_{0N} which can be obtained from the recursive relations $V^*_{NN} = (1+r)^{-N} Y_N$ and for $n = 0, 1, \dots, N-1$*

$$V^*_{nN} = \min((1+r)^{-n} X_n, \max((1+r)^{-n} Y_n, E^*(V^*_{n+1N}|\mathcal{F}_n))). \tag{2.7}$$

Moreover, for $n = 0, 1, \dots, N$,

$$\begin{aligned} V^*_{nN} &= \min_{\sigma \in \mathcal{J}_{nN}} \max_{\tau \in \mathcal{J}_{nN}} E^*\left((1+r)^{-\sigma\wedge\tau} R(\sigma,\tau)\middle|\mathcal{F}_n\right) \\ &= \max_{\tau \in \mathcal{J}_{nN}} \min_{\sigma \in \mathcal{J}_{nN}} E^*\left((1+r)^{-\sigma\wedge\tau} R(\sigma,\tau)\middle|\mathcal{F}_n\right). \end{aligned} \tag{2.8}$$

Furthermore, for each $n = 0, 1, \dots, N$ the stopping times

$$\begin{aligned} \sigma^*_{nN} &= \min\{k \geq n : (1+r)^{-k} X_k = V^*_{kN} \text{ or } k = N\} \quad \text{and} \\ \tau^*_{nN} &= \min\{k \geq n : (1+r)^{-k} Y_k = V^*_{kN}\} \end{aligned} \tag{2.9}$$

belong to $\mathcal{J}_{nN}$ (since $V^*_{NN} = (1+r)^{-N} Y_N$) and they satisfy

$$E^*\left((1+r)^{-\sigma^*_{nN}\wedge\tau} R(\sigma^*_{nN},\tau)\middle|\mathcal{F}_n\right) \leq V^*_{nN} \leq E^*\left((1+r)^{-\sigma\wedge\tau^*_{nN}} R(\sigma,\tau^*_{nN})\middle|\mathcal{F}_n\right) \tag{2.10}$$

for any $\sigma, \tau \in \mathcal{J}_{n,N}$. Finally, there exists a self-financing portfolio strategy π^* such that (σ^*_{0N}, π^*) is a hedge against this GCC with the initial capital $Z_0^{\pi^*} = V^*_{0N}$ and such strategy is unique up to the time $\sigma^*_{0N} \wedge \tau^*_{0N}$.

Proof. Let $\pi = (\pi_1, ..., \pi_N)$, $\pi_n = (\beta_n, \gamma_n)$ be a self-financing portfolio strategy with $Z_0^\pi = z > 0$ then $M_n^\pi = (1+r)^{-n} Z_n^\pi$ satisfies for $n = 1, \dots, N$, (see [SKKM1])

$$M_n^\pi = z + \sum_{k=1}^n (1+r)^{-k} \gamma_k S_{k-1}(\rho_k - r). \tag{2.11}$$

Since $E^*(\rho_k - r) = 0$ it follows that M_n^π, $n = 0, \dots, N$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{0\leq n\leq N}$ and the probability P^*. Observe that Ω is a finite space with P^* giving positive probability to any point so any "with probability one" statement is true everywhere. Suppose that (σ, π) is a hedge then by the Optional Sampling Theorem (see [Ne], Theorem II-2-13) for any $\tau \in \mathcal{J}_{0N}$,

$$Z_0^\pi = E^*\left((1+r)^{-\sigma\wedge\tau} Z^\pi_{\sigma\wedge\tau}\right) \geq E^*\left((1+r)^{-\sigma\wedge\tau} R(\sigma,\tau)\right). \tag{2.12}$$

Since, by the definition, V^* is the infimum of such initial capitals Z_0^π then

$$V^* \geq \min_{\sigma\in\mathcal{J}_{0N}} \max_{\tau\in\mathcal{J}_{0N}} E^*\Big((1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)\Big). \tag{2.13}$$

In order to prove the inequality in the other direction, for any $\sigma\in\mathcal{J}_{0N}$ set

$$V_n^\sigma = \max_{\tau\in\mathcal{J}_{nN}} E^*(U_\tau^\sigma|\mathcal{F}_n) \tag{2.14}$$

where $U_k^\sigma = (1+r)^{-\sigma\wedge k}R(\sigma,k)$, $k=0,1,\dots,N$. Observe that U_k^σ is $\mathcal{F}_{\sigma\wedge k}$-measurable (and so, $\mathcal{F}_k-$measurable) since both $\sigma\wedge k$ and $R(\sigma,k)=X_{\sigma\wedge k}\mathbb{I}_{\sigma\wedge k<k}+Y_{\sigma\wedge k}\mathbb{I}_{\sigma\wedge k=k}$ are $\mathcal{F}_{\sigma\wedge k}$-measurable. It is easy to check directly and follows from general theorems (see [Ne], Proposition VI-1-2) that $\{V_n^\sigma\}_{0\leq n\leq N}$ is a minimal supermartingale with respect to the filtration $\{\mathcal{F}_n\}_{0\leq n\leq N}$ such that $V_n^\sigma\geq U_n^\sigma$, $n=0,1,\dots,N$. Hence, in view of the Doob decomposition (see [Ne], Proposition VIII-1-1),

$$V_n^\sigma = \tilde{M}_n^\sigma - A_n^\sigma, \quad n=0,1,\dots,N, \quad A_0^\sigma=0 \tag{2.15}$$

where $\{\tilde{M}_n^\sigma\}_{0\leq n\leq N}$ is a martingale with respect to the filtration $\{\mathcal{F}_n\}_{0\leq n\leq N}$ and $\{A_n^\sigma\}_{0\leq n\leq N}$ is a nondecreasing process such that A_n^σ is $\mathcal{F}_{n-1}$-measurable, $n=1,\dots,N$. In fact, one can write explicitly

$$\tilde{M}_n^\sigma = V_0^\sigma + \sum_{k=1}^n (V_k^\sigma - E^*(V_k^\sigma|\mathcal{F}_{k-1})) \text{ and } A_n^\sigma = \sum_{k=1}^n E^*(V_{k-1}^\sigma - V_k^\sigma|\mathcal{F}_{k-1}). \tag{2.16}$$

Each such martingale $\{\tilde{M}_n^\sigma\}_{0\leq n\leq N}$ can be represented in the form (see [SKKM1], §§2,3),

$$\tilde{M}_n^\sigma = \tilde{M}_0^\sigma + \sum_{k=1}^n (1+r)^{-k}\gamma_k^\sigma S_{k-1}(\rho_k - r), \tag{2.17}$$

where γ_k^σ is $\mathcal{F}_{k-1}$-measurable, and so

$$V_n^\sigma = V_0^\sigma + \sum_{k=1}^n (1+r)^{-k}\gamma_k^\sigma S_{k-1}(\rho_k - r) - A_n^\sigma. \tag{2.18}$$

Construct a self-financing portfolio strategy $\pi^\sigma = (\pi_1^\sigma,\dots,\pi_N^\sigma)$, $\pi_n^\sigma = (\beta_n^\sigma,\gamma_n^\sigma)$ with the initial capital V_0^σ setting $Z_0^{\pi^\sigma} = V_0^\sigma$ and inductively for $n=1,\dots,N$,

$$\beta_n^\sigma = \frac{Z_{n-1}^{\pi^\sigma} - \gamma_n^\sigma S_{n-1}}{B_{n-1}} \text{ and } Z_n^{\pi^\sigma} = \beta_n^\sigma B_n + \gamma_n^\sigma S_n. \tag{2.19}$$

Since $\tilde{M}_0^\sigma = V_0^\sigma = Z_0^{\pi^\sigma}$ and $Z_n^{\pi^\sigma} = (1+r)Z_{n-1}^{\pi^\sigma} + \gamma_n^\sigma S_{n-1}(\rho_n - r)$, it follows inductively that

$$(1+r)^{-n}Z_n^{\pi^\sigma} = \tilde{M}_n^\sigma, \text{ and so } \tilde{M}_n^\sigma = M_n^{\pi^\sigma}, \quad n = 0,1,\ldots,N. \tag{2.20}$$

For a stopping time $\eta \le N$ denote by $\mathcal{J}_{\eta N}$ the set of stopping times with values from η to N. It is easy to check (see Lemma VI-1-5 in [Ne]) that (2.14) implies also that

$$V_\eta^\sigma = \max_{\tau \in \mathcal{J}_{\eta N}} E^*(U_\tau^\sigma | \mathcal{F}_\eta).$$

This together with (2.14), (2.15), and (2.20) yield that for $n = 0,1,\ldots,N$,

$$\begin{aligned} Z_{\sigma\wedge n}^{\pi^\sigma} &= (1+r)^{\sigma\wedge n}\tilde{M}_{\sigma\wedge n}^\sigma = (1+r)^{\sigma\wedge n}(V_{\sigma\wedge n}^\sigma + A_{\sigma\wedge n}^\sigma) \\ &\ge (1+r)^{\sigma\wedge n}V_{\sigma\wedge n}^\sigma = (1+r)^{\sigma\wedge n}\max_{\tau\in\mathcal{J}_{\sigma\wedge n N}} E^*(U_\tau^\sigma|\mathcal{F}_{\sigma\wedge n}) \\ &\ge (1+r)^{\sigma\wedge n}E^*(U_n^\sigma|\mathcal{F}_{\sigma\wedge n}) = (1+r)^{\sigma\wedge n}U_n^\sigma = R(\sigma,n), \end{aligned} \tag{2.21}$$

i.e. (σ,π^σ) is a hedge. Therefore, I showed that for any $\sigma \in \mathcal{J}_{0N}$ there exists a self-financing portfolio strategy π^σ with the initial capital V_0^σ given by (2.14) such that (σ,π^σ) is a hedge.

Since the results on optimal stopping games are usually formulated for infinite horizon N, I define first the σ-algebras $\mathcal{F}_n$ and the processes X_n, Y_n for all n setting $\mathcal{F}_n = \mathcal{F}_N$ and $X_n = Y_n = 0$ provided $n > N$. Set $\mathcal{J}_{0\infty} = \bigcup_{N=1}^\infty \mathcal{J}_{0N}$ which is the set of all finite stopping times with respect to the filtration $\{\mathcal{F}\}_{0\le n<\infty}$. Consider a game between two players **I** and **II** with the payoff processes $(1+r)^{-n}X_n$ and $(1+r)^{-n}Y_n$ so that if **I** chooses a stopping time σ and **II** chooses a stopping time τ then **I** pays to **II** the amount

$$(1+r)^{-\sigma}X_\sigma\mathbb{I}_{\sigma<\tau} + (1+r)^{-\tau}Y_\tau\mathbb{I}_{\tau\le\sigma} = (1+r)^{-\sigma\wedge\tau}R(\sigma,\tau). \tag{2..22}$$

Of course, **I** tries to minimize his payment to **II** and, on the other hand, **II** tries to maximize it.

It follows from [Oh1] that any such game starting at time $n \ge 0$ has a value

$$\begin{aligned} \tilde{V}_n^* &\stackrel{\text{def}}{=} \inf_{\sigma\in\mathcal{J}_{n\infty}}\sup_{\tau\in\mathcal{J}_{n\infty}} E^*\left((1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)\middle|\mathcal{F}_n\right) \\ &= \sup_{\tau\in\mathcal{J}_{n\infty}}\inf_{\sigma\in\mathcal{J}_{n\infty}} E^*\left((1+r)^{-\sigma\wedge\tau}R(\sigma,\tau)\middle|\mathcal{F}_n\right) \end{aligned} \tag{2.23}$$

and the sequence $\tilde{V}_n^*$, $n = 0, 1, \dots$ satisfies the recursive relations (2.7) (with $\tilde{V}_n^*$ in place of V_{nN}^*). Since I set $X_n = Y_n = 0$ for all $n > N$ and $X_N = Y_N \geq 0$ then $\tilde{V}_n^* = 0$ for all $n > N$ and $\tilde{V}_N^* = (1+r)^{-N} Y_N$. Hence, $\tilde{V}_n^* = V_{nN}^*$ for all $n = 0, 1, \dots, N$, and so (2.8) holds true. These together with [Oh1] yields also that the stopping times σ_{nN}^* and τ_{nN}^* given by (2.9), which, clearly, belong to $\mathcal{J}_{nN}$, satisfy (2.10).

Now take $\sigma^* = \sigma_{0N}^* \in \mathcal{J}_{0N}$ and construct the corresponding self-financing portfolio strategy $\pi^* = \pi^{\sigma^*}$, as above, which yields the hedge (σ^*, π^*) with the initial capital

$$V_0^{\sigma^*} = \max_{\tau \in \mathcal{J}_{0N}} E^*((1+r)^{-\sigma^* \wedge \tau} R(\sigma, \tau)) = V_{0N}^* \tag{2.24}$$

where the last equality in (2.24) follows from (2.10). Since the fair price V^* of the GCC is the minimal initial capital for which hedging is possible I conclude that $V^* \leq V_{0N}^*$. On the other hand, by (2.8), which is proved already, the right hand side of (2.13) is equal to V_{0N}^* which gives $V^* \geq V_{0N}^*$, and so, in fact, the equality $V^* = V_{0N}^*$, holds true.

It remains to obtain the uniqueness. Set $\tau^* = \tau_{0N}^*$. Then by (2.20), (2.21), and (2.24),

$$\begin{aligned} M_0^{\pi^{\sigma^*}} &= V_0^{\sigma^*} = E^*((1+r)^{-\sigma^* \wedge \tau^*} R(\sigma^*, \tau^*)) \\ &\leq E^*((1+r)^{-\sigma^* \wedge \tau^*} Z_{\sigma^* \wedge \tau^*}^{\pi^{\sigma^*}}) = E^* M_{\sigma^* \wedge \tau^*}^{\pi^{\sigma^*}} = M_0^{\pi^{\sigma^*}} \end{aligned} \tag{2.25}$$

since $M_n^{\pi^{\sigma^*}}$ is a martingale. This together with (2.21) implies that $Z_{\sigma^* \wedge \tau^*}^{\pi^{\sigma^*}} = R(\sigma^*, \tau^*)$. Let now $\pi = (\pi_1, ..., \pi_N)$, $\pi_n = (\beta_n, \gamma_n)$ be another self-financing portfolio strategy with $Z_0^\pi = V^* = V_0^{\sigma^*}$. Then according to the first part of the proof of Theorem 2.1 $M_n^\pi = (1+r)^{-n} Z_n^\pi$ is a martingale and I again have (2.25) with M_n^π and Z_n^π in place of $M_n^{\pi^{\sigma^*}}$ and $Z_n^{\pi^{\sigma^*}}$, respectively. Hence $Z_{\sigma^* \wedge \tau^*}^\pi = R(\sigma^*, \tau^*) = Z_{\sigma^* \wedge \tau^*}^{\pi^{\sigma^*}}$, and so $M_{\sigma^* \wedge \tau^*}^\pi = M_{\sigma^* \wedge \tau^*}^{\pi^{\sigma^*}}$. Since both M_n^π and $M_n^{\pi^{\sigma^*}}$ are martingales it follows that

$$M_n^\pi = M_n^{\pi^{\sigma^*}} \text{ and } Z_n^\pi = Z_n^{\pi^{\sigma^*}} \text{ for all } n \leq \sigma^* \wedge \tau^*. \tag{2.26}$$

Since the representation (2.11) is unique, $S_n > 0$ and $\rho_n \neq r$ for all n then $\gamma_n = \gamma_n^{\sigma^*}$ for all $n \leq \sigma^* \wedge \tau^*$. This together with (2.4), (2.19), and (2.26) yield that $\beta_n = \beta_n^{\sigma^*}$ for all $n \leq \sigma^* \wedge \tau^*$, completing the proof of Theorem 2.1. $\square$

2.2 Remark. Consider a bit more general set up where the payoff function $R(\sigma,\tau)$ given by (2.6) is replaced by

$$\hat{R}(\sigma,\tau)=X_\sigma \mathbb{I}_{\sigma<\tau}+Y_\tau \mathbb{I}_{\tau<\sigma}+W_\sigma \mathbb{I}_{\sigma=\tau} \tag{2.27}$$

where W_n is $\mathcal{F}_n$-measurable, $Y_n \le W_n \le X_n$, $n=0,1,\ldots,N$ and $W_N=Y_N$. It follows from [Oh1] that if $\hat{V}^*_{nN}$, $n=0,1,\ldots,N$ are given by (2.8) with $\hat{R}$ in place of R then they will still satisfy the recursive relations (2.7) and since $\hat{V}^*_{NN}=(1+r)^{-N}Y_N$ I conclude that $\hat{V}^*_{nN}=V^*_{nN}$ for all $n=0,1,\ldots,N$. By [Oh1] the stopping times $\hat{\sigma}^*_{nN}=\min\{k\ge n:(1+r)^{-k}X_k=\hat{V}^*_k \text{ or } k=N\}=\sigma^*_{nN}$ and $\hat{\tau}^*_{nN}=\min\{k\ge n:(1+r)^{-k}Y_k=\hat{V}^*_k\}=\tau^*_{nN}$ satisfy (2.10) with R replaced by $\hat{R}$. Next, I define again (σ,π) to be a hedge if $Z^\pi_{\sigma\wedge n}\ge \hat{R}(\sigma,n)$. Now the same proof as in Theorem 2.1 shows that the fair price $\hat{V}^*$ of the GCC with the payoff function (2.27) equals $\hat{V}^*_{0N}$, and so by above, $\hat{V}^*=V^*$. This is rather interesting since $\hat{R}(\sigma,n)\ge R(\sigma,n)$ and the strict inequality is also possible when $\sigma=n$. So, sometimes (take, for instance, $N=1$, $\sigma=0$, $X_0>Y_0$) a hedging portfolio requires less initial capital when the payoff function is R than when it is $\hat{R}$. Still, the fair prices of the corresponding GCC's are the same is both cases.

2.3 Remark. Theorem 2.1 can be extended to the infinite horizon case $N=\infty$ with the same proof relying on results from [Oh1] provided that with P^*-probability one

$$\lim_{n\to\infty} e^{-rn}X_n=\lim_{n\to\infty} e^{-rn}Y_n=0. \tag{2.28}$$

In this case Ω becomes the space of sequences and all statements above will be true now with probability one with respect to the probability $P^*=\{p^*,1-p^*\}^\infty$. Since P^* is singular with respect to any other probability $\{p,1-p\}^\infty$ with $p\ne p^*$ and there is no reason why the stock fluctuations should be connected with this particular probability p^*, it seems that this extension of Theorem 2.1 to the case $N=\infty$ may have financial applications only as an approximation of a very large N case. Still, observe that (2.28) always holds true in the game put option case $Y_n=(K-S_n)^+$ provided the penalty δ_n does not grow too fast so that with probability one $\lim_{n\to\infty} e^{-nr}\delta_n=0$. Moreover, then (2.28) holds also true for the game call option case $Y_n=(S_n-K)^+$ since by Jensen's inequality $\log(1+r)>p^*\log(1+b)+(1-p^*)\log(1+a)$, and so by the law of large numbers with P^*-probability one

$$\lim_{n\to\infty} n^{-1}\log((1+r)^{-n}S_n) = \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n}\log(1+\rho_k) - \log(1+r) < 0.$$

2.4 Remark. Similarly to [SKKM1] Theorem 2.1 can be generalized to the case when consumption or infusion of capital is also possible. In this case the price of a portfolio $\pi = (\pi_1, \dots, \pi_N)$, $\pi_n = (\beta_n, \gamma_n)$ after new stock prices at time n were announced is $Z_n^\pi = \beta_n B_n + \gamma_n S_n$ but immediately before that $Z_{n-1}^\pi = \beta_n B_{n-1} + \gamma_n S_{n-1} + g_n$ (see [SKKM1]) where g_n is $\mathcal{F}_{n-1}$-measurable. An easy modification of the above proof gives the following formula for the fair price V^* of the corresponding GCC

$$V^* = \min_{\sigma\in\mathcal{J}_{0N}} \max_{\tau\in\mathcal{J}_{0N}} E^*\left((1+r)^{-\sigma\wedge\tau}R(\sigma,\tau) + \sum_{k=1}^{\sigma\wedge\tau}(1+r)^{-(k-1)}g_k\right) \tag{2.29}$$

and this minmax equals maxmin of the same expression. Other, correspondingly modified, assertions of Theorem 2.1 remain true in this case, as well.

2.5 Remark. It is easy also to generalize the above set up allowing dependence of r, a and b on time, i.e. assuming that $\rho_k(\omega) = \frac{1}{2}(a_k + b_k + \omega_k(b_k - a_k))$ and $B_n = B_0 \prod_{k=1}^{n}(1+r_k)$ where $r_k, a_k, b_k;\ k = 1, \dots, N$ are nonrandom sequences satisfying $-1 < a_k < r_k < b_k$. Setting $p_k^* = \frac{r_k - a_k}{b_k - a_k}$ and $P^* = \prod_{k=1}^{N}\{p_k^*, 1 - p_k^*\}$, an easy modification of the proof leads to the corresponding statements of Theorem 2.1 with $(1+r)^{-n}$ replaced by $\prod_{k=1}^{n}(1+r_k)^{-1}$.

2.6. Remark. The formula (2.7) gives a recursive way of computation of the fair price $V^* = V_{0N}^*$ of the GCC. Still, this requires to deal with conditional expectations and functions V_N^* which needs a lot of computations and computer memory since a $\mathcal{F}_n$-measurable function can take on 2^n values.

3. Continuous time

I adopt here a popular model of a financial market consisting of a savings account with the time evolution

$$B_t = B_0 e^{rt}, \quad B_0 > 0, \quad r \geq 0 \tag{3.1}$$

and of a stock whose price S_t is the geometric Brownian motion

$$S_t = S_0 \exp\left((\mu - \frac{\kappa^2}{2})t + \sigma W_t \right) \tag{3.2}$$

where $\{W_t\}_{t\geq 0}$ is the standard one dimensional Wiener process starting at zero and $\kappa > 0$, μ are some numbers. In the differential form

$$dB_t = rB_t dt \text{ and } dS_t = S_t(\mu dt + \kappa dW_t), \tag{3.3}$$

where the second equation is the Ito stochastic differential equation. Again, r is interpreted as the interest rate on the savings account and the term $\mu dt + \kappa dW_t$ is responsible for random "risky" fluctuations of the stock price where σ and μ are called volatility and appreciation rate, respectively. Let $(\Omega, \mathcal{F}, P)$ be the probability space corresponding to the Wiener process, i.e. Ω is the space of continuous functions $\omega = (\omega_t)_{t\geq 0}$, $\omega_0 = 0$, $\mathcal{F}$ is the Borel σ-field generated by cylinder sets, and P is the Wiener measure on $(\Omega, \mathcal{F})$. Then $W_t(\omega) = \omega_t$, $t \geq 0$. Denote by $\mathcal{F}_t$ the complete σ-algebra generated by $\{W_u, u \leq t\}$. Then

$$S_t = S_0 e^{\mu t} Q_t \tag{3.4}$$

where $Q_t = e^{\kappa W_t - (\kappa^2/2)t}$, $t \geq 0$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Recall, (see, for instance, [SKKM2]) that a self-financing portfolio strategy π with an initial capital $Z_0^\pi = z > 0$ and a horizon $T < \infty$ is a process $\pi = (\pi_t)_{0\leq t\leq T}$ of pairs $\pi_t = (\beta_t, \gamma_t)$ with $\mathcal{F}_t$-measurable β_t and γ_t, $t \geq 0$ such that

$$\int_0^T e^{rt} |\beta_t| dt < \infty \text{ and } \int_0^T (\gamma_t S_t)^2 dt < \infty \tag{3.5}$$

and the portfolio price Z_t^π at time $t \in [0, T]$ is given by

$$Z_t^\pi = z + \int_0^t \beta_u dB_u + \int_0^t \gamma_u dS_u \tag{3.6}$$

where B_u and S_u are the same as in (3.1)–(3.3). I call π $\{\mathcal{F}_t\}_{0\leq t\leq T}$-progressively measurable if the processes β_t and γ_t are progressively measurable with respect to this filtration (see [KS], Section 1.1).

Denote by $\mathcal{J}_{tT}$ the set of stopping times $\mathcal{J}$ with respect to the filtration $\{\mathcal{F}_u\}_{0\leq u\leq T}$ with values in $[t, T]$. A game contingent claim (GCC) is a contract between investors A and B consisting of a maturity date $T < \infty$, of selection of a cancellation time $\sigma \in \mathcal{J}_{0T}$ by A, of selection of an exercise time $\tau \in \mathcal{J}_{0T}$ by B and of $\mathcal{F}_t$-adapted right continuous with left limits (RCLL) payoff processes $\infty > X_t \geq Y_t \geq 0$, so that A pledges to pay to B at time $\sigma \wedge \tau = \min(\sigma, \tau)$ the sum

$$R(\sigma, \tau) = X_\sigma \mathbb{I}_{\sigma < \tau} + Y_\tau \mathbb{I}_{\tau \leq \sigma}. \tag{3.7}$$

As in the discrete time case, the exchange between $\leq$ and $<$ in (3.7) does not influence the final result (see Remark 3.2) provided $X_T = Y_T$.

A hedge against such GCC with a maturity date T is a pair (σ, π) of a stopping time $\sigma \in \mathcal{J}_{0T}$ and a $\{\mathcal{F}_t\}_{0 \leq t \leq T}$-progressively measurable self-financing portfolio strategy π such that $Z^\pi_{\sigma \wedge t} \geq R(\sigma, t)$ with probability one for each $t \in [0, T]$. Again the fair price V^* of a GCC is the infimum of $V \geq 0$ such that there exists a hedge (σ, π) against this GCC with $Z^\pi_0 = V$.

Set

$$W^{\mu - r}_t = W_t + \frac{\mu - r}{\kappa} t \tag{3.8}$$

then the process $\{W^{\mu-r}_t\}_{t \geq 0}$ is the standard Wiener process with respect to the probability $P^{\mu - r}$ whose restrictions $P^{\mu-r}_t$ to $\mathcal{F}_t$ are equivalent to restrictions P_t of P to $\mathcal{F}_t$ and

$$\frac{dP^{\mu-r}_t}{dP_t}(\omega) = \exp\left\{ -\frac{\mu - r}{\kappa} W_t(\omega) - \frac{1}{2}\left(\frac{\mu - r}{\kappa}\right)^2 t \right\} \tag{3.9}$$

(see, for instance, [KS], Section 3.5).

By (3.3) and (3.8),

$$dS_t = S_t(r\,dt + \kappa dW^{\mu-r}_t) \tag{3.10}$$

which together with (3.6) gives

$$dZ^\pi_t = rZ^\pi_t dt + \kappa \gamma_t S_t dW^{\mu-r}_t \tag{3.11}$$

for any self-financing portfolio strategy. It is important to observe that both stochastic differential equations (3.10) and (3.11) do not depend explicitly on μ which is usually not known.

Assume that

$$E^{\mu - r} \sup_{0 \leq t \leq T} (e^{-rt} X_t) < \infty, \tag{3.12}$$

where $E^{\mu-r}$ is the expectation corresponding to the probability $P^{\mu-r}$.

3.1. Theorem. *The fair price V^* of the above GCC equals V^*_{0T} where $\{V^*_{tT}\}_{0\le t\le T}$ is the right continuous process such that with $P^{\mu-r}$-probability one,*

$$V^*_{tT}=\operatorname*{essinf}_{\sigma\in\mathcal{J}_{tT}}\operatorname*{esssup}_{\tau\in\mathcal{J}_{tT}}E^{\mu-r}\left(e^{-r\sigma\wedge\tau}R(\sigma,\tau)\Big|\mathcal{F}_t\right)\tag{3.13}$$
$$=\operatorname*{esssup}_{\sigma\mathcal{J}_{tT}}\operatorname*{essinf}_{\tau\in\mathcal{J}_{tT}}E^{\mu-r}\left(e^{-r\sigma\wedge\tau}R(\sigma,\tau)\Big|\mathcal{F}_t\right).$$

Moreover, for each $t\in[0,T]$ and $\varepsilon>0$ the stopping times

$$\sigma^\varepsilon_{tT}=\inf\{u\ge t: e^{-ru}X_u\le V^*_{uT}+\varepsilon\ \text{ or } u=T\}\ \text{ and}\tag{3.14}$$
$$\tau^\varepsilon_{tT}=\inf\{u\ge t: e^{-ru}Y_u\ge V_{uT}-\varepsilon\}$$

belong to $\mathcal{J}_{tT}$ (since $V^*_{TT}=e^{-rT}Y_T$) and with $P^{\mu-r}$-probability one they satisfy

$$E^{\mu-r}\left(e^{-r\sigma^\varepsilon_{tT}\wedge\tau}R(\sigma^\varepsilon_{tT},\tau)\Big|\mathcal{F}_t\right)-\varepsilon\le V^*_{tT}\le E^{\mu-r}\left(e^{-r\sigma\wedge\tau^\varepsilon_{tT}}R(\sigma,\tau^\varepsilon_{tT})\Big|\mathcal{F}_t\right)+\varepsilon\tag{3.15}$$

for any $\sigma,\tau\in\mathcal{J}_{t,T}$. Furthermore, for each $\varepsilon>0$ there exists a self-financing portfolio strategy π^ε such that $(\sigma^\varepsilon_{0T},\pi^\varepsilon)$ is a hedge against this GCC with the initial capital $Z_0^{\pi^\varepsilon}\le V^*_{0T}+\varepsilon$. Suppose, in addition, that the processes Y_t and $-X_t$ are upper semicontinuous from the left, i.e. in our circumstances they may have only positive jumps at points of discontinuity. Then the stopping times $\sigma^*_{tT}=\lim_{\varepsilon\downarrow 0}\sigma^\varepsilon_{tT}$ and $\tau^*_{tT}=\lim_{\varepsilon\downarrow 0}\tau^\varepsilon_{tT}$, which are well defined since σ^ε_{tT} and τ^ε_{tT} are monotone in ε, with $P^{\mu-r}$-probability one satisfy

$$E^{\mu-r}\left(e^{-r\sigma^*_{tT}\wedge\tau}R(\sigma^*_{tT},\tau)\Big|\mathcal{F}_t\right)\le V^*_{tT}\le E^{\mu-r}\left(e^{-r\sigma\wedge\tau^*_{tT}}R(\sigma,\tau^*_{tT})\Big|\mathcal{F}_t\right)\tag{3.16}$$

for any $\sigma,\tau\in\mathcal{J}_{t,T}$. Moreover, $\sigma^*_{tT}\wedge\tau^*_{tT}=\sigma^0_{tT}\wedge\tau^0_{tT}$, where σ^0_{tT} and τ^0_{tT} are defined by (3.14) with $\varepsilon=0$, and so with $P^{\mu-r}$-probability one,

$$V^*_{tT}=E^{\mu-r}\left(e^{-r\sigma^0_{tT}\wedge\tau^0_{tT}}R(\sigma^0_{tT},\tau^0_{tT})\Big|\mathcal{F}_t\right).\tag{3.17}$$

Furthermore, there exists a self-financing portfolio strategy π^* such that (σ^*_{0T},π^*) is a hedge against this GCC with the initial capital $Z_0^{\pi^*}=V^*_{0T}$ and with $P^{\mu-r}$–probability one such strategy is unique in this case up to the time $\sigma^0_{tT}\wedge\tau^0_{tT}$.

Proof. Let π be a self-financing portfolio strategy with $Z_0^\pi=z>0$ then in view of (3.11), $M_t^\pi=e^{-rt}Z_t^\pi$ satisfies

$$M_t^\pi=M_0^\pi+\kappa\int_0^t e^{-ru}\gamma_uS_u\,dW_u^{\mu-r},\quad t\le T<\infty,\tag{3.18}$$

and so $\{M_t^\pi\}_{0\le t\le T}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$. Suppose that (σ,π) is a hedge then by the Optimal Sampling Theorem (see [KS], Section 1.3) for any $\tau\in\mathcal{J}_{0T}$,

$$Z_0^\pi = E^{\mu-r}(e^{-r\sigma\wedge\tau}Z^\pi_{\sigma\wedge\tau}) \ge E^{\mu-r}(e^{-r\sigma\wedge\tau}R(\sigma,\tau)). \tag{3.19}$$

If follows that

$$V^*\ge \inf_{\sigma\in\mathcal{J}_{0T}}\sup_{\tau\in\mathcal{J}_{0T}} E^{\mu-r}(e^{-r\sigma\wedge\tau}R(\sigma,\tau)). \tag{3.20}$$

In order to prove the inequality in the other direction for any $\sigma\in\mathcal{J}_{0T}$ set

$$V_t^\sigma = \operatorname*{esssup}_{\tau\in\mathcal{J}_{tT}} E^{\mu-r}(U_\tau^\sigma|\mathcal{F}_t) \tag{3.21}$$

where $U_t^\sigma = e^{-r\sigma\wedge t}R(\sigma,t)$ and I observe that U_t^σ is $\mathcal{F}_{\sigma\wedge t}-$measurable. In the same way as in Lemma from §6 of [SKKM2] I conclude that $\{V_t^\sigma\}_{0\le t\le T}$ is a supermartingale. Still, since U_t^σ is not, in general, right continuous I cannot use this lemma directly to conclude that $\{V_t^\sigma\}_{0\le t\le T}$ has a RCLL modification. It is known that, in order to establish the latter assertion it suffices to show that the function

$$\varphi_t = E^{\mu-r}V_t^\sigma = \sup_{\tau\in\mathcal{J}_{tT}} E^{\mu-r}U_\tau^\sigma, \qquad t\in[0,T]$$

in right continuous (see [KS], Section 1.3). Since V_t^σ is a supermartingale it follows that $\lim_{s\downarrow t}\varphi_s\le\varphi_t$. For the opposite inequality, I can still employ the argument from the lemma cited above relying just on the right lower semicontinuity of U_t^σ,

$$\lim_{s\downarrow t} U_s^\sigma = e^{-r\sigma\wedge t}(X_\sigma\mathbb{I}_{\sigma\le t} + Y_t\mathbb{I}_{t<\sigma}) \ge U_t^\sigma$$

which follows since $X_u\ge Y_u$, $u\in[0,T]$ and both X_u and Y_u are right continuous.

Thus I can and do assume that $\{V_t^\sigma\}_{0\le t\le T}$ is a RCLL supermartingale. Moreover, in view of (3.12) the family $\{V_\tau^\sigma\}_{\tau\in\mathcal{J}_{0T}}$ is uniformly integrable with respect to $P^{\mu-r}$ (see Lemma 5.5 in [Ka1]). Hence by the Doob-Meyer decomposition theorem (see [KS], Section 1.4) I can write

$$V_t^\sigma = \tilde{M}_t^\sigma - A_t^\sigma, \quad t\in[0,T], \quad A_0^\sigma = 0 \tag{3.22}$$

where $\{\tilde{M}_t^\sigma\}_{0\le t\le T}$ is a RCLL martingale with respect to the filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ and $\{A_t\}_{0\le t\le T}$ is a RCLL nondecreasing process such that A_t is $\mathcal{F}_t$-measurable. In view of the martingale representation theorem (see [KS], Section 3.4) there exists a $\{\mathcal{F}_t\}_{0\le t\le T}$ progressively measurable process $\{\gamma_t^\sigma\}_{0\le t\le T}$ such that

$$\tilde{M}_t^\sigma = \tilde{M}_0^\sigma + \kappa \int_0^t e^{-ru}\gamma_u^\sigma S_u dW_u^{\mu-r}, \tag{3.23}$$

and so for all $t\in[0,T]$,

$$V_t^\sigma = V_0^\sigma + \kappa \int_0^t e^{-ru}\gamma_u^\sigma S_u dW_u^{\mu-r} - A_t^\sigma. \tag{3.24}$$

Set $\beta_t^\sigma = (\tilde{M}_t^\sigma - e^{-rt}\gamma_t^\sigma S_t)B_0^{-1}$, $t\in[0,T]$ and

$$Z_t^{\pi^\sigma} = e^{rt}\tilde{M}_t^\sigma = \beta_t^\sigma B_t + \gamma_t^\sigma S_t, \tag{3.25}$$

where $\pi^\sigma = (\pi_t^\sigma)_{0\le t\le T}$ and $\pi_t^\sigma = (\beta_t^\sigma, \gamma_t^\sigma)$ is the $\{\mathcal{F}_t\}_{0\le t\le T}$-progressively measurable portfolio strategy with the initial capital $Z_0^\pi = V_0^\sigma$, then by (3.10), (3.23), and (3.25),

$$dZ_t^\pi = Z_t^\pi dt + \kappa\gamma_t^\sigma S_t dW_t^{\mu-r} = \beta_t^\sigma dB_t + \gamma_t^\sigma dS_t, \tag{3.26}$$

i.e. (3.6) holds true, and so π^σ is self-financing. Choose a sequence of stopping times $\eta_n \downarrow \sigma$ as $n\uparrow\infty$ taking on only finitely many values then it is easy to check as in Lemma VI-1-5 from [Ne] that (3.21) implies also

$$V_{\eta_n\wedge t}^\sigma = \operatorname*{esssup}_{\tau\in\mathcal{J}_{\eta_n\wedge tT}} E^{\mu-r}(U_\tau^\sigma|\mathcal{F}_{\eta_n\wedge t}),$$

where, again, for a stopping time η I denote by $\mathcal{J}_{\eta T}$ the set of stopping times with values between η and T. This together with (3.21) and (3.22) yield that for any $t\in[0,T]$ with $P^{\mu-r}$-probability one,

$$\begin{aligned} Z_{\eta_n\wedge t}^{\pi^\sigma} &= e^{r\eta_n\wedge t}\tilde{M}_{\eta_n\wedge t}^\sigma = e^{r\eta_n\wedge t}(V_{\eta_n\wedge t}^\sigma + A_{\eta_n\wedge t}^\sigma) \ge e^{r\eta_n\wedge t}V_{\eta_n\wedge t}^\sigma \\ &= e^{r\eta_n\wedge t} \operatorname*{esssup}_{\tau\in\mathcal{J}_{\eta_n\wedge tT}} E^{\mu-r}(U_\tau^\sigma|\mathcal{F}_{\eta_n\wedge t}) \ge e^{r\eta_n\wedge t}E^{\mu-r}(U_t^\sigma|\mathcal{F}_{\eta_n\wedge t}) \\ &= e^{r\eta_n\wedge t}U_t^\sigma = e^{r(\eta_n\wedge t-\sigma\wedge t)}R(\sigma,t) \ge R(\sigma,t) \end{aligned} \tag{3.27}$$

since U_t^σ is $\mathcal{F}_{\sigma\wedge\, t}$–measurable and $\mathcal{F}_{\sigma\wedge\, t}\subset\mathcal{F}_{\eta_n\wedge\, t}$. The right continuity of $Z_s^{\pi^\sigma} = e^{rs}\tilde{M}_s^\sigma$ in s enables me to pass to the limit in (3.27) as $n\uparrow\infty$ yielding $Z_{\sigma\wedge t}^{\pi^\sigma}\ge R(\sigma,t)$, and so (σ,π^σ) is a hedge.

Next, I proceed similarly to the discrete time case. Extend the payoff processes X_t and Y_t beyond T by $X_t = X_T$ and $Y_t = Y_T$ for all $t > T$ so that the right continuity, existence of left limits, and upper simicontinuity from the left are preserved. Denote by $\mathcal{J}_{0\infty}$ the set of stopping times with values in $[0, \infty]$ with respect to the Wiener process filtration $\{\mathcal{F}_t\}_{t\geq 0}$ which is defined for all $t \geq 0$ anyway. Consider a game between two players **I** and **II** with the payoff processes $e^{-rt}X_t$ and $e^{-rt}Y_t$ so that if **I** chooses a stopping time σ and **II** chooses a stopping time τ then **I** pays to **II** the sum

$$e^{-r\sigma}X_\sigma \mathbb{I}_{\sigma<\tau} + e^{-r\tau}Y_\tau \mathbb{I}_{\tau\leq\sigma} = e^{-r\sigma\wedge\tau}R(\sigma,\tau). \tag{3.28}$$

Next, I intend to apply to this game the results from [LM] which were stated there for bounded payoff processes but they remain true for $X_t \geq Y_t \geq 0$ satisfying (3.12). It follows from [LM] that

$$\begin{aligned}\tilde{V}_t^* &\overset{\text{def}}{=} \operatorname*{essinf}_{\sigma\in\mathcal{J}_{t\infty}} \operatorname*{esssup}_{\tau\in\mathcal{J}_{t\infty}} E^{\mu-r}\left(e^{-r\sigma\wedge\tau}R(\sigma,\tau)\middle|\mathcal{F}_t\right)\\ &= \operatorname*{esssup}_{\tau\in\mathcal{J}_{t\infty}} \operatorname*{essinf}_{\sigma\in\mathcal{J}_{t\infty}} E^{\mu-r}\left(e^{-r\sigma\wedge\tau}R(\sigma,\tau)\middle|\mathcal{F}_t\right)\end{aligned} \tag{3.29}$$

and for each $\varepsilon > 0$ the stopping times

$$\tilde{\sigma}_t^\varepsilon = \inf\{u \geq t : e^{-ru}X_u \leq V_u^* + \varepsilon\} \text{ and } \tilde{\tau}_t^\varepsilon = \inf\{u \geq t : e^{-ru}Y_u \geq \tilde{V}_u^* - \varepsilon\} \tag{3.30}$$

satisfy

$$E^{\mu-r}\left(e^{-r\tilde{\sigma}_t^\varepsilon\wedge\tau}R(\tilde{\sigma}_t^\varepsilon,\tau)\middle|\mathcal{F}_t\right) - \varepsilon \leq \tilde{V}_t^* \leq E^{\mu-r}\left(e^{-r\sigma\wedge\tilde{\tau}_t^\varepsilon}R(\sigma,\tilde{\tau}_t^\varepsilon)\middle|\mathcal{F}_t\right) + \varepsilon \tag{3.31}$$

for any $\sigma, \tau \in \mathcal{J}_{t\infty}$.

¿From the definition of X_t and Y_t beyond T it follows easily that $\tilde{V}_t^* = V_{tT}^*$ for all $t \in [0,T]$ since the player **II** may only decrease his gain if he stops the game later than T. Then by (3.14) and (3.30), $\tau_{tT}^\varepsilon = \tilde{\tau}_t^\varepsilon \in \mathcal{J}_{tT}$ for all $t \in [0,T]$. But then $\sigma_{tT}^\varepsilon = \tilde{\sigma}_t^\varepsilon \wedge T$ also satisfies (3.31), and so (3.13) and (3.15) follow from (3.29) and (3.31).

Now take $\sigma^\varepsilon = \sigma_{0T}^\varepsilon \in \mathcal{J}_{0T}$ and construct the corresponding self-financing portfolio strategy $\pi^\varepsilon = \pi^{\sigma^\varepsilon}$, as above, which yields the hedge $(\sigma^\varepsilon, \pi^\varepsilon)$ with the initial capital

$$V_0^{\sigma^\varepsilon} = \sup_{\tau\in\mathcal{J}_{0T}} E^{\mu-r}(e^{-r\sigma^\varepsilon\wedge\tau}R(\sigma,\tau)) \leq V_{0T}^* + \varepsilon \tag{3.32}$$

where the last inequality in (3.32) follows from (3.15). Since the fair price V^* of the GCC is the infimum of initial capitals for which hedging is possible it follows that $V^* \leq V^*_{0T} + \varepsilon$. This being true for any positive ε yields $V^* \leq V^*_{0T}$. On the other hand, by (3.13) and (3.20), $V^* \geq V^*_{0T}$, i.e. in fact, $V^* = V_{0T}$, as required.

Next, suppose that $-X_t$ and Y_t are left upper semicontinuous. Since σ^ε_t and τ^ε_t may only grow when ε decreases then

$$\sigma^*_{tT} \stackrel{\text{def}}{=} \lim_{\varepsilon \downarrow 0} \sigma^\varepsilon_{tT} \in \mathcal{J}_{tT} \text{ and } \tau^*_{tT} \stackrel{\text{def}}{=} \lim_{\varepsilon \downarrow 0} \tau^\varepsilon_{tT} \in \mathcal{J}_{tT}.$$

Letting $\varepsilon \downarrow 0$ in (3.15) one arrives at (3.16) and $\sigma^0_{tT} \wedge \tau^0_{tT} = \sigma^*_{tT} \wedge \tau^*_{tT}$ follows easily too (see Theorem 15 in [LM]).

Let now $\sigma^* = \sigma^*_{0T}$ and $\pi^* = \pi^{\sigma^*}$ so that (σ^*, π^*) is the corresponding hedge. Then by (3.16) it follows similarly to (3.31) that $V_0^{\sigma^*} \leq V^*_{0T}$. Since I already proved that $V^* = V^*_{0T}$ and by the definition $V_0^{\sigma^*} \geq V^*$, it follows that $V_0^{\sigma^*} = V^*$.

Finally, I obtain the uniqueness assertion in the same way as in the discrete time case. Namely, as in (2.25) I derive using (3.27) that if $\pi = (\pi_t)_{0 \leq t \leq T}$, $\pi_t = (\beta_t, \gamma_t)$ is another self-financing portfolio strategy with $Z_0^\pi = V^* = V_0^{\sigma^*}$ then $Z_t^\pi = Z_t^{\pi^{\sigma^*}}$ for all $t \leq \sigma^0_{tT} \wedge \tau^0_{tT}$. Now the pair β_t and γ_t is uniquely defined by (3.6) and (3.11), completing the proof of Theorem 3.1. □

3.2 Remark. Suppose that the payoff function $R(\sigma, \tau)$ given by (3.7) is replaced by

$$\hat{R}(\sigma, \tau) = X_\sigma \mathbb{I}_{\sigma \leq \tau} + Y_\tau \mathbb{I}_{\tau < \sigma}. \tag{3.33}$$

Assume also that $X_T = Y_T$. Then by Lemma 5 from [LM], (3.13) will remain true when R is replaced by $\hat{R}$ with the same process V^*_{tT}. Then (3.14) and (3.15) will hold true, as well, with $\hat{R}$ in place of R. As in the discrete time case, it follows that the fair price $\hat{V}^*$ of the GCC with the payoff function (3.33) equals $V^* = V^*_{0T}$ with V^*_{0T} given by (3.13) if hedging pairs (σ, π) are required to satisfy $Z^\pi_{\sigma \wedge t} \geq \hat{R}(\sigma, t)$ with $P^{\mu - r}$-probability one for each $t \in [0, T]$. The proof is the same as in Theorem 3.1 (and even a bit easier since $\hat{R}(\sigma, t)$ is right continuous).

3.3 Remark. Theorem 3.1 can be extend to the infinite horizon case $T = \infty$ (perpetual claims) similarly to [Ka1] but in order to apply [LM] to the corresponding optimal stopping game one needs

$$\lim_{t \to \infty} e^{-rt} X_t = \lim_{t \to \infty} e^{-rt} Y_t = 0. \tag{3.34}$$

If $r > 0$ then (3.34) always holds true in the game put option case $Y_t = (K - S_t)^+$ provided $\delta_t = X_t - Y_t$ satisfies $\lim_{t\to\infty} e^{-rt}\delta_t = 0$ with $P^{\mu-r}$-probability one. Since $e^{-rt}S_t = S_0 \exp(-\frac{\kappa^2}{2}t + \kappa W_t^{\mu-r})$ then also in the game call option case $Y_t = (S_t - K)^+$ one has $\lim_{t\to\infty} e^{-rt}Y_t = 0$ $P^{\mu-r}$-almost surely.

3.4 Remark. Similarly to [SKKM2] and [Ka1] Theorem 3.1 can be generalized to the case when consumption is also possible. If $\{g_t\}_{0\le t\le T}$ is a $\{\mathcal{F}_t\}_{0\le t\le T}$ adapted consumption process with $E^{\mu-r}(\int_0^T |g_t|dt) < \infty$ then the fair price V^* of the GCC in this case will be given by

$$V^* = \inf_{\sigma\in\mathcal{J}_{0T}} \sup_{\tau\in\mathcal{J}_{0T}} E^{\mu-r}(e^{-r\sigma\wedge\tau}R(\sigma,\tau) + \int_0^{\sigma\wedge\tau} e^{-ru}g_u du). \tag{3.35}$$

3.5. Remark. It is not difficult to generalize the set up considering $r = r_t$, $\mu = \mu_t$, and $\kappa = \kappa_t$ in (3.3) depending on t (but deterministic). In the usual American contingent claim case this was done in [Ka1]. In fact, one can deal with r_t, μ_t, and σ_t being stochastic processes independent of the driving Wiener process W_t, i.e. to consider a (B, S)-market in a random dynamical environment.

3.6 Remark. In the general continuous time case effective computations of the fair price V^* are hardly possible. One possibility is to discretize time, i.e. consider only stopping times σ and τ taking on, say, only values $k2^{-n}T$, $k = 0, 1, \dots, 2^n$ and to obtain the corresponding values $V_{k2^n}^{*(n)}$ of games starting at $k2^{-n}T$. As in Theorem 2.1 this values satisfy the recursive relations

$$V_{k2^n}^{*(n)} = \min(X_{k2^{-n}T}, \max(Y_{k2^{-n}T}E^{\mu-r}(V_{(k+1)2^n}|\mathcal{F}_{k2^{-n}T})))$$

$V_{2^n 2^n}^{*(n)} = Y_T$, and so, in principle, one can compute $V_{02^n}^{*(n)}$ which is the price of the GCC when A and B can cancel and exercise only at times $k2^{-n}T$, $k = 0, 1, \dots, 2^n$. If X_t and Y_t, $t \in [0, T]$ are continuous then one can see that $V^* = V_{0T}^* = \lim_{n\to\infty} V_{02^n}^{*(n)}$.

4. Markov case

Taking into account that the stock fluctuation processes $\{S_n\}_{n\ge 0}$ and $\{S_t\}_{t\ge 0}$ given by (2.2) and (3.2), respectively, are Markov processes one can employ other methods of computations of the fair price V^* of a GCC if X_n and Y_n or X_t and Y_t depend only on S_n or S_t, correspondingly.

I start with the discrete time case. Let $X_n = \beta^n X(S_n)$ and $Y_n = \beta^n Y(S_n)$, $n = 0, 1, 2, \dots$, $0 < \beta \le 1$ for some Borel functions X and Y on $(0, \infty)$. Particular cases of this situation are $Y_n = \beta^n (K - S_n)^+$ and $Y_n = \beta^n (S_n - K)^+$, which are discounted put and call game options, provided that the penalty process have the form $\delta_n = \beta^n \delta(B_n)$ of just $\delta_n = \beta^n \delta$ for some constant $\delta > 0$. In this case it follows from (2.7), (2.8), and the Markov property that there exist Borel functions $v_k = v_k(x)$, $k = 0, 1, \dots$ on $(0, \infty)$ such that $V^*_{nN} = (\alpha\beta)^{-n} v_{N-n}(S_n)$, where $\alpha = (1+r)^{-1}$, and for $n = 0, 1, \dots$

$$v_{n+1}(x) = U v_n(x), \quad v_0(x) = Y(x), \tag{4.1}$$

where the operator U acts by the formula

$$Ug(x) = \min(X(x), \max(Y(x), \alpha\beta E^*_x g(S_1))) \tag{4.2}$$

and E^*_x is the expectation for P^* provided $S_0 = x$. This provides recursive formulas for computation of the fair price $V^*(x) = v_N(x)$ of the GCC with the horizon $N < \infty$ given S_0.

By (4.1) and (4.2) the sequence v_n, $n = 0, 1 \dots$ is monotone nondecreasing, $Y \le v_n \le X$ and the limit

$$v(x) = \lim_{n\to\infty} v_n(x) = \lim_{n\to\infty} U^n Y(x) \tag{4.3}$$

satisfies the equation $Uv = v$. Moreover, it follows from (2.8) and the equality $v_N(x) = V^*_{0N}$, $S_0 = x$ that v equals the value of the infinite game between the players **I** and **II** described in Section 2 when only finite stopping times are allowed (cf. [EL] and [Oh1]), and so the fair price $V^* = V^*(x)$ of the GCC with the infinite horizon $N = \infty$ given $S_0 = x$ equals $v(x)$. By (2.9) the corresponding optimal (or rational) stopping times (saddle point) for the GCC with $N < \infty$ are given by

$$\begin{aligned} &\sigma^*_{nN} = \min\{0 \le n \le N : X(S_n) = v_{N-n}(S_n) \text{ or } n = N\} \text{ and} \\ &\tau^*_{nN} = \min\{0 \le n \le N : Y(S_n) = v_{N-n}(S_n)\} \end{aligned} \tag{4.4}$$

and for the $N = \infty$ case

$$\sigma^* = \min\{n \ge 0 : X(S_n) = v(S_n)\} \text{ and } \tau^* = \min\{n \ge 0 : Y(S_n) = v(S_n)\} \tag{4.5}$$

provided that with P^*-probability one σ^* and τ^* are finite.

Consider, next, the case $\beta = 1$ and $Y(x) = (x-K)^+$. Since

$$E^*(S_{n+1}|\mathcal{F}_n) = S_n(1+p^*b+(1-p^*)a) = (1+r)S_n = \alpha^{-1}S_n \tag{4.6}$$

then $\alpha^n S_n$ is a martingale, and so $\alpha^n Y(S_n) = (\alpha^n S_n - \alpha^n K)^+$ is a submartingale (with respect to the probability P^*). Thus, in view of the Optional Sampling Theorem, for the game call option case with a horizon $N < \infty$ the fair price is given by

$$V^* = \min_{\sigma \in \mathcal{J}_{0N}} E^*((\alpha^\sigma(S_\sigma - K)^+ + \delta_\sigma)\mathbb{I}_{\sigma<N} + \alpha^N(S_N-K)^+\mathbb{I}_{\sigma=N}). \tag{4.7}$$

This corresponds to the well known fact that American call options with an expiration date $N < \infty$ coincide with the corresponding European call options. In the game call option case it follows that the buyer B should exercise as late as possible, i.e. at the expiration date N if $N < \infty$. On the other hand, the seller A should choose an optimal cancellation stopping time which minimizes in (4.7) and it is easy to give examples when this stopping time is nontrivial (i.e. nonconstant).

For each $m = 0, 1 \ldots$, set

$$C_m^A = \{x : v_m(x) < X(x)\}, \quad C_m^B = \{x : v_m(x) > Y(x)\}, \quad C_m = C_m^A \cap C_m^B \tag{4.8}$$

and

$$D_m^A = \{x : v_m(x) = X(x)\}, \quad D_m^B = \{x : v_m(x) = Y(x)\}, \quad D_m = D_m^A \cap D_m^B \tag{4.9}$$

so that $C_m \cup D_m = \mathbb{R}$ since $Y \le v_m \le X$.

Since the sequence v_n is nondecreasing then assuming that $X(x) > Y(x)$ for all x one has

$$C_n^A \subset C_{n-1}^A \subset \ldots \subset C_0^A = \mathbb{R}, \quad \emptyset = C_0^B \subset \ldots \subset C_{n-1}^B \subset C_n^B, \tag{4.10}$$

$$D_n^A \supset D_{n-1}^A \supset \ldots \supset D_0^A = \emptyset, \quad \mathbb{R} = D_0^B \supset \ldots \supset D_{n-1}^B \supset D_n^B, \tag{4.11}$$

$n = 0, 1, \ldots$. By (4.4),

$$\begin{aligned} \sigma_{nN}^* &= \min\{0 \le n \le N : S_n \in D_{N-n}^A \text{ or } n = N\} \text{ and} \\ \tau_{nN}^* &= \min\{0 \le n \le N : S_n \in D_{N-n}^B\}, \end{aligned} \tag{4.12}$$

so that A or B should stop when the stock price S_n gets to the domain D^A_{N-n} or D^B_{N-n}, respectively.

More specific results can be obtained for a particular case considered in [SKKM1] where $1+a=\lambda^{-1}$ and $1+b=\lambda$ for some $\lambda>1$, and so $S_n(\omega)=S_0\lambda^{\omega_1+\omega_2+\cdots+\omega_n}$, $\omega=(\omega_1,\omega_2,\dots,\omega_N)$, $\omega_i=\pm 1$. Thus, in this case one has to study geometric random walk on the set $E=\{\lambda^k, k=0,\pm 1,\pm 2,\dots\}$. Let $Y(x)=(x-1)^+$ and $X(x)=(x-1)^++\delta$ for some $\delta>0$. If $\beta=1$ then, as explained above, B should not exercise before the expiration time N and if $N=\infty$ he does not have a finite optimal stopping time though by [Oh1] ε-optimal stopping times exist for any $\varepsilon>0$. The optimal cancellation time of A can be evaluated in this case explicitly. An analysis of this and $\beta<1$ cases can carried out in the same way as in §6 of [SKKM1]. This analysis shows that

$$D^B_n=\{\lambda^k : k<0 \text{ and } k\geq k^B_n\} \text{ and } D^A_n=\{\lambda^k : k^A_n\leq k\leq K^A_n\}\cup D^0_n$$

where k^A_n, k^B_n, K^A_n depend on parameters $\alpha,\beta,\lambda,\delta$; D^0_n is either empty or contains one or both points 1 and λ, and $1<k^A_n\leq K^A_n\leq k^B_n$. Now $\bigcap_n D^B_n=D^B=[\lambda^{k^B},\infty)$ where $k^B=\lim_{n\to\infty}k^B_n$ and $\bigcup_n D^A_n=D^A=[\lambda^{k^A},\lambda^{K^A}]\cup D_0$ where $k^A=\lim_{n\to\infty}k^A_n$, $K^A=\lim_{n\to\infty}K^A_n$, and D_0 may be either empty or may contain one or both points 1 and λ. On $C=\mathbb{R}\setminus(D^A\cup D^B)$ the function v given by (4.3) satisfies the equation

$$v(x)=\alpha\beta E^*_x v(S_1) \tag{4.13}$$

which can be solved similarly to [SKKM1] and the boundary points of the above domains can be determined via the smooth fit principle.

In the continuous time case the stock price fluctuations S_t form the Markov diffusion process solving the stochastic differential equation (3.10). Suppose that $X_t=e^{-\beta t}X(S_t)$, $Y_t=e^{-\beta t}Y(S_t)$, $\beta>0$ and $T=\infty$. In particular one can take $Y_t=e^{-\beta t}(K-S_t)^+$ or $Y_t=e^{-\beta t}(S_t-K)^+$ which are discounted put or call game options, respectively, with some penalty process $\delta_t=e^{-\beta t}\delta(S_t)$ or even $\delta_t=e^{-\beta t}\delta$ for some constant $\delta>0$. Though (3.10) is a degenerate at zero equation but considering instead the stochastic differential equation for $L_t=\log S_t$ one can deal with nondegenerate diffusions. Then the fair price $V^*=V^*(x)$ of the GCC given $S_0=x$ being the value of the optimal stopping game with the payoff processes $e^{-(r+\beta)t}X(S_t)$ and $e^{-(r+\beta)t}Y(S_t)$ can be described via certain variational inequalities (see [Fr] and [BF1]). Some computational algorithms for such variational inequalities were justified in [JLL]. Another characterization of the value of this game was given in [Bi] so that $V^* = v - \tilde{v}$ where v and $\tilde{v}$ are minimal $(r+\beta)$-excessive majorants of $\tilde{v} + Y$ and $v - X$, respectively. A free boundary approach to even a more general problem for nonzero sum games was given in [BF2].

Observe that discretizing time one obtains recursive relations of the form (2.7) and the value of the corresponding game can be obtained similarly to (2.30). Then letting the discretization step to zero one obtains the value of the continuous time game, and so of the corresponding GCC. Namely, let

$$U_t g(x) = \min(X(x), \max(Y(x), e^{-(r+\beta)t} E_x^{\mu-r} g(S_t))) \tag{4.14}$$

then one can show that

$$V^*(x) = \lim_{n\to\infty} \lim_{k\to\infty} U^k_{2^{-n}} Y(x). \tag{4.15}$$

Again $Y \le V^* \le X$ and there are 4 domains $D^A = \{x : X(x) = V^*(x)\}$, $D^B = \{x : Y(x) = V^*(x)\}$, $C^A = \{x : X(x) > V^*(x)\}$, and $C^A = \{x : Y(x) < V^*(x)\}$. On $C = C^A \cap C^B$ the function $V^*(x)$ satisfies the equation

$$\frac{1}{2}\kappa^2 x^2 \frac{d^2V^*(x)}{dx^2} + rx\frac{dV^*(x)}{dx} = (r+\beta)V^*(x) \tag{4.16}$$

with the free boundary conditions

$$V^*|_{\partial D^A} = X \text{ and } V^*|_{\partial D^B} = Y. \tag{4.17}$$

A more specific analysis of this problem for the case $Y(x) = (x-K)^+$ or $Y(x) = (K-x)^+$ and $X(x) = Y(x) + \delta$ can be carried out, in principle, along the lines of §8 from [SKKM2] though it is more complicated here. Still, when $\beta = 0$ one conclusion follows easily in the game call option case. Namely, (3.11) implies that $e^{-rt}S_t$ is a martingale, and so in the game call option case $e^{-rt}Y_t = (e^{-rt}S_t - e^{-rt}K)^+$ is a submartingale. Thus, by the Optional Sampling Theorem the fair price V^* for the game call option with a finite horizon $T < \infty$ is given by

$$V^* = \inf_{\sigma\in\mathcal{J}_{0T}} E^{\mu-r}((e^{-r\sigma}(S_\sigma - K)^+ + \delta_\sigma)\mathbb{I}_{\sigma<T} + e^{-rT}(S_T-K)^+\mathbb{I}_{\sigma=T}) \tag{4.18}$$

and the buyer should not exercise before the expiration date though the seller has to find an optimal cancelation time.

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