HOMOTOPY COENDS*

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Preprint No. 3
1997/98

Notations:
$\mathcal{S}$ = simplicial sets $\mathcal{S}_*$ pointed simplicial set.
$\mathcal{J}, \mathcal{D}, \mathcal{C}$, ?? denote (small) categories
$\mathcal{S}^D$ denote the functor category: Namely the category of functors $\mathcal{D} \longrightarrow \mathcal{S}$ or $\mathcal{D}$-diagram of spaces.

*Partially supported by BSF grant # ??
Partially supported by the Edmund Landau Center for Research in Mathematical Analysis and Related Areas,sponsored by the Minerva Foundation (Germany).

Typeset by $\mathcal{AMS}$-TEX

§1. Definitions and basic properties

The principal use of homotopy end and homotopy coend constructions is the presentation of a space X or of a diagram of spaces X or made out of well defined pieces, each of which contains certain partial information of the final object X.

In this particular reconstruction or decomposition of X each elementary piece is given by two factors that "complements each other in some way" or more generally each piece depends functionally on two variables $(c,d) \in \mathcal{C}^{op} \times \mathcal{C}$ where $\mathcal{C}$ is the indexing category that organizes these piece into a whole structure. A typical example is the reconstruction [Elmendorf], up to homotopy equivalence of a G-space (i.e. a space with an action by a group G) out of the collection of fixed points subspace X^H for all subgroups $H \subseteq G$. Here the two factors mentioned above will be:

- $X^H = \{x \in X | hx = x \text{ for all } h \in H\}$ taken, as we vary H, as a diagram over the opposite category $\mathcal{O}_G^{op}$ to the orbit category $\mathcal{O}_G = \{G/H | H \subseteq G\}$ of G.

•• The orbits of G i.e. the G-spaces G/H for all $H \subseteq G$, taken here as a diagram over the orbit category itself.

Together they form a diagram $X : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \{\text{Spaces}\}$ with $X(G/H, G/K) = X^H times G/K$ a formula that makes sense in view of the identity $X^H = \mathrm{map}_G(G/H, X)$. Thus "the basic building blocks" for a G space X are the orbits G/K of G and each one is taken with "multiplicity" $\mathrm{map}_G(G/K, X)$ i.e. multiply G/K by "the number of times that it appears in the G space X". If fact we will have a formula $\underset{\mathcal{O}_G}{\mathrm{hocoend}}\{G/K \times X^H\} \simeq X$, which is a homotopy version of Elmendorf's functor.

On the other hand suppose we want to "decompose and build" a homotopically meaningful space of mappings of the map $A \longrightarrow B$ to $X \longrightarrow Y$ i.e. map $\begin{pmatrix} \begin{smallmatrix} A & X \\ \downarrow, & \downarrow \\ B & Y \end{smallmatrix} \end{pmatrix}$. If $A \longrightarrow B$ is not a cofibration or $X \longrightarrow Y$ not a fibration then the space of strictly commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow \\ B & \longrightarrow & Y \end{array}$$

is not "homotopically meaningful" in as much as we can replace $A \longrightarrow B$ by an equivalent $A' \longrightarrow B'$ with

$$\begin{tikzcd} A' \arrow[r, "\simeq"] \arrow[d] & B \arrow[d] \\ B' \arrow[r, "\simeq"] & B \end{tikzcd}$$

a commutative diagram where the horizontal maps are homotopy equivalence, and the space of strict maps as above $map(A' \longrightarrow B', X \longrightarrow Y)$ will not have the same homotopy type as $\mathrm{map}(A \longrightarrow B, X \longrightarrow Y)$. To get a meaningful mapping space we may of course replace $A \longrightarrow B$ by $A^c \longrightarrow B^c$ an equivalent cofibration and $X \longrightarrow Y$ by $X^f \longrightarrow Y^f$ an equivalent fibration and then take the strict mapping space. We can achieve the same thing by constructing the desirable mapping space directly from the pieces $\mathrm{map}(A, X), \mathrm{map}(B, Y), \mathrm{map}(A, Y)$ by means of a homotopy **end**. In the case at hand this amounts simply to taking the homotopy pull back of the diagram

$$\begin{tikzcd} & \mathrm{map}(A, X) \arrow[d] \\ \mathrm{map}(B, Y) \arrow[r] & \mathrm{map}(A, Y) \end{tikzcd}$$

Example. Let $f : A \longrightarrow B$ be $\{0, 1\} \longrightarrow \{*\}$ and $g : X \longrightarrow Y$ be any map then $\mathrm{map}(A \longrightarrow B, X \longrightarrow Y)$ is the strict pull back of $X \xrightarrow{g} Y \xleftarrow{g} X$ whose homotopy type can, of course, change upon changing g by a homotopy. While for the desired "homotopy invariant" complex function we replace f by the cofibration $\{0, 1\} \hookrightarrow [0, 1]$, i.e. the inclusion map $\partial I \subseteq I$ where I denote the unit interval, and then $\mathrm{map}(A' \longrightarrow B', X \longrightarrow Y)$ is the homology pull back of $X \longrightarrow Y \longleftarrow X$.

In the present framework this desired homotopy meaningful complex function appears as a homotopy coend of the diagram $\mathrm{Hom}(V_\alpha, W_\beta)$ over the small category $\mathcal{J}^{op} \times \mathcal{J}$ where $\mathcal{J} = \{\cdot \longrightarrow \cdot\}$ is the "two object one arrow" category, when V_α, W_α are the spaces A, B, X, Y.

Notice that for two diagrams X, Y over $\mathcal{C}$ the spaces $\mathrm{Hom}(X_\alpha, Y_\beta)$ form a diagram over $\mathcal{C}^{op} \times \mathcal{C}$.

Homotopy coends and homotopy ends

Definition. (Compare [], []) Let $X : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{H}$ be a functor to a category $\mathcal{H}$ in which the notion of homotopy colimit is well defined. The homotopy coend of X denoted by $\underset{\mathcal{C}}{\mathrm{hocoend}}\, X$ is defined as the homotopy colimit of the simplicial object

BX defined as follows: In dimension $k \geq 0$ we have the sum in $\mathcal{C}$

$$\coprod_{c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \xrightarrow{f_k} c_k} X(c_0, c_k) = B_k X$$

indexed by the k-simplices $c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \xrightarrow{f_k} c_k$ of the nerve of $\mathcal{C}$. The face and degeneracy maps in BX are the usual ones: if $(a; f_n, \ldots, f_1)$ is an element of $X(\alpha, \beta)$ where $f_n \circ f_{n-1} \circ \cdots \circ f_1 : \alpha \longrightarrow \beta$ we have

$$d^i x(f_n, \ldots, f_1) = \begin{cases} (X(id, f_n)a; f_1, \ldots, f_{n-1} & i = 0 \\ (a; f_1, f_i \circ f_{i+1}, \ldots, f_n) & 1 \leq i \leq n-1 \\ (X(f_1, id)(a; f_2, \ldots, f_n) & i = n \end{cases}$$

$$\alpha = c_0 \xrightarrow{f_1} c_1 \cdots \xrightarrow{f_k} c_k = \beta$$

$$X(c_0, c_{k-1}) \xleftarrow{X(id, f_n)} X(c_0, c_k) \xleftarrow{X(f_1, id)} X(c_1, c_k)$$

Definition. Let $X : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{H}$ be a functor to a category $\mathcal{H}$ in which homotopy limits are defined. The homotopy end denoted by hoend X is the total space (or homotopy limit) of the cosimplicial object $\mathcal{L}^{\bullet} X$ defined as follows. In codimension $k \geq 0$ we have the product in $\mathcal{H}$

$$\prod_{c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \xrightarrow{f_k} c_k} X(c_0, c_k) = \mathcal{L}^k X$$

indexed by the k-simplices of the reverse of $\mathcal{C}$. The coface and codegeneracies are defined as usual. Namely $k+1$ maps d_i for $0 \leq i \leq k$ into a typical factor $X(c_0, c_k)$ of $\mathcal{L}_{\bullet}^{k+1} X$ associated to $c_0 \xrightarrow{f_1} c_1 \longrightarrow \cdots \xrightarrow{f_k} c_k$ is the projections followed induced maps

?????

where for $0 < i < k$ we compose and otherwise use the opposite variance of X : $\mathcal{C}^{op} \times \mathcal{C}$ to get elements in c_1 or c_{k-1} for d_0 and d_k.

Special cases: The most common occurrence of ends and coends arise when one starts with two similarly shaped diagrams and take certain functors of two variables such as products, joins, smash products or mapping spaces. These are examples of homotopy bar and cobar constructions compare [??]. For example a typical homotopy coend construction occurs as a 'homotopy tensor product' of two functors of opposite variance as we saw above in the theory of G-spaces. Namely one starts with two functors: $X_1 : \mathcal{C}^{op} \longrightarrow \mathcal{S}$ and $X_2 : \mathcal{C}^{op} \longrightarrow \mathcal{S}$ are given and so we may define $X(c,d) = X_1(c) \times X_2(d)$ and thus $X : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{S}$ is a functor as needed. Alternatively another $Y_1 : \mathcal{C} \longrightarrow \mathcal{S}$ is given and one defines $U : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{S}$ by $U(c,d) = \mathrm{map}(X_1(c), Y_1(d))$.

definition. The homotopy tensor product or homotopy coend of X_1 and X_2 is the coend of $X_1 \times X_2$ and is denoted by $X_1 \otimes_c alc^h X_2$. Dually the homotopy end of the functor $map_C(X_2, Y_2) : calc^{op} \times \mathcal{C} \longrightarrow \mathcal{S}$ will be denoted by $\mathrm{map}^h_{\mathcal{C}}(X_2, Y_2)$

Example. $F^c \otimes^h_{\mathcal{C}} \bar{F}^d \simeq *$ when $F^c : \mathcal{C} \longrightarrow S$, $\bar{F}^d : \mathcal{C}^{op} \longrightarrow \mathcal{S}$. In this case F^d, F^c are free so th homotopy coend is equivalent to the coend $F^c \otimes_{\mathcal{C}} \bar{F}^d$.

Basic properties.

Following [Mac p. 221 (3)(4)] the basic properties of hoend and hocoend are

(1) homotopy invariance. Any weak homotopy equivalence of diagrams yields a weak equivalence of their homotopy coends and homotopy ends.

(2) Adjunction. $\mathrm{map}(\underset{\mathcal{C}}{\mathrm{hocoend}}\, X, W) = \underset{\mathcal{C}^{op}}{\mathrm{hocoend}}\, map(X, W)$ where W is in $\mathcal{S}$, $X \in \mathcal{S}^{\mathcal{C}^{op} \times \mathcal{C}}$.

(3) Commutation. $\mathrm{map}(W, \underset{\mathcal{C}}{\mathrm{hoend}}\, X) = \underset{\mathcal{C}}{\mathrm{hoend}}\, \mathrm{map}(W, X)$.

Proof. All these follow formally from corresponding properties of holim and hocolim.

In the situation above for example property (2) has the form

$$\mathrm{map}(A \otimes^h_{\mathcal{C}} B, W) = \mathrm{map}^h_{\mathcal{C}}(A, \mathrm{map}(B, W))$$

,

for any diagrams of opposite variance A and B and a fixed space W.

Yoneda's lemma, strict and homotopy versions.

Here we recapitulate a well-known generalization of the two natural isomorphisms $R \otimes_R M \simeq \mathrm{Hom}_R(R, M) \simeq M$ for any (left) module M over a ring with unit R. Notice that in these equations the symbol R denotes three different objects: a ring, a right R-module in $R \otimes_R M$ and a left R-module in $\mathrm{Hom}_R(R, M)$. Given a category $\mathcal{D}$ we denote by $I_{\mathcal{D}}$ the $\mathcal{D}^{op} \times \mathcal{D}$ set with $\mathcal{D}(d, d') = \mathrm{Hom}(d, d') = \mathcal{D}(d, d')$. Then we have:

Yoneda's lemma. *For any functor $G : \mathcal{D} \longrightarrow \mathcal{E}ns$ there are natural isomorphisms and homotopy equivalences:*

(1) $I_{\mathcal{D}} \otimes^h_{\mathcal{D}} G \simeq G \stackrel{\simeq}{=} G \otimes_{\mathcal{D}} I_{\mathcal{D}}$.

(2) $\mathrm{Hom}^h_{\mathcal{D}}(I_{\mathcal{D}}, G) \simeq G \stackrel{\simeq}{=} \mathrm{Hom}_{\mathcal{D}}(I_{\mathcal{D}}, G)$.

or alternatively for any $A \in \mathcal{D}$ one has isomorphism of sets: for any $A \in \mathcal{D}$

(1) $\mathrm{Hom}_{\mathcal{D}}(-, A) \otimes^h_{\mathcal{D}} G \simeq G(A) \stackrel{\simeq}{=} G \otimes_{\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(A, -)$.

(2) $(\mathrm{Hom}_{\mathcal{D}}(A, -), G) \simeq G(A)$.

Proof. The isomorphism $\mathrm{Hom}_{\mathcal{D}}(-, A) \otimes_{\mathcal{D}} G \xrightarrow{\simeq} G(A)$ is gotten as the adjoint of the map

$$\mathrm{Hom}_{\mathcal{D}}(x, A) \longrightarrow \mathrm{Hom}(G(x), G(A))$$

which is gotten by applying the functor G to a given arrow $x \longrightarrow A$ in $\mathcal{D}$ whereas the ?? is defined as "evaluation on $id_A \in Hom_{\mathcal{D}}(A, A)$. Namely to each morphism of functors $\mathrm{Hom}_{\mathcal{D}}(A, -) \longrightarrow G$ we associate its value in $G(A)$ on the element $id_A \in \mathrm{Hom}_{\mathcal{D}}(A, A)$.

Similarly the map $\bar{G} \otimes_{\mathcal{D}} \mathrm{Hom}_{\mathcal{D}}(A, -) \longrightarrow \bar{G}(A)$ for a contravariant $\bar{G} : \mathcal{D}^{op} \longrightarrow \mathcal{E}ns$ is the adjoint of natural transformation $\bar{G} \longrightarrow \mathrm{map}(\mathrm{Hom}_{\mathcal{D}}(A, -), \bar{G}(A))$ between two the contravariant functors. This transformation sends $g_x \in \bar{G}(x)$ to the map that assigns to an arrow $a : A \longrightarrow X$ the value $\bar{G}(a)(g_x) \in \bar{G}(A)$ where $\bar{G}(a) : G(x) \longrightarrow G(A)$ is the map induced by a.

Proof for the homotopy coend see section (-) bellow..

The homotopy version is proved similarly by a diagonal argument.

§2. Homotopy coend for free diagrams

Orbits, free orbits..

Often it is helpful to calculate homotopy (co)ends via free resolutions. In fact we shall see that if X (or Y)is a dimensionwise free diagram (see below) then $X \otimes^h_{\mathcal{D}} Y$ is homotopy equivalent to the strict coend $X \otimes_{\mathcal{D}} Y$ and similarly for $\mathrm{map}^h_{\mathcal{D}}(X, W)$. We recall from [] [] that a $\mathcal{D}$-space $e : \mathcal{D} \to \mathcal{S}$ is called a $\mathcal{D}$-**orbit** if $\mathrm{colim}_{\mathcal{D}}\, e \simeq *$. Since we are working with simplicial diagrams over a discrete indexing category $\mathcal{D}$ we will be concerned with $\mathcal{D}$-orbit that are $\mathcal{D}$-sets, namely $e : \mathcal{D} \to \mathcal{E}ns$. With every object $d \in \mathcal{D}$ one can associate a canonical orbit set namely $F^d : \mathcal{D} \longrightarrow \mathcal{E}ns$ with $F^d(d') = \mathrm{Hom}_{\mathcal{D}}(d, d') \equiv \mathcal{D}(d, d')$. Since every member x of $F^d(d')$ is the image of $id \in F^d(d)$ under x itself we see that $\operatorname*{colim}_{\mathcal{D}} F^d = *$ so that F^d is a $\mathcal{D}$-orbit set. This special orbits are called the *free orbits* of $\mathcal{D}$ it clearly satisfies Yoneda's lemma $\mathrm{map}_{\mathcal{D}}(F^d, X) = X(d)$. We have defined a functor $F : \mathcal{D}^{op} \longrightarrow \mathcal{S}^{\mathcal{D}}$ that associate to each $d \in \mathcal{D}^{op}$ the $\mathcal{D}$-set F^d. Since for each $d \longrightarrow d'$ there is a functor $F^{d'} \longrightarrow F^d$ the association $d \longmapsto F^d$ is contravariant. In other words we can rewrite F as a functor $F_{\mathcal{D}} \equiv F : \mathcal{D}^{op} \times \mathcal{D} \longrightarrow \mathcal{E}ns$ by adjunction. Notice that F is a representable functor to which one might apply Yoneda's lemma, notice also that with the previous notations $F = I_{\mathcal{D}}$.

Free $\mathcal{D}$-sets.

Now given an arbitrary $\mathcal{D}$-set $S : \mathcal{D} \longrightarrow \mathcal{E}ns$ than clearly one can decompose S into its $\mathcal{D}-$orbits via $S = \coprod_{\alpha} S_\alpha$ where S_α is a $\mathcal{D}$-orbit, $\alpha \in \mathrm{colim}\, S$. If all these orbits are free $\mathcal{D}$-orbits then we say that S is a free $\mathcal{D}$-set. An orbit E is free if there is $d \in \mathrm{obj}\mathcal{D}$ and an element $i \in E(d)$ which freely generates E namely the correspondence $F^d(d') \longrightarrow E(d')$ that carries $f : d \longrightarrow d' \in F^d(d')$ to $\mathcal{D}(f)(i) \in E(d')$ is an isomorphism $F^d \xrightarrow{\cong} E$ of $\mathcal{D}$-sets. More precisely a free $\mathcal{D}$-set is a $\mathcal{D}$-set $\Phi : \mathcal{D} \longrightarrow \mathcal{E}ns$ together with a generating (subset) natural transformation: $G \xrightarrow{g} \Phi$ where $G : \mathcal{D}^{dis} \longrightarrow \mathcal{E}ns$ where $\mathcal{D}^{dis}$ the discrete category of $\mathcal{D}$ and g is a map of functors such that g has the obvious universal property:

$$\begin{tikzcd}
G \arrow[r, "g"'] \arrow[d, "\forall m"'] & \Phi \\
S \arrow[ur, "\exists ! \tilde{m}"'] &
\end{tikzcd}$$

Free $\mathcal{D}$-spaces.

Definition. A $\mathcal{D}$-space $X : \mathcal{D} \longrightarrow \mathcal{S}$ is called dimensionwise free if for all $n \geq 0$ the $\mathcal{D}$-set $X_n : \mathcal{D} \to \mathcal{E}ns$ is free.

Definition. A $\mathcal{D}$-space $X : \mathcal{D} \longrightarrow \mathcal{S}$ is called free $\mathcal{D}$-space X has a graded $\mathcal{D}^{dis}$-set of generators which is closed under degeneracies.

The following is not hard to prove using induction on dimension.

Theorem. *For any $\mathcal{D}$-space X there exists a free $\mathcal{D}$-space resolution $r : X^{free} \longrightarrow X$ where X^{free} is a free $\mathcal{D}$-space and $r(d)\ X^{free}(d) \longrightarrow X(d)$ is a weak homotopy equivalence for all $d \in \mathrm{obj}\mathcal{D}$.*

Example. If X is a space with a discrete group action $G \times X \longrightarrow X$ we can take $X^{free} \longrightarrow X$ to be the projection $EG \times X \longrightarrow X$. If $A \longrightarrow B$ is the diagram from example above then its free resolutions the diagram $\partial I \hookrightarrow I$.

One way to define homotopy colimit for diagrams of simplicial sets i.e. objects of $S^{\mathcal{C}}$ is first resolving them as above by free diagrams and then using the following:

Theorem. *If W is a dimensionwise free $\mathcal{D}$-space then the natural map*

$$\operatorname*{hocolim}_{\mathcal{D}} W \longrightarrow \operatorname*{colim}_{\mathcal{D}} W$$

is a weak equivalence.

The proof proceeds by a diagonal argument using the special case of discrete spaces:

Proposition. *If $E : \mathcal{D} \longrightarrow \mathcal{E}ns$ is a free discrete $\mathcal{D}$-set then the natural map*

$$\operatorname*{hocolim}_{\mathcal{D}} E \longrightarrow \operatorname{colim} E$$

is a weak equivalence.

Proof. Since $\mathrm{hocolim}_{\mathcal{D}}$ and $\operatorname{colim} E$ commutes with disjoint union it is sufficient to consider the case $E = F^d$ the free orbit generated at d. In that case $\operatorname*{hocolim}_{\mathcal{D}} F^d$ is easily seen to be contractible.

Proof of Theorem. We use the equivalence $\mathrm{hocolim}_{\mathcal{D}} X = \mathrm{diag}(\mathrm{hocolim}_{\mathcal{D}} X_n)$.

In a similar way one proves:

Theorem. *If X is dimensionwise free $\mathcal{D}$-space then for any $\mathcal{D}^{op}$-space $\bar{Y}$ and fibrant $\mathcal{D}$-space W the natural maps:*

(i) $\bar{Y} \otimes^h_{\mathcal{D}} X \xrightarrow{\simeq} \bar{Y} \otimes_{\mathcal{D}} X$

(ii) $\mathrm{map}^h_{\mathcal{D}}(X, W) \xrightarrow{\simeq} \mathrm{map}_{\mathcal{D}}(X, W)$

are weak equivalences.

Case I. $X, \bar{Y}$ are discrete spaces. In that case we can assume $X, \bar{Y}$ are free $\mathcal{D}, \mathcal{D}^{op}$-sets. Since $- \otimes_{\mathcal{D}} -$ commutes with disjoint union we can take X to be free orbits and $\bar{Y}$ any orbit

$$\bar{Y} \otimes F^d = \bar{Y}(d)$$
$$\bar{Y} \otimes^h F^d \simeq \bar{Y}(d)$$

which is just the Yoneda's lemma.

Now if X is a free $\mathcal{D}$-space we get similar result by a diagonal (or total space) argument on the (co)-simplicial space that defines $- \otimes^h_{\mathcal{D}} -$ and $\mathrm{map}^h_{\mathcal{D}}(-, -)$.

Remark. More generally: given a functor $W : C^{op} \times \mathcal{D} \longrightarrow \mathcal{S}$, assume that its adjoint $\tilde{W} : \mathcal{D}^{op} \longrightarrow \mathcal{S}^{\mathcal{D}}$ or $\tilde{W} : \mathcal{D} \longrightarrow \mathcal{S}^{op}$ gives a $\mathcal{D}^{op}$ (or $\mathcal{D}$-diagram) of free $\mathcal{D}$-spaces (free $\mathcal{D}^{op}$-spaces) then the map

$$\mathrm{hocoend}\, W \xrightarrow{\simeq} \mathrm{coend}\, W$$

is a weak equivalence.

§3. Fixed point diagram for $\mathcal{D}$-spaces

One of the main tools of the theory of G-space for a group G is the decomposition of the space X to its subspaces of fixed point $X^H \subset X$ for $H \subset G$ and the reconstruction of X from the collection $\{X^H\}$ compare [Elmendorf], [Segal]. This technique generalized to more general situation that includes $\mathcal{D}$-spaces in [DF], [D-K]. After briefly recalling this technique we combine it with the homotopy coend colimit functor so as to write various spaces associated to a $\mathcal{D}$-space as homotopy coends. In particular we will write the (strict) colimit of X as a homotopy colimit over a category of "orbits" associated to $\mathcal{D}$. In what follows we will be concerned with discrete small categories so that in case of G-spaces for a group G we assume here that the group G is a discrete group. However, most of our results can be generalized to simplicial and topological small categories $\mathcal{D}$.

We first recall the assemblage construction of Elmendorf: (compare ...)

Let G be a discrete group acting on a simplicial set X. Consider the category $\mathcal{O}^{op} = \mathcal{O}_G^{op}$ of all the G-orbits i.e. the G-sets G/H for $H \subseteq G$.

Form the $\mathcal{O}^{op}$-space $\Phi_X : \mathcal{O}^{op} \longrightarrow \mathcal{S}$ defined by $\Phi_X(G/H) = \mathrm{Hom}_G(G/H, X) = X^H$ = the points in X fixed by H, with the obvious induced map $X^K \longrightarrow X^H$ for any G-map $G/H \longrightarrow G/K$. Conversely given an arbitrary diagram $\Phi : \mathcal{O}^{op} \longrightarrow \mathcal{S}$ of spaces one can "assemble" them into a G-space X_Φ with $X_\Phi^H \simeq \Phi(G/H)$ by the coend functor as given by Elmendorf so that the H-fixed point space of X_Φ is homotopy equivalent to the value $\Phi(G/H)$ prescribed the arbitrary diagram Φ. Namely $X_\Phi = \Phi \otimes_{\mathcal{O}} I$

Orbit-set functor for a general diagram.

Our purpose here is to generalized the above procedure in the spirit of [D-K,DF] for a diagram of simplicial sets over any small discrete category. One of the main benefits that will be a way to present the strict colimits of any diagram $X : \mathcal{D} \longrightarrow \mathcal{S}$ as a homotopy colimit of an associated diagram Φ_X over a category of orbits $\mathcal{O}$. This presentation which extends also to strict coends allows us to compare the strict colimit and with the homotopy colimit, since it will present both as homotopy coends over the same category of orbits. The main innovation here is the replacement of the fixed point space $X^k = \mathrm{map}_G(G/K, X)$ for the G-space case by the orbit-points space namely $X^e = \mathrm{map}_{\mathcal{D}}(e, X)$ for any orbit e and $\mathcal{D}$-space X.

Construction of the diagram of spaces of orbit points.

Given a small category $\mathcal{D}$ we associate it with the (large) category all $\mathcal{D}$-orbits $\mathcal{O} = \mathcal{O}_{\mathcal{D}}$ which is the full subcategory of $\mathcal{S}^{\mathcal{D}}$ consisting of discrete orbits $\mathcal{O} \subseteq \mathcal{E}ns^{\mathcal{D}}$ (see §2 above). In fact there is a natural analogue to the diagram of fixed point mentioned above $\Phi_X = \{X^H\}$ from above: Namely to any $\mathcal{D}$-space Y we associate the functor $\Phi_Y : \mathcal{O}_{\mathcal{D}} \longrightarrow \mathcal{S}$ by $\Phi_Y(e) = \mathrm{map}_{\mathcal{D}}(e, Y)$. This is a representable functor on $\mathcal{O}_{\mathcal{D}}$ and therefore by Yoneda's lemma ??? we can write an identity $\Phi_Y \otimes_O I \cong Y$, where $I : \mathcal{O} \hookrightarrow \mathcal{S}^{\mathcal{D}}$ is the inclusion functor.

Notice however, that $\mathcal{O}$ is not a small category (see example ... below) and we will not deal with the precise meaning of this coend over large categories instead we proceed to consider small subcategories of $\mathcal{O}_{\mathcal{D}}$.

Example. Let $\mathcal{J} = \{\bullet \longrightarrow \bullet\}$ be the small category of two object with one non-identity map between them. Then any map $V \longrightarrow W$ is a $\mathcal{J}$-space and any map $W \longrightarrow *$ to a point is a $\mathcal{J}$-orbits. Therefore the category $\mathcal{O}_{\mathcal{J}}$ of discrete $\mathcal{J}$-orbits is precisely the category of all sets.

Let $\mathcal{E} \subset \mathcal{O}_{\mathcal{D}}$ be a full small subcategory of $\mathcal{O}_{\mathcal{D}}$. With each $\mathcal{D}$-space X we associate an $\mathcal{E}$-space $X^{\mathcal{E}} : \mathcal{E} \longrightarrow \mathcal{S}$ given by $X^{\mathcal{E}}(e) = \mathrm{map}_{\mathcal{D}}(e, X)$ for $e \in \mathcal{E}$, with $X^{\mathcal{E}}(e \longrightarrow e')$ given by the induced $\mathrm{map}_{\mathcal{D}}(e', X) \longrightarrow \mathrm{map}_{\mathcal{D}}(e, X)$.

For example, let $\mathcal{G}$ be the full subcategory of $\mathcal{O}_{\mathcal{J}}$ from above consisting of orbits $S \longrightarrow *$ where S is a finite set, then for any $\mathcal{J}$-diagram $X \longrightarrow Y$ the associated $\mathcal{G}$-diagram consists of the finite fibred powers of X over Y namely $\prod_Y X$ with various projection as maps.

First properties of $X^{\mathcal{E}}$.

Proposition. *For any $\mathcal{D}$-set X, if $\mathcal{E}$ is a full small subcategory of $\mathcal{D}$-orbits containing all the orbits of X then there is a natural isomorphism of sets*

$$\operatorname*{colim}_{\mathcal{E}^{op}} X^{\mathcal{E}} \xrightarrow{\cong} \operatorname*{colim}_{\mathcal{D}} X.$$

In the proof we need the following lemma:

Lemma. *For any $\mathcal{D}$-set X and any $\mathcal{D}$-orbit e there is a natural decomposition*

$$\mathrm{map}_{\mathcal{D}}(e, X) \cong \coprod_{\alpha} \mathrm{map}(e, X_\alpha)$$

where $X = \coprod\{X_\alpha : \alpha \in \mathrm{colim}_{\mathcal{D}} X\}$ is the decomposition of X into $\mathcal{D}$-orbits X_α.

Proof of Lemma. Any map $e \longrightarrow X$ factor through some orbit in X namely the orbit over $\mathrm{colim}_{\mathcal{D}}\, e = * \longrightarrow \mathrm{colim}_{\mathcal{D}} X$.

Proof of Proposition. By the lemma the $\mathcal{E}^{op}$ set $X^{\mathcal{E}}$ is isomorphic to $\coprod\{X^{\mathcal{E}}_\alpha : \alpha \in \mathrm{colim}_{\mathcal{D}} X\}$. So it is sufficient to consider the case when $X = X_\alpha$ is itself an orbit. But for a $\mathcal{D}$-orbit X_α the $\mathcal{E}^{op}$-space $X^{\mathcal{E}}_\alpha$ is a $\mathcal{E}^{op}$-orbit since $\mathcal{E}$ contains X_α itself by assumption. Therefore any map $f : e \longrightarrow X_\alpha$ factors through $e \xrightarrow{f} X_\alpha \xrightarrow{id} X_\alpha$ so that $X^{\mathcal{E}}_\alpha(f)(id) = f$.

Φ_X is free.

In some way the deepest property of the diagram Φ_X that distinguishes it from X itself is that it is $\mathcal{O}^{op}$-free while containing, as we shall see, all the homotopy information in X since we can recover X from Φ_X a thing that cannot done with X^{free}, the free $\mathcal{D}$-space approximating of X.

Theorem. *For any $\mathcal{D}$-space X the corresponding $\Phi_X : \mathcal{O}^{op} \longrightarrow \mathcal{S}$ is a dimension-wise free $\mathcal{O}^{op}$-space.*

As usual the main lemma here is that the same is true for a discrete X i.e. for any $\mathcal{D}$-set.

Lemma. *If $X : \mathcal{D} \longrightarrow \mathcal{E}ns$ is any $\mathcal{D}$-set then $\Phi_X : \mathcal{O}^{op} \longrightarrow \mathcal{E}ns$ is a free $\mathcal{D}$-set.*

Proof. Since by lemma ?? above $\Phi_X = \coprod_\alpha \Phi_{X_\alpha}$ where $\alpha \in \operatorname{colim} X$ and $X = \coprod X_\alpha$ is the decomposition of X into $\mathcal{D}$-orbits, it is sufficient to consider the case when $X = E$ is a $\mathcal{D}$-orbit. Now by assumption $E \in \mathcal{O}^{op}$. By lemma ?? above Φ_E is itself an $\mathcal{O}^{op}$-orbit i.e. $\operatorname{colim}_{\mathcal{O}^{op}} \Phi_E = *$. We claim it is a free $\mathcal{O}^{op}$ orbit generated by the element $(id : E \longrightarrow E) \in \Phi_E(E)$. To see this we write $F^E(e)$ and compare it to $\Phi_E(e)$:

$$F^E_{\mathcal{O}^{op}}(e) \equiv \operatorname{Hom}_{\mathcal{O}^{op}}(E, e) = \operatorname{Hom}_{\mathcal{O}}(e, E) =$$
$$= \operatorname{map}_{\mathcal{D}}(e, E) \equiv \Phi_E(e).$$

Proof of theorem. Since we only claim that in each dimension $n \geq 0$ the $\mathcal{D}$-space Φ_X is a free $\mathcal{D}$-set we consider $(\Phi_X)_n(e) = \operatorname{map}_{\mathcal{D}}(e^x\Delta[n], X) = \operatorname{map}_{\mathcal{D}}(e, X_n)$, $e \in \mathcal{O}_{\mathcal{D}}$. Thus the theorem follows directly from lemma ?? above for the $\mathcal{D}$-set X_n.

Assemblage. Given a small category $\mathcal{E} \subseteq \mathcal{O}$ of $\mathcal{D}$-orbits and a diagram over $\mathcal{E}$ namely $W : \mathcal{E} \longrightarrow \mathcal{S}$ one can "realize" W as a $\mathcal{D}$-diagram by assembling the spaces $\{\mathcal{E}(e) | e \in \mathcal{E}\}$ into an $\mathcal{D}$-space $|W|_{\mathcal{D}}$ "with the same $\mathcal{E}$-information as W"". In order to do that we recall that if in a typical homotopy coend $A \otimes^h_{\mathcal{C}} B$ the functor B, in addition to being a $\mathcal{C}$-space, may also be a $\mathcal{U}$-space for some (small) category $\mathcal{U}$ then $A \otimes^h_e B$ is also a $\mathcal{U}$-space: Explicitly the value of $A \otimes^h_{\mathcal{C}} B$ on an element v of $\mathcal{U}$ (be it a object or a morphism) is given by $(A \otimes^h_e B)(u) = A \otimes^h_e (B(u))$. With this reminder we define the "$\mathcal{D}$-realization" of W:

$$|W|_{\mathcal{D}} = W \otimes^h_{\mathcal{E}} I$$

where $I : \mathcal{E} \longrightarrow \mathcal{S}^{\mathcal{D}}$ is the inclusion functor that assigns to each $\mathcal{D}$-orbit $e \in \mathcal{E} \subset \mathcal{S}^{\mathcal{D}}$ that orbit itself as a member of $\mathcal{S}^{\mathcal{D}}$. Thus $|W|_{\mathcal{D}} \in \mathcal{S}^{\mathcal{D}}$ and we claim:

Proposition. *For each orbit $e \in \mathcal{E}$ there is a natural equivalence*

$$\mathrm{map}_{\mathcal{D}}(e, |W|_{\mathcal{D}}) \simeq W(e).$$

In other words $|W|_{\mathcal{D}}$ has the prescribed space $W(e)$ as its space of e-orbit points.

Proof. This is proven in [D-K] for the strict coend but the same argument goes through here too.

Example. Let $G = \mathbb{Z}/p\mathbb{Z}$. In this case we have only two orbits $(\mathbb{Z}/p\mathbb{Z}, \{e\})$. The diagram $\mathcal{O}$ has the shape $\bullet \longrightarrow \underset{\mathbb{Z}/p}{\bullet} \curvearrowright$ has two object and p different maps: Any $\mathcal{O}$-diagram is a pair of spaces $X_0 \longrightarrow X_1$ where X_1 has $\mathbb{Z}/p$ action and X_0 maps **into** the space of fixed points. We can then form the pushout W in the diagram:

$$\begin{tikzcd}
BG \times X_0 \arrow[r] \arrow[d] & E \times_G X_1 \arrow[d] \\
BG \times * \arrow[r] & W
\end{tikzcd}$$

Claim. *There are natural maps $X_1 \longrightarrow W$ and $X_0 \longrightarrow W^{\mathbb{Z}/p\mathbb{Z}}$ which are homotopy equivalence commuting up to homotopy with the map $X_0 \longrightarrow X_1$, $W^{\mathbb{Z}/p\mathbb{Z}} \hookrightarrow W$.*

In order to be able to recover the weak $\mathcal{D}$-homotopy type of X from a diagram $X^{\mathcal{E}}$ associated to some category of orbits $\mathcal{E}$ it is necessary and sufficient that the spaces $X(d)$ themselves will appear in $X^{\mathcal{E}}$ as spaces of fixed points. Since for a **free** $\mathcal{D}$ orbit F^d one has

$$X^{F^d} = \mathrm{map}_{\mathcal{D}}(F^d, X) \simeq X(d)$$

we get:

Proposition. *If for all $d \in \mathrm{obj}\mathcal{D}$ the $\mathcal{D}$-free orbit F^d is a member of $\mathcal{E} \subseteq \mathcal{O}_{\mathcal{D}}$ then there is a weak homotopy equivalence*

$$|X^{\mathcal{E}}|_{\mathcal{D}} \xrightarrow{\simeq} X.$$

Remark. Notice that this is an analogue of the equivalence $|\mathrm{Sing}X| \longrightarrow X$ between the realization of the simplicial set $\mathrm{Sing}X$ associated to a space X.

Remark. In general for any $\mathcal{E} \subseteq \mathcal{O}_{\mathcal{D}}$ the $\mathcal{E}$-space $X^{\mathcal{E}}$ as defined above contains all the information about X that is "seen" by the orbits in $\mathcal{E}$. If $\mathcal{E}$ contains a single orbit say $e_*(d) = *$ then $X^{\{e_*\}}$ is precisely the inverse limit of X e.g. if $\mathcal{D}$ is a group G this is the fixed set X^G. The space $|X^{\mathcal{E}}|_{\mathcal{D}}$ is an approximation to the space X by a $\mathcal{D}$-space which is as free as possible while containing all the information about X about orbit of type $\mathcal{E}$. In the extreme case if $\mathcal{E}$ contains all the orbits that actually appear in X, i.e. all the $\mathcal{D}$-orbits that appear in the decomposition of any $X_n : \mathcal{D} \to \mathcal{E}ns$ into orbits for all $n \geq 0$, then the $\mathcal{D}$-homotopy type of $|X^{\mathcal{E}}|_{\mathcal{D}}$ is independent of $\mathcal{E}$ and

Proposition. *For any two $\mathcal{E}^1, \mathcal{E}^2 \subseteq \mathcal{O}_{\mathcal{D}}$ that contains any orbit that appear in a given fibrant $\mathcal{D}$-space X one has a $\mathcal{D}$-homotopy equivalence*

$$|X^{\mathcal{E}^1}|_{\mathcal{D}} \rightleftarrows |X^{\mathcal{E}^2}|_{\mathcal{D}}.$$

Notation. In this case we denote $X^{\mathcal{E}}$ by Φ_X (fixed point set diagram of X).

§4 Chain complex

Homotopy coend and ends can be considered profitable in the category of chain complexes over some ring with unit R. In this case one gets a version of the usual definitions of the derived functors of tensor product and the Hom functors.

Theorem. *If L, M, are left R-modules and N is a right R-module then there are natural isomorphisms:*

(i) $H_k(N \otimes^h_R L) \cong \mathrm{Tor}^k_R(N, L)$.

(ii) $H^k(\mathrm{map}^h_R(L, M)) \cong \mathrm{Ext}^k_R(L, M)$.

where in these formulae we denote by L, M, N the chain complexes bounded below which are non-zero only in dimension zero.

Proof. We consider the ring R as an additive category with one object where the elements of R are the morphisms of that category. Then a free orbit over this category is a copy of R and a free R-diagram of abelian groups is a free R-module in the usual sense i.e. a module isomorphic to a direct sum $\oplus_\alpha R$.

From now on we will assume that $\mathcal{E} \subseteq \mathcal{O}_\mathcal{D}$ contains all the $\mathcal{D}$-orbits that appear in the spaces involved.

§5. Three formulae

Orbit operations. We now rewrite several orbit functors on $\mathcal{D}$-diagrams in terms of corresponding orbit functors on an associated category of orbits.

By the term orbit functor we mean a right adjoint functor on $\mathcal{D}$-spaces whose value on the typical $\mathcal{D}$-space is determined by its value on $\mathcal{D}$-orbits. For example the colimit functor sends each orbit to a single point and a $\mathcal{D}$-space to its space of orbits, i.e. the colimit.

We re-write both the homotopy colimit and the strict colimit of X in terms of homotopy coends on Φ_X.

Proposition. *The following natural maps are weak equivalences in $\mathcal{S}$ for any $\mathcal{D}$-space $X : \mathcal{D} \longrightarrow \mathcal{S}$.*

(1) $\Phi_X \otimes^h_\mathcal{O} I \xrightarrow{\simeq} \Phi_X \otimes_\mathcal{O} I \xrightarrow{\simeq} X$.

(2) $\Phi_X \otimes^h_\mathcal{O} * \xrightarrow{\simeq} \Phi_X \otimes_\mathcal{O} * = \mathrm{colim}_\mathcal{D} X$.

(3) $\Phi_X \otimes^h_\mathcal{O} I_{h\mathcal{D}} \xrightarrow{\simeq} \mathrm{hocolim}_\mathcal{D} X$.

where $I_{h\mathcal{D}}$ is the $\mathcal{O}$-space defined by $I_{h\mathcal{D}}(e) = \mathrm{hocolim}_{\mathcal{D}}\, e = e_{h\mathcal{D}}$. Here $\mathcal{O}$ denote any small category of orbits that includes all the free orbits and all the orbits in X.

Proof. To show $\Phi_X \otimes^h_{\mathcal{O}} I \simeq X$ we first notice that since Φ_X is a ?? of free $\mathcal{O}^{op}$ set it is sufficient to show the strict coend isomorphism: $\Phi_X \otimes_{\mathcal{D}} I \simeq X$. Next we use the fact that $(\otimes_{\mathcal{D}} I)$ commutes with disjoint union and since $\Phi_{X \coprod Y} \simeq \Phi_X \coprod \Phi_Y$ we can assume w.l.o.g. that X is a $\mathcal{D}$-orbit, $\mathrm{colim}\, X \simeq *$. In that case $X = e$ $\Phi_X(e') = \mathrm{map}_{\mathcal{D}}(e', e)$. So that we have $\Phi : \mathcal{O}^{op} \times \mathcal{O} \longrightarrow \mathcal{E}ns$ with $\Phi(e, e') = \Phi_e(e') = \mathrm{map}_{\mathcal{D}}(e', e)$, and by Yoneda's lemma $\Phi \otimes_{\mathcal{O}} I = I$, in particular $\Phi_e \otimes_{\mathcal{O}} I = I(e) = e$ as needed.

Note if X is a diagram of spaces $X : \mathcal{D} \longrightarrow \mathcal{S}$ since Φ_X is defined dimensionwise $(\Phi_x)^{(e)}_n = \mathrm{map}_{\mathcal{D}}(e \times \mathcal{D}[n], X) = \mathrm{map}_{\mathcal{D}}(e, X_n)$ and same goes for the coend $(\Phi_X \otimes_{\mathcal{D}} Y)_n = (\Phi_X)_n \otimes_{\mathcal{D}} Y_n$, we have a natural isomorphism for any X:

$$\Phi_X \otimes_{\mathcal{D}} I \simeq X.$$

But again since Φ_X is dimensionwise free $\Phi_X \otimes_X (-) \simeq \Phi_X \otimes^h_{\mathcal{D}} (-)$.

As for (3) we rewrite by 1)

$$\mathop{\mathrm{hocolim}}_{\mathcal{D}} X \simeq X \otimes^h_{\mathcal{D}^{op}} * \simeq (\Phi_X \otimes^h_{\mathcal{O}} I) \otimes^h_{\mathcal{D}^{op}} *$$

which by associativity we rewrite as

$$\Phi_X \otimes^h_{\mathcal{O}} (I \otimes^h_{\mathcal{O}^{op}} *) \equiv \Phi_X \otimes^h_{\mathcal{O}} I_{h\mathcal{D}}.$$

In order to show (2) we use three statements. **First**, the diagram Φ_X is dimensionwise free, **second** for any dimensionwise free diagram Y the map $\mathrm{hocolim}\, Y \longrightarrow \mathrm{colim}\, Y$ is a weak equivalence and **third** that $\mathrm{colim}_{\mathcal{O}}\, \Phi_X \simeq \mathrm{colim}_{\mathcal{D}}\, X$. Each one of these statements was proved above and (2) follows directly from these three statements.

§6.. An application: Cellular inequalities

Much of the above development was motivated by our attempt to understand from a different point of view Bousfield's key lemma about symmetric products. Recall:

Bousfiled's key Lemma. *Let Y be a 1-connected fibrant space. If the function complex $\mathrm{map}_*(X, \Omega^2 Y) \simeq *$ is contractible for some X then the induced map $\mathrm{map}_*(X, \Omega Y) \simeq \mathrm{map}_*(SP^k X, \Omega Y)$ is a homotopy equivalence, for any $1 \leq k \leq \infty$.*

A weaker version of this lemma follows from the cellular inequality: $\Sigma X \ll SP^k X/X$. For example for $k = \infty$, if $X = S^1 \vee S^1$, then $SP^\infty(S^1 \vee S^1) = S^1 \times S^1$ and $SP^\infty(S^1 \vee S^1)/S^1 \vee S^1 \simeq S^2$ which of course is $\Sigma(S^1 \vee S^1)$-cellular. For the full statement of the key lemma as well as the above inequality we consider here the cofibre of the map of pointed $\operatorname{hocolim}_{*\,\mathcal{D}} X \longrightarrow \operatorname{colim}_{\mathcal{D}} X$ for an arbitrary pointed $\mathcal{D}$-space $X : \mathcal{D} \longrightarrow \mathcal{S}_*$.

We will now work in the category of pointed spaces $\mathcal{S}_*$ since we want to consider cellular inequalities.

Theorem. *If X is a $\mathcal{D}$-space with $A \ll \mathrm{map}_{\mathcal{D}}(e, X)$ for any $\mathcal{D}$-orbit $e : \mathcal{D} \longrightarrow \mathcal{E}ns$ that appear in X, then the following cellular inequality holds:*

$$\Sigma^2 A \ll \textit{cofibre}(\underset{\mathcal{D}}{\operatorname{hocolim}_*} X \longrightarrow \underset{\mathcal{D}}{\operatorname{colim}} X)$$

Example. $\Sigma^2 X \ll \mathrm{cofibre}(E\Sigma_n \times_{\Sigma_n} X^n \longrightarrow SP^n X)$.

Corollary. *For any diagram $X : \mathcal{D} \longrightarrow \mathcal{S}_*$ the cofibre of*

$$\underset{\mathcal{D}}{\operatorname{hocolim}_*} X \longrightarrow \underset{\mathcal{D}}{\operatorname{colim}} X$$

is one connected.

Proof. Since $S^0 \ll Y$ for any Y we have $S^2 \ll$ cofibre which is equivalent to being one connected.

Corollary. *For any space X we have $\Sigma X \ll SP^k X/X$ where $k \geq 0$ and SP^k is the symmetric power of X.*

Proof. The proof proceeds, as outlined in [DF-4.1.6] without full justification, by expressing the desired cofibre as a pointed homotopy coends of certain functor K :

$\mathcal{D} \longrightarrow \mathcal{S}_*$. We first rewrite both $\operatorname{hocolim}_{\mathcal{D}} X$ and $\operatorname{colim}_{\mathcal{D}} X$ as a homotopy coend over the orbit category $\mathcal{O}^{op}$ associated to the diagram X. We assume of course that $\mathcal{O}$ is a small full category of $\mathcal{D}$-orbits that contains all the orbits that appear in X. According to formulae (??) above the map $\operatorname*{hocolim}_{\mathcal{D}}{}_* X \longrightarrow \operatorname*{colim}_{\mathcal{D}} X$ is equivalent to $\operatorname*{hocoend}_{\mathcal{D}} \Phi_X \rtimes I_{h\mathcal{D}} \longrightarrow \operatorname{hocoend} \Phi \rtimes *$, with cofibre C. Since taking cofibre commutes with taking homotopy coend (both are homotopy colimit operations) we can calculate the cofibre C as a pointed homotopy coend

$$
\begin{aligned}
C &\cong \operatorname{coend}(\operatorname{cofibre}\Phi_X \rtimes I_{h\mathcal{D}} \longrightarrow \Phi_X \rtimes t) \\
&= \operatorname{coend} \Phi_X * I_{h\mathcal{D}}
\end{aligned}
$$

where $*$ denotes here the join operation $A * B \simeq \Sigma(A \wedge B)$ between two pointed spaces A, B.

Here we use the fact that $\operatorname{colim} e = *$ for each orbit of $\mathcal{D}$ so that $e_{h\mathcal{D}} = \operatorname{hocolim}_{\mathcal{D}} e$ is connected. Therefore for each e

$$
\Phi_X(e) * I_{h\mathcal{D}}(e) \simeq \Sigma(\Phi_X(e) \wedge I_{h\mathcal{D}}(e)) \geq \Sigma(A \wedge S^1) = \Sigma^2 A.
$$

More generally

Theorem. *Let X satisfy the condition of () above at $Y : \mathcal{D}^{op} \longrightarrow \mathcal{S}_*$ then the cofibre of $Y \otimes^h_{\mathcal{D}} X \longrightarrow Y \otimes_C X$ is $\Sigma^2 A$-cellular.*

§7. A spectral sequence for the generalized homology of $\operatorname{hocoend}_{\mathcal{D}} X\bullet$

Using the dimension fibration on a bifunctor $F : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{S}$ i.e. a diagram of spaces over $\mathcal{C}^{op} \times \mathcal{C}$ or a $\mathcal{C}^{op}$-diagram of $\mathcal{C}$-diagrams of spaces, we build a spectral sequence that starts with a kind of Hochschild homology associated with $\mathcal{C}$ and abuts to $h_*(\operatorname{hocoend} F)$. For this we recall that if $A \in R^{op}$-mod and $B \in R$-mod then $A \otimes B \in R^{op} \times R$-mod and $\operatorname{Tor}_*^{R^{op} \times R}(R, A \times B) \cong \operatorname{Tor}_*^R(A, B)$. As we saw above we can write

$$
\operatorname{hocoend} F \cong C(-,-) \otimes^h_{C^{op} \times C} F
$$

which for chain complexes gives exactly the usual construction of HH.

This leads to a spectral sequence

$$
\operatorname{Hoch} H_j(\mathcal{C}, h_i F) \Rightarrow h_{i+j} \operatorname{hocoend} F.
$$

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