“RANDOM” RANDOM MATRIX PRODUCTS

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Abstract. The paper deals with compositions of independent random bundle maps whose distributions form a stationary process which leads to study of Markov processes in random environments. A particular case of this situation is a product of independent random matrices with stationarily changing distributions. I obtain results concerning invariant filtrations for such systems, positivity and simplicity of the largest Lyapunov exponent, as well as the central limit theorem type results. An application to random harmonic functions and measures is also considered. Continuous time versions of these results are also discussed which yield applications to linear stochastic differential equations in random environments.

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1. Introduction

Starting from the beginning of sixties a lot of work has been done on products of independent identically distributed random matrices. This paper yields that many of these results can be extended to products of independent random matrices whose distributions evolve according to a stationary process. Actually, the paper deals with a more general case of compositions of independent random bundle maps whose distributions form a stationary process. I shall generalize to this situation the result from [FK] and [Ki1] on invariant filtrations, derive conditions which ensure positivity and simplicity of the biggest Lyapunov exponent, obtain a central limit theorem type result, and exhibit applications to continuous time models such as solutions of linear stochastic differential equations in a random stationary in time environments. Some results concerning random harmonic functions and measures for products of independent random matrices with stationarily changing distributions will be derived, as well. I am trying to implement here the general ideology saying that many results concerning products of independent or Markov dependent random matrices remain true in some form in the more general situation of stationary matrix sequences (processes) which can be represented as sufficiently nondegenerate independent or Markov matrix sequences (processes) conditioned to another stationary process. I do not discuss here an interesting question when such representations are possible.

The set up consists of a complete probability space $(\Omega, \mathcal{A}, P)$ with an invertible P-preserving ergodic map θ of Ω into itself and of another measurable space $(\mathcal{X}, \mathcal{B})$ where $\mathcal{X}$ is a Borel subset of a Polish space (i.e. of a complete separable metric space) and $\mathcal{B}$ is the Borel σ-algebra on $\mathcal{X}$. A pair $F = (f_F, T_F)$ is called a (vector) bundle map of the direct product $E = \mathcal{X} \times \mathbb{R}^d$ (where $\mathbb{R}^d$ is the d-dimensional Euclidean space) over a Borel map $f_F : \mathcal{X} \to \mathcal{X}$ if $T_F = T_F(x)$ is a Borel function of x with values in the group $GL(d, \mathbb{R})$ of real $d \times d$ invertible matrices and

$$F(x, a) = (f_F x, T_F(x)a), \quad x \in \mathcal{X}, \quad a \in \mathbb{R}^d. \tag{1.1}$$

Denote by $\mathcal{T}$ the space of all vector bundle maps endowed with a measurable structure such that the map $\mathcal{T} \times E \to E$ sending (F, u) to Fu, $u \in E$ is measurable with respect to the product measurable structure in $\mathcal{T} \times E$. Set $\Xi = \mathcal{T}^{\mathbb{Z}_+} = \{\xi = (\xi_0, \xi_1, ...), \xi_i \in \mathcal{T}\}$. Given a measurable in ω family of probability measures μ^ω on $\mathcal{T}$ denote by Π^ω the product measure $\prod_{i=0}^{\infty} \mu^{\theta^i \omega}$ on Ξ. Let $F_0 : \Omega \times \Xi \to \mathcal{T}$ be the measurable map $F_0^\omega(\xi) = \xi_0 \in \mathcal{T}$. Set $F_i^\omega(\xi) = \xi_i = F_0^{\theta^i \omega}(\sigma^i \xi)$, where σ is the left shift on Ξ acting by $(\sigma\xi)_i = \xi_{i+1}$. Then F_i^ω, $i \in \mathbb{Z}$ is a sequence of independent random bundle maps with distributions $\mu^{\theta^i \omega}$.

The actions on $\mathcal{X}$ yield time inhomogeneous Markov chains $X_n^\omega(\xi) = X_n^\omega(\xi, x) = f_{\xi_{n-1}} \circ \cdots \circ f_{\xi_1} \circ f_{\xi_0} x$, $X_0^\omega(\xi) = x$ such that $X_{n+1}^\omega(\xi) \in \Gamma$ with probability $\mu^{\theta^n \omega}\{F : f_F y \in \Gamma\}$ provided $X_n^\omega(\xi) = y$. Let ρ be an ergodic probability invariant measure of the skew product Markov chain $(\theta^n \omega, X_n^\omega(\xi), \sigma^n \xi)$ having marginals Π^ω on Ξ and P on Ω, i.e. $d\rho(\omega, x, \xi) = d\rho^\omega(x) d\Pi^\omega(\xi) dP(\omega)$. Set $T(n, \omega, x) = T(n, \omega, x, \xi) = T_{\xi_{n-1}}(X_{n-1}^\omega(\xi)) \ldots T_{\xi_1}(X_1^\omega(\xi)) T_{\xi_0}(x)$ and assume

$$\int (\log^+ \|T_{\xi_0}(x)\| + \log^+ \|T_{\xi_0}^{-1}(x)\|) d\rho(\omega, x, \xi) < \infty.$$

Then by Kingman's subadditive ergodic theorem (see, for instance, [Ki1] Section A.2) ρ-a.s. the limit

$$\beta_0(\rho) = \lim_{n\to\infty} \frac{1}{n} \log \|T(n, \omega, x)\| \tag{1.2}$$

exists and it is finite and nonrandom.

A more precise result follows from Oseledec's "multiplicative ergodic theorem" (see, for instance, [Ar], Ch. 4) which yields that for any vector $v \in \mathbb{R}^d$ the limit

$$\beta^\omega(\rho, \xi, v) = \lim_{n\to\infty} \frac{1}{n} \log \|T(n, \omega, x, \xi) v\| \tag{1.3}$$

exists ρ-a.s. but, in general, it may depend on ω, ξ, and v. Still, in the ergodic situation $\beta^\omega(\rho, \xi, v)$ may take on only a finite number of values $\infty > \lambda_0(\rho) \geq \lambda_1(\rho) \geq \ldots \geq \lambda_{d-1}(\rho) > -\infty$ called the Lyapunov exponents and the biggest such value coincides with $\beta_0(\rho)$. In the next section I shall show that for P-almost all (a.a.) $\omega \in \Omega$ the number $\beta^\omega(\rho, \xi, v)$ depends only on ω and v but not on ξ and its dependence on v can be described by a filtration of subspaces of $\mathbb{R}^d$ which depend on ω but not on ξ. When Ω degenerates to one point one has the situation considered in [FK] and [Ki1] and if $\mathcal{T}$ is a point then my arguments provide another proof of a part of Oseledec's theorem.

Clearly, $\lambda_0(\rho) \geq d^{-1} \int \log |\det T_F(x)| d\rho^\omega(x) d\mu^\omega(F) dP(\omega)$ and I provide conditions when this inequality is strict. Under certain nondegeneracy conditions on distributions μ^ω I derive also the simplicity of $\lambda_0(\rho)$, i.e. that all other Lyapunov exponents are strictly less than $\lambda_0(\rho)$, which yields the contraction of actions of $T(n, \omega, x, \xi)$ on the projective space. Under some conditions I also show that for P-a.a.ω the distribution of $n^{-1/2}(\log \|T(n, \omega, x, \xi)\| - \lambda_0^\omega(\rho))$ is asymptotically (as $n \to \infty$) Gaussian (in ξ), where $\lambda_0^\omega(\rho)$ are certain centralizing random variables satisfying $\int \lambda_0^\omega(\rho) dP(\omega) = \lambda_0(\rho)$.

In the last section I consider random harmonic functions and measures for Markov chains with stationary changing transition probabilities. More specific results are obtained for random harmonic measures of products of independent random matrices with stationarily changing distributions which I apply to random continued fractions.

The set up above enables me to treat also a seemingly more general following situation which provides also a continuous time version. Let now θ^t, $t \in \mathbb{Z}$ or $t \in \mathbb{R}$ be a group of P-preserving maps of Ω into itself and $Q^\omega(t, (x, M), \cdot)$, $\omega \in \Omega$, $t \geq 0$, $x \in \mathcal{X}$, $M \in GL(d, \mathbb{R})$, be a measurable family of probability measures on $\mathcal{G} = \mathcal{X} \times GL(d, \mathbb{R})$. By Kolmogorov's extension theorem this yields a time inhomogeneous Markov process Y_t^ω evolving according to $\{Q^{\theta^t \omega}\}_{t \geq 0}$, i.e. $Y_t^\omega \in \Gamma \subset \mathcal{G}$ with probability $Q^{\theta^s \omega}(t - s, y, \Gamma)$ provided $Y_s^\omega = y \in \mathcal{G}$, $s < t$. In particular, the Chapman-Kolmogorov formula holds true:

$$Q^\omega(t, y, \Gamma) = \int Q^\omega(s, y, dz) Q^{\theta^s \omega}(t - s, z, \Gamma). \tag{1.4}$$

Such Markov process Y_t^ω is called multiplicative if

$$Q^\omega(t, (x, M), U \times V) = Q^\omega(t, (x, \mathrm{Id}), U \times V M^{-1}) \tag{1.5}$$

for all $t \geq 0$, $x \in \mathcal{X}$, and $M \in GL(d, \mathbb{R})$. The process Y_t^ω is the pair (X_t^ω, M_t^ω) with $X_t^\omega \in \mathcal{X}$ and $M_t^\omega \in GL(d, \mathbb{R})$. If $q^\omega(t, x, U) = Q^\omega(t, (x, \mathrm{Id}), U \times \mathbb{R}^d)$ then by (1.4) and (1.5),

$$q^\omega(t, x, U) = \int q^\omega(s, x, dy) q^{\theta^s \omega}(t - s, y, U), \tag{1.6}$$

i.e. X_t^ω is also a Markov process on $\mathcal{X}$ with transition probabilities $q^\omega(t,x,\cdot)$. I call processes like X_t^ω and Y_t^ω Markov processes in random environments with Ω interpreted as an environments space. The multiplicative Markov processes with $Q^\omega(t,y,\Gamma)$ independent of ω were considered in [Bo1,2].

Let $F_i^\omega(\xi)=\xi_i,\ i=0,1,\dots$ be independent random bundle maps and $X_n^\omega = f_{\xi_{n-1}}\circ\cdots\circ f_{\xi_0}x$. Set $M_n^\omega=M_n^\omega(\xi)=T(n,\omega,x,\xi)$. Then $Y_n^\omega=(X_n^\omega,M_n^\omega)$ with $Y_0^\omega=(x,\mathrm{Id})$ becomes a multiplicative Markov process in random environments with transition probabilities

$$Q^\omega((x,M),U\times V)=\mu^\omega\{F: f_Fx\in U, T_F(x)M\in V\} \tag{1.7}$$

for all $x\in\mathcal{X}$, $M\in GL(d,\mathbb{R})$, $U\subset\mathcal{X}$, $V\subset GL(d,\mathbb{R})$. Thus the asymptotic results for random bundle maps described above can be studied via multiplicative Markov processes. On the other hand, by [Ki1], Section 1.1 any Markov chain can be considered as a composition of independent Borel maps which yields (see Proposition 2.4) that any discrete time multiplicative Markov process can be represented via independent random bundle maps as above, and so, essentially, these setups are equivalent. Observe, that considering the skew product multiplicative Markov chain $Z_n=((\theta^n\omega,X_n^\omega),M_n^\omega)$ one can formally eliminate random environments but this helps only for basic results when a strong nondegeneracy of matrix products is not required.

It turns out that the asymptotic behavior of $\|M_t^\omega v\|$, $v\in\mathbb{R}^d$ as $t\to\infty$ for a multiplicative Markov process $Y_t^\omega=(X_t^\omega,M_t^\omega)$ can be studied considering only discrete times $t\in\mathbb{Z}_+$ which, as explained above, leads to compositions of random bundle maps. This enables me to apply results to the specific continuous time example when M_t^ω is a solution of the matrix linear stochastic differential equation

$$dM_t^\omega=A_0(X_t^\omega,\theta^t\omega)M_t^\omega dt+\sum_{i=1}^m A_i(X_t^\omega,\theta^t\omega)M_t^\omega dB_t^u,\quad M_0^\omega=M \tag{1.8}$$

where $B_t^1,\dots,B_t^m$ are independent one dimensional Brownian motions independent of a Markov process X_t^ω on $\mathcal{X}$ as above and of a stationary process $\{\theta^t\omega\}_{t\in\mathbb{R}}$.

2. Invariance, ergodicity and i.i.d. representations

Throughout this paper I assume that $(\Omega,\mathcal{A},P)$ is a Lebesgue space, i.e. that it is measurably mod 0 isomorphic to an interval $[a,b)$ (may be empty) with the completion of the Borel σ-algebra and the Lebesgue measure on it together with countably many atoms. It is known (see [Ro]) that if Ω is a Borel subset of a Polish space and $\mathcal{A}$ is the completion of the Borel σ-algebra with respect to P then $(\Omega, \mathcal{A}, P)$ is a Lebesgue space. Note also that any Borel subset of a Polish space is Borel measurably isomorphic to a Borel subset of the unit interval (see [Ku], §36–37).

I shall deal in this paper with different Markov chains Z_n^ω in random environments on a Borel subset of a Polish space $\mathcal{V}$ having some transition probabilities $R^\omega(v, \cdot)$ measurably depending on $(\omega, v) \in \Omega \times \mathcal{V}$ and whose n-step transition probabilities have the form

(2.1)
$$R^\omega(n, v, U) = \int \cdots \int$$
$$R^\omega(v, dv_1) R^{\theta\omega}(v_1, dv_2) \cdots R^{\theta^{n-2}\omega}(v_{n-2}, dv_{n-1}) R^{\theta^{n-1}\omega}(v_{n-1}, U).$$

In particular, $Z_{k+1}^\omega \in U$ with probability $R^{\theta^k\omega}(v, U)$ provided $Z_k^\omega = v$. For each fixed ω this defines an inhomogeneous in time Markov chain whose transition operator R^ω acts by the formula $R^\omega h(v) = \int h(y) R^\omega(v, dy)$. Denote by $\mathcal{P}(\mathcal{V})$ the space of probability measures on $\mathcal{V}$ considered with the topology of weak convergence. A measurable in ω family $\nu^\omega \in \mathcal{P}(\mathcal{V})$, $\omega \in \Omega$ is called R^ω-stationary if $\nu^\omega R^\omega = \nu^{\theta\omega}$ P-a.s., i.e. $\int d\nu^\omega(v) R^\omega(v, U) = \nu^{\theta\omega}(U)$ for any Borel U. If ν is defined by $d\nu(\omega, v) = d\nu^\omega(v) dP(\omega)$ then ν is an invariant measure of the skew product Markov chain $\mathcal{Z}_n = (\theta^n\omega, Z_n^\omega)$ on $\Omega \times \mathcal{V}$. Conversely, any probability invariant measure ν of $\mathcal{Z}_n$ whose marginal on Ω is P has the above desintegration with ν^ω, $\omega \in \Omega$ being an $R^\omega-$stationary family.

An R^ω-stationary family ν^ω, $\omega \in \Omega$ will be called ergodic if the corresponding invariant measure ν of $\mathcal{Z}_n$ is ergodic. This means that any bounded measurable function $h = h_\omega(v)$ such that

$$R^\omega h_{\theta\omega}(v) = \int h_{\theta\omega}(w) R^\omega(v, dw) = h_\omega(v) \tag{2.2}$$

for ν-almost all (a.a.) v, satisfies $h \equiv \text{const}$ ν-almost surely (a.s.). In view of Lemma I.2.4. from [Ki1] such a family ν^ω is ergodic if and only if for any family of

Borel sets $A_\omega \subset \mathcal{V}$ such that $A = \{(\omega, v), v \in A_\omega\}$ is measurable and

$$R^\omega(v, A_{\theta\omega}) = 1 \text{ for } \nu^\omega\text{-a.a. } v \text{ and } P\text{-a.a.}\omega, \tag{2.3}$$

one has either $\nu^\omega(A_\omega) = 1$ P-a.s. or $\nu^\omega(A_\omega) = 0$ P-a.s.

Next, let Φ be the space of Borel maps of $\mathcal{V}$ into itself and let μ^ω, $\omega \in \Omega$ be a measurable family of probability measures on Φ. This determines a Markov process Z_n^ω in random environments on $\mathcal{V}$ with transition probabilities

$$R^\omega(v, \mathcal{U}) = \mu^\omega * \delta_v(\mathcal{U}) = \mu^\omega\{\varphi \in \Phi : \varphi v \in \mathcal{U}\} \tag{2.4}$$

where $v \in \mathcal{V}$, $\mathcal{U} \subset \mathcal{V}$ is Borel, δ_v is the Dirac measure at v and, as usual, for any $\nu \in \mathcal{P}(\mathcal{V})$ I set

$$\mu^\omega * \nu = \int \varphi\nu d\mu^\omega(\varphi), \tag{2.5}$$

i.e. $\int g d\mu^\omega * \nu = \int g(\varphi v) d\mu^\omega(\varphi) d\nu(v)$ for any bounded Borel function g on $\mathcal{V}$. In particular, I can write now

$$R^\omega(n, v, \mathcal{U}) = \mu^{\theta^{n-1}\omega} * \cdots * \mu^{\theta\omega} * \mu^\omega * \delta_v(\mathcal{U}) \tag{2.6}$$

and a family $\nu^\omega \in \mathcal{P}(\mathcal{V})$, $\omega \in \Omega$ is R^ω-stationary if and only if

$$\mu^\omega * \nu^\omega = \nu^{\theta\omega} \qquad P\text{-a.s.} \tag{2.7}$$

in which case I call ν^ω a μ^ω-stationary family.

Set $\Xi = \Phi^{\mathbb{Z}_+} = \{\xi = (\xi_0, \xi_1, \dots), \xi_i \in \Phi\}$ and $\Pi^\omega = \prod_{i=0}^\infty \mu^{\theta^i\omega}$. Let $\tau : \Omega \times \mathcal{V} \times \Xi \to \Omega \times \mathcal{V} \times \Xi$ be the skew product transformations acting by $\tau(\omega, v, \xi) = (\theta\omega, \xi_0 v, \sigma\xi)$, $\xi \in \Xi$ where σ is the left shift on Ξ. Let $\varphi_0 : \Omega \times \Xi \to \Phi$ be the measurable map $\varphi_0^\omega(\xi) = \xi_0 \in \Phi$. Then $\varphi_i^\omega(\xi) = \xi_i = \varphi_0^{\theta^i\omega}(\sigma^i\xi)$ yield a sequence $\varphi_0^\omega, \varphi_1^\omega, \dots$ of independent random Borel maps of $\mathcal{V}$ such that φ_i^ω has the distribution $\mu^{\theta^i\omega}$. Now the Markov chain Z_n^ω can be written in the form $Z_n^\omega = \varphi_{n-1}^\omega \circ \cdots \circ \varphi_1^\omega \circ \varphi_0^\omega v$ provided $Z_0^\omega = v$. The following result relates μ^ω-stationary families and τ-invariant measures.

2.1. Lemma. *Given a measurable in ω family $\nu^\omega \in \mathcal{P}(\mathcal{V})$, $\omega \in \Omega$ the following properties are equivalent*

(i) $\nu^\omega, \omega \in \Omega$ *is a μ^ω-stationary family;*

(ii) $\nu_\Pi \in \mathcal{P}(\Omega \times \mathcal{V} \times \Xi)$, *defined by* $d\nu_\Pi(\omega, v, \xi) = d\Pi^\omega(\xi) d\nu^\omega(v) dP(\omega)$, *is τ-invariant.*

Proof. For any bounded measurable function g on $\Omega \times \mathcal{V} \times \Xi$,

$$\begin{aligned} \int g d\tau\nu_\Pi &= \int g(\theta\omega, \xi_0 v, \sigma\xi) d\Pi^\omega(\xi) d\nu^\omega(v) dP(\omega) \\ &= \int g(\theta\omega, \varphi v, \xi') d\Pi^{\theta\omega}(\xi') d\mu^\omega(\varphi) d\nu^\omega(v) dP(\omega) \\ &= \int g(\omega, \varphi v, \xi') d\Pi^\omega(\xi') d\mu^{\theta^{-1}\omega}(\varphi) d\nu^{\theta^{-1}\omega}(v) dP(\omega), \end{aligned} \tag{2.8}$$

and so $\tau\nu_\Pi = \nu_\Pi$ if and only if $\mu^{\theta^{-1}\omega} * \nu^{\theta^{-1}\omega} = \nu^\omega$ P-a.s. Thus (i) and (ii) are equivalent. □

The following result which generalizes Theorem I.2.1. in [Ki1] and which may be called the "random" random ergodic theorem enables me to employ ergodic theorems, in particular, the subadditive ergodic theorem which yields (1.2).

2.2. Proposition. *Given a measurable in ω family $\nu^\omega \in \mathcal{P}(\mathcal{V})$, $\omega \in \Omega$ the following properties are equivalent*

(i) $\nu^\omega, \omega \in \Omega$ *is a μ^ω-stationary ergodic family;*

(ii) ν_Π, *defined in (ii) of Lemma 2.1, is a τ-invariant ergodic measure.*

Proof. Call any measurable function h on $\Omega \times \mathcal{V}$ μ-harmonic if

$$R^\omega h_{\theta\omega}(v) = \int h_{\theta\omega}(\varphi v) d\mu^\omega(\varphi) = h_\omega(v) \tag{2.9}$$

for $P-$a.a.ω and ν^ω-a.a.v. Assuming that ν_Π is ergodic I shall show that all bounded μ-harmonic functions are a.s. constants. So let h be bounded and μ-harmonic. Let $\mathbb{I}_A$ denotes the indicator of a set A, i.e. $\mathbb{I}_A(v) = 1$ if $v \in A$ and $= 0$ otherwise. Considering the skew product Markov chain $\mathcal{Z}_n = (\theta^n\omega, Z_n^\omega)$ I derive in the same way as in the proof of Lemma I.2.4 from [Ki1] that for any C the function $\mathbb{I}_{A_C^\omega}(v)$, where $A_C^\omega = \{v : h_\omega(v) \geq C\}$, is μ-harmonic. But $\int \mathbb{I}_{A_C^{\theta\omega}}(\varphi v) d\mu^\omega(\varphi) = \mathbb{I}_{A_C^\omega}(v)$ means that $v \in A_C^\omega$ if and only if $\varphi v \in A_C^{\theta\omega}$ for μ^ω-a.a.φ. Therefore $\mathbb{I}_{A_C^{\theta\omega}}(\xi_0 v) = \mathbb{I}_{A_C^\omega}(v)$ ν_Π-a.s., and since ν_Π is ergodic with respect to τ then $\mathbb{I}_{A_C^\omega}(v) = \text{const}$ for

$P-$a.a.ω and $\nu^\omega-$a.a.v. It follows that h is constant too, and so (i) follows from (ii).

Next, assume that ν^ω is an ergodic μ^ω-stationary family and let a bounded measurable function $g = g_\omega(v,\xi)$ on $\Omega \times \mathcal{V} \times \Xi$ satisfies $g \circ \tau = g$ ν_Π-a.s. Then

$$
\begin{aligned}
(2.10) \qquad g_\omega^{(0)}(v) &\overset{\text{def}}{=} \int g_\omega(v,\xi)d\Pi^\omega(\xi) = \int g_{\theta\omega}(\xi_0 v, \sigma\xi)d\Pi^\omega(\xi)\\
&= \int g_{\theta\omega}(\varphi v, \xi')d\mu^\omega(\varphi)d\Pi^{\theta\omega}(\xi') = \int g_{\theta\omega}^{(0)}(\varphi v)d\mu^\omega(\varphi).
\end{aligned}
$$

Hence $g^{(0)}$ is μ-harmonic, and so $g_\omega^{(0)}(v) = \text{const}$ for $P-$a.a.ω and $\nu^\omega-$a.a.v since ν^ω is an ergodic family. Let $\mathcal{F}_n^\omega$ be the σ-algebra generated by $\varphi_0^\omega, \varphi_1^\omega, \dots, \varphi_n^\omega$. Set

$$
g_\omega^{(n)}(v;\xi_0,\dots,\xi_n) = E(g_\omega(v,\cdot)|\mathcal{F}_n^\omega)(\xi)
$$

then the τ-invariance of g yields

(2.11)

$$
\begin{aligned}
g_\omega^{(n)}(v;\xi_0,\dots,\xi_n) &= \int g_{\theta^{n+1}\omega}(\xi_n \circ \cdots \circ \xi_1 \circ \xi_0 v, \sigma^n\xi)d\Pi^\omega(\xi)\\
&= \int g_{\theta^{n+1}\omega}(\varphi_n \circ \cdots \circ \varphi_1 \circ \varphi_0 v, \xi')d\mu^\omega(\varphi_0)\dots d\mu^{\theta^n\omega}(\varphi_n)d\Pi^{\theta^{n+1}\omega}(\xi')\\
&= \int g_{\theta^{n+1}\omega}^{(0)}(\varphi_n \circ \cdots \circ \varphi_0 v)d\mu^\omega(\varphi_0)\dots d\mu^{\theta^n\omega}(\varphi_n) = \text{const}
\end{aligned}
$$

for $P-$a.a.ω, $\nu^\omega-$a.a.v, Π^ω-a.a.ξ. It follows that $g_\omega(v,\cdot)$ depends only on the tail σ-field $\bigcap_{n=1}^\infty \sigma\{\varphi_n^\omega, \varphi_{n+1}^\omega, \dots\}$ and since $\varphi_0^\omega, \varphi_1^\omega, \dots$ are independent then the zero-one law yields that g is constant ν_Π-a.s. $\square$

Consider another Markov chain

$$
\tilde{Z}_n^\omega = (Z_n^\omega, \varphi_n^\omega), \quad n = 1,2,\dots, \quad \tilde{Z}_0^\omega = (v, \varphi_0^\omega)
$$

with the state space $\mathcal{V} \times \Phi$ and with the transition probabilities

$$
(2.12) \qquad \tilde{R}^\omega((v,\varphi), U \times \Gamma) = \delta_{\varphi v}(U)\mu^{\theta\omega}(\Gamma), \quad U \subset \mathcal{V}, \quad \Gamma \subset \Phi
$$

so that the corresponding transition operator acts by the formula

$$
(2.13) \qquad \tilde{R}^\omega g(v,\varphi) = \int g(\varphi v, \psi)d\mu^{\theta\omega}(\psi).
$$

2.3. Proposition. *a) One has a one-to-one correspondence between R^ω-stationary families $\nu^\omega \in \mathcal{P}(\mathcal{V})$, $\omega \in \Omega$ and $\tilde{R}^\omega$-stationary families $\lambda^\omega \in \mathcal{P}(\mathcal{V}\times\Phi)$, $\omega \in \Omega$ which is given by*

$$d\lambda^\omega(v,\varphi) = d\nu^\omega(v)d\mu^\omega(\varphi); \tag{2.14}$$

(b) A R^ω-stationary family $\nu^\omega, \omega \in \Omega$ is ergodic if and only if the corresponding $\tilde{R}^\omega$-stationary family $\lambda^\omega, \omega \in \Omega$ is ergodic.

Proof. a) Let $\lambda^\omega \tilde{R}^\omega = \lambda^{\theta\omega}$ and set $\nu^\omega(U) = \int \mathbb{I}_U(\varphi v) d\lambda^{\theta^{-1}\omega}(v,\varphi)$ for any Borel $U \subset \mathcal{V}$. Then for any Borel $U \subset \mathcal{V}$ and $\Gamma \subset \Phi$,

$$\lambda^{\theta\omega}(U\times\Gamma) = \lambda^\omega \tilde{R}^\omega(U\times\Gamma) = \int d\lambda^\omega(v,\varphi)\tilde{R}^\omega((v,\varphi),U\times\Gamma)$$
$$= \mu^{\theta\omega}(\Gamma)\int d\lambda^\omega(v,\varphi)\mathbb{I}_U(\varphi v) = \mu^{\theta\omega}(\Gamma)\nu^{\theta\omega}(U),$$

and so $d\lambda^\omega(v,\varphi) = d\nu^\omega(v)d\mu^\omega(\varphi)$. But in this case

$$\int d\lambda^\omega(v,\varphi)\tilde{R}^\omega((v,\varphi),U\times\Gamma) = \mu^{\theta\omega}(\Gamma)\int \mathbb{I}_U d\mu^\omega * \nu^\omega.$$

Thus $\lambda^\omega\tilde{R}^\omega = \lambda^{\theta\omega}$ if and only if (2.14) holds true and $\mu^\omega * \nu^\omega = \nu^{\theta\omega}$.

b) Let λ^ω, $\omega \in \Omega$ be an ergodic $\tilde{R}^\omega$-stationary family satisfying (2.14). Let $A \subset \Omega\times\mathcal{V}$ be a measurable set such that its sections $A_\omega = \{v \in \mathcal{V} : (\omega, v) \in A\}$ satisfy (2.3) which for $R^\omega(v,\cdot)$ given by (2.4) means that $\varphi v \in A_{\theta\omega}$ for μ^ω-a.a.φ, ν^ω-a.a.$v \in A_\omega$, and $P-$a.a.ω. But then for λ^ω-a.a.(v,φ) and $P-$a.a.ω,

$$\tilde{R}^\omega \mathbb{I}_{A_{\theta\omega}\times\Phi}(v,\varphi) = \int \mathbb{I}_{A_{\theta\omega}\times\Phi}(\varphi v,\psi)d\mu^{\theta\omega}(\psi) = \mathbb{I}_{A_{\theta\omega}}(\varphi v) = \mathbb{I}_{A_\omega\times\Phi}(v,\varphi).$$

Since λ^ω, $\omega \in \Omega$ is an ergodic family one must have either $\lambda^\omega(A_\omega\times\Phi) = \nu^\omega(A_\omega) = 1$ P-a.s. or $\lambda^\omega(A_\omega\times\Phi) = \nu^\omega(A_\omega) = 0$ P-a.s., and so ν^ω, $\omega \in \Omega$ is an ergodic family.

It remains to show that if a R^ω-stationary family ν^ω, $\omega \in \Omega$ is ergodic then the $\tilde{R}^\omega$-stationary family λ^ω, $\omega \in \Omega$ determined by (2.14) is also ergodic. Let $h = h_\omega(v,\varphi)$ be a bounded measurable function on $\mathcal{V}\times\Phi$ satisfying for λ^ω-a.a.(v,φ) and $P-$a.a.ω,

$$\tilde{R}^\omega h_{\theta\omega}(v,\varphi) = \int h_{\theta\omega}(\varphi v,\psi)d\mu^{\theta\omega}(\psi) = h_\omega(v,\varphi). \tag{2.15}$$

Set $\tilde h_\omega(v)=\int h_\omega(v,\varphi)d\mu^\omega(\varphi)=E_{\Pi^\omega}h_\omega(v,\varphi_0^\omega)$ where, again, $\Pi^\omega=\prod_{i=0}^\infty\mu^{\theta^i\omega}$ and E_{Π^ω} is the corresponding expectation. By (2.15), for $\nu^\omega-$a.a.v and $P-$a.a.ω,

$$\tilde h_\omega(v)=\int\tilde h_{\theta\omega}(\varphi v)d\mu^\omega(\varphi)=R^\omega\tilde h_{\theta\omega}(v) \tag{2.16}$$

which together with the ergodicity of the family ν^ω, $\omega\in\Omega$ imply that $\tilde h_\omega(v)=C=$ const for $\nu^\omega-$a.a.v and $P-$a.a.ω. This means that if $\Gamma_\omega=\{v\in V:\tilde h_\omega(v)=C\}$ then $\nu^\omega(\Gamma_\omega)=1$ for $P-$a.a.ω. Thus for $P-$a.a.ω,

$$1=\int\mathbb{I}_{\Gamma_{\theta\omega}}d\nu^{\theta\omega}=\int\mathbb{I}_{\Gamma_{\theta\omega}}d\mu^\omega*\nu^\omega=\int\mathbb{I}_{\Gamma_{\theta\omega}}(\varphi v)d\mu^\omega(\varphi)d\nu^\omega(v),$$

i.e. $\tilde h_{\theta\omega}(\varphi v)=C$ for $\nu^\omega-$a.a.v and μ^ω-a.a.φ. This together with (2.15) yield that $h_\omega(v,\varphi)=C$ for $P-$a.a.ω, ν^ω-a.a v, μ^ω-a.a.φ, completing the proof of Proposition 2.3. $\square$

Next, I consider a multiplicative Markov process $Y_n^\omega=(X_n^\omega,M_n^\omega)$, $X_n^\omega\in\mathcal{X}$, $M_n^\omega\in GL(d,\mathbb{R})$ in random environments with the discrete time $n\in\mathbb{Z}_+$ and transition probabilities $Q^\omega((x,M),U\times V)=Q^\omega(1,(x,M),U\times V)$ satisfying (1.5). The skew product Markov chain $Y_n=(\theta^n\omega,X_n^\omega,M_n^\omega)$ is a multiplicative Markov process (in the deterministic environment) with transition probabilities

$$Q((\omega,x,M),\Gamma\times U\times V)=\delta_{\theta\omega}(\Gamma)Q^\omega((x,M)),U\times V). \tag{2.17}$$

Let Φ and Ψ be the spaces of Borel maps of $\mathcal{X}\times GL(d,\mathbb{R})$ and of $\Omega\times\mathcal{X}\times GL(d,\mathbb{R})$, respectively, into itself and let $\pi:\Omega\times\mathcal{X}\times GL(d,\mathbb{R})\to\mathcal{X}\times GL(d,\mathbb{R})$ be the natural projection on two last factors. Denote by $\pi_\omega:\Psi\to\Phi$ the corresponding projections acting by $(\pi_\omega G)(x,M)=\pi(G(\omega,x,M))$. Let $\mathcal{T}_\Phi$ and $\mathcal{T}_\Psi$ be the subsets of maps from Φ and Ψ, respectively, acting by $F(x,M)=(f_F(x),T_F(x)M)$ and $G(\omega,x,M)=(\theta\omega,g_G^\omega(x),T_G^\omega(x)M)$ for some Borel maps f_F and g_G^ω of $\mathcal{X}$ into itself and for some Borel measurable $GL(d,\mathbb{R})$-valued functions $T_F(x)$ and $T_G^\omega(x)$.

2.4. Proposition. *(cf. Lemma 2.6 in [Bo1]) There exists a probability measure μ on Ψ such that $\mu(\mathcal{T}_\Psi)=1$ and for any measurable $\Gamma\subset\Omega$, $U\subset\mathcal{X}$, $V\subset GL(d,\mathbb{R})$,*

$$\mu\{G:G(\omega,x,M)=(\theta\omega,g_G^\omega(x),T_G^\omega(x)M)\in\Gamma\times U\times V\}=Q((\omega,x,M),\Gamma\times U\times V), \tag{2.18}$$

and so $\mu^\omega=\pi_\omega\mu$ *satisfies* $\mu^\omega(\mathcal{T}_\Phi)=1$ *and*

$$\mu^\omega\{F: F(x,M)=(f_F(x),T_F(x)M)\}=Q^\omega((x,M),U\times V). \tag{2.19}$$

It follows that if $\{G_i\}_{i\geq 0}$ *is a sequence of independent random maps from* $\mathcal{T}_\Psi$ *all having the same distribution* μ *then* $\tilde Y_n=G_{n-1}\circ\cdots G_1\circ G_0(\omega,x,M)$ *is a version of the Markov chain* Y_n *(i.e. both processes have the same distributions) provided* $Y_0=\tilde Y_0=(\omega,x,M)$. *Finally, if* $F_i^\omega=\pi_{\theta^i\omega}G_i$ *then* $\tilde Y_n^\omega=F^\omega_{n-1}\circ\cdots\circ F_1^\omega\circ F_0^\omega(x,M)$ *is a version of the Markov chain (in random environments)* Y_n^ω.

Proof. By Theorem I.1.1 form [Ki1] there exists a probability measure m on the space Ψ such that for any Borel $U\subset\mathcal{X}$, $V\subset GL(d,\mathbb{R})$, $\Gamma\subset\Omega$ and $x\in\mathcal{X}$, $M\in GL(d,\mathbb{R})$, $\omega\in\Omega$,

$$Q((\omega,x,M)\in\Gamma\times U\times V)=m\{g\in\Psi: g(\omega,x,M)\in\Gamma\times U\times V\}. \tag{2.20}$$

Denote by $\mathbb{P}$ the product measure $m^{\mathbb{Z}_+}$ and let $g_0,g_1,g_2,\dots$ be a sequence of independent random maps all having the same distribution m.. Then the Markov chain $Z_n=g_{n-1}\circ\cdots\circ g_1\circ g_0(\omega,x,M)$ is a version of the skew product Markov chain Y_n. In view of (2.17) one can write $g_k(\omega,x,\mathrm{Id})=(\theta\omega,g_k^\omega(x),T_k^\omega(x))$ with $g_k^\omega(x)\in\mathcal{X}$ and $T_k^\omega(x)\in GL(d,\mathbb{R})$.

For $n=0,1,\dots$ and each $x\in\mathcal{X}$, $u\in\mathbb{R}^d$, $\omega\in\Omega$ define independent random bundle maps G_n and F_n^ω by $G_n(\omega,x,u)=(\theta\omega,g_n^\omega(x),T_n^\omega(x)u)$ and $F_n^\omega(x,u)=(g_n^{\theta^n\omega}(x),T_n^{\theta^n\omega}(x)u)$. By construction the distribution of G_n does not depend on n and I denote it by μ. Then F_n^ω has the distribution $\mu^{\theta^n\omega}$ where $\mu^\omega=\pi_\omega\mu$ for each $\omega\in\Omega$. Set $F(n,\omega)=F^\omega_{n-1}\circ\cdots\circ F_1^\omega\circ F_0^\omega$ then $F(n,\omega)(x,u)=(g(n,\omega,x),T(n,\omega,x)u)$ where $g(0,\omega,x)=x$, $T(0,\omega,x)=\mathrm{Id}$ and for $n\geq 1$,

$$\begin{aligned}&g(n,\omega,x)=g_{n-1}^{\theta^{n-1}\omega}\circ\cdots\circ g_1^{\theta\omega}\circ g_0^\omega(x)\ \text{ and}\\ &T(n,\omega,x)=T_{n-1}^{\theta^{n-1}\omega}(g(n-2,\omega,x))\cdots T_1^{\theta\omega}(g(1,\omega,x))T_0^\omega(x).\end{aligned} \tag{2.21}$$

Set $W_n^\omega=(g(n,\omega,x),T(n,\omega,x)M)$, $x\in\mathcal{X}$, $M\in GL(d,\mathbb{R})$ and denote by $\mathcal{F}_n^\omega$, $n=0,1,\dots$ the σ-algebra generated by $\{W_k^\omega, k=0,1,\dots,n-1\}$. Then for any

Borel $U \subset \mathcal{X}$ and $V \subset GL(d,\mathbb{R})$ assuming that $W_n^\omega = (y, M)$ one has by (1.5), (2.17), and (2.20),

(2.22)
$$\begin{aligned}
&\mathbb{P}\{W_{n+1}^\omega \in U \times V | \mathcal{F}_n^\omega\} = \mathbb{P}\{g_n^{\theta^n\omega}(y) \in U, T_n^{\theta^n\omega}(y)M \in V\}\\
&= \mathbb{P}\{g_n(\theta^n\omega, y, \mathrm{Id}) \in \Omega \times U \times VM^{-1}\} = m\{g \in \Psi : g(\theta^n\omega, y, \mathrm{Id})\\
&\in \Omega \times U \times VM^{-1}\} = Q^{\theta^n\omega}((y, M), U \times V).
\end{aligned}$$

Thus $(\theta^n\omega, W_n^\omega)$ and W_n^ω are versions of the Markov chains Y_n and Y_n^ω, respectively, completing the proof of Proposition 2.4. $\square$

Proposition 2.4 says, essentially, that the behavior of $M_n^\omega u$, $u \in \mathbb{R}^d$ for a multiplicative Markov process $Y_n^\omega = (X_n^\omega, M_n^\omega)$, $X_0^\omega = x$ is the same as the behavior of $T(n, \omega, x)u$ given by (2.21) for some independent random bundle map $F_0^\omega, F_1^\omega, \dots,$ and so I can deal only with the latter set up.

3. Invariant filtration

Let $\mathbb{P}^{d-1}$ be the $(d-1)$-dimensional projective space whose points can be identified with lines passing through the origin $\mathbb{R}^d$. Since all matrices from the group $GL(d,\mathbb{R})$ send these lines to themselves, one has a natural action of $GL(d,\mathbb{R})$ on $\mathbb{P}^{d-1}$ which induces the action of $\mathcal{T}$ on $\mathbb{P}E = \mathcal{X} \times \mathbb{P}^{d-1}$ by the formula (1.1), only now $a \in \mathbb{P}^{d-1}$.

Note that the space of probability measures ν on $\Omega \times \mathbb{P}E$ having desintegrations $d\nu(\omega, x, u) = d\nu_x^\omega(u)d\rho(\omega, x) = d\nu_x^\omega(u)d\rho^\omega(x)dP(\omega)$ is compact with respect to the topology determined by duality with the space $L_\rho^1(\Omega\times\mathcal{X}, C(\mathbb{P}^{d-1}))$ which consists of measurable maps $\varphi : \Omega \times \mathcal{X} \to C(\mathbb{P}^{d-1})$ such that $\int \sup_{u\in\mathbb{P}^{d-1}} |\varphi_{\omega,x}(u)|d\rho(\omega, x) < \infty$. Those of such ν whose desintegrations ν^ω are μ^ω-stationary families form a closed nonempty subset.

The following result was proved as Theorem III.1.2. in [Ki1] for the partial case when Ω consists just of one point or, in other words, for the case of identically distributed independent random bundle maps.

3.1. Theorem. *Let μ^ω, $\omega \in \Omega$ be a measurable family of probability measures on the space $\mathcal{T}$ of bundle maps $F = (f_F, T_F)$ acting on $E = \mathcal{X} \times \mathbb{R}^d$ by the formula*

(1.1). Suppose that $\rho^\omega \in \mathcal{P}(\mathcal{X})$, $\omega \in \Omega$ is a μ^ω-stationary ergodic family such that

$$\int (\log^+ \|T_F(x)\| + \log^+ \|T_F^{-1}(x)\|) d\rho^\omega(x) d\mu^\omega(F) dP(\omega) < \infty. \tag{3.1}$$

Then there exist Borel sets $\mathcal{X}_\rho^\omega \subset \mathcal{X}$ such that P-a.s.

$$\rho^\omega(\mathcal{X}_\rho^\omega) = 1, \quad f_F \mathcal{X}_\rho^\omega = \mathcal{X}_\rho^{\theta\omega} \ \text{for } \mu^\omega\text{-a.a.}F \tag{3.2}$$

and for any $x \in \mathcal{X}_\rho^\omega$ there exists a sequence of linear subspaces

$$0 = \mathcal{L}_{x,\omega}^{r(\rho)+1} \subset \mathcal{L}_{x,\omega}^{r(\rho)} \subset \ldots \subset \mathcal{L}_{x,\omega}^1 \subset \mathcal{L}_{x,\omega}^0 = \mathbb{R}^d \tag{3.3}$$

and a sequence of numbers $\beta_i(\rho) = \beta_i(P, \mu, \rho)$, $-\infty < \beta_{r(\rho)}(\rho) < \ldots < \beta_1(\rho) < \beta_0(\rho) < \infty$ such that Π^ω-a.s. $\beta_0(\rho)$ is given by (1.2) and if $u \in \mathcal{L}_{x,\omega}^i \setminus \mathcal{L}_{x,\omega}^{i+1}$, $i = 0, 1, \ldots, r(\rho)$ then Π^ω-a.s.,

$$\lim_{n\to\infty} \frac{1}{n} \log \|T(n, \omega, x)u\| = \beta_i(\rho) \tag{3.4}$$

where $T(n, \omega, x)$ is the same as in (1.2).

The numbers $\beta_i(\rho)$ are the values which the integrals

$$\gamma(\nu) \stackrel{def}{=} \int \log \frac{\|T_F(x)\hat{u}\|}{\|\hat{u}\|} d\nu_x^\omega(u) d\rho^\omega(x) d\mu^\omega(F) dP(\omega) \tag{3.5}$$

take on for different μ^ω-stationary ergodic families $\nu^\omega \in \mathcal{P}(\mathbb{P}E)$, $\omega \in \Omega$ having marginal ρ^ω on $\mathcal{X}$, where $\hat{u} \in \mathbb{R}^d$ is a nonzero vector on the line corresponding to $u \in \mathbb{P}^{d-1}$. Furthermore, P-a.s. the dimensions of $\mathcal{L}_{x,\omega}^i$ do not depend on x and ω, provided $x \in \mathcal{X}_\rho^\omega$, and $\mathcal{L}_\omega^i = \{\mathcal{L}_{x,\omega}^i\}$ form Borel measurable subbundles of $\mathcal{X}_\rho^\omega \times \mathbb{R}^d$ which satisfy

$$T_F \mathcal{L}_{x,\omega}^i = \mathcal{L}_{f_F x, \theta\omega}^i \tag{3.6}$$

for P-a.a.ω, μ^ω-a.a.F, and ρ^ω-a.a.x.

Proof. Let $\mathcal{Z}_n$ be a Markov chain on $\Omega \times \mathcal{X} \times GL(d, \mathbb{R})$ with transition probabilities

$$Q((\omega, x, M), \Gamma \times U \times V) = \delta_{\theta\omega}(\Gamma) Q^\omega((x, M), U \times V) \tag{3.7}$$

for Borel $\Gamma \subset \Omega$, $U \subset \mathcal{X}$, and $V \subset GL(d, \mathbb{R})$ where $Q^\omega(\cdot, \cdot)$ is defined by (1.7). Then $\mathcal{Z}_n$ is a multiplicative Markov process on $\tilde{\mathcal{X}} \times GL(d, \mathbb{R})$, where $\tilde{\mathcal{X}} = \Omega \times \mathcal{X}$, and so by Proposition 2.4 there exists a probability measure μ satisfying (2.18). Thus there exists a sequence of independent random bundle maps G_i, $i = 0, 1, \dots$ of $\tilde{E} = \tilde{\mathcal{X}} \times \mathbb{R}^d$ into itself over $g_{G_i} : \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}$ all having the same distribution μ and acting by $G_n(\omega, x, u) = (g_{G_n}(\omega, x), T_{G_n}(\omega, x)u)$ with $g_{G_n}(\omega, x) = (\theta\omega, g^{\omega}_{G_n}(x))$ where $\omega \in \Omega$, $x \in \mathcal{X}$, $u \in \mathbb{R}^d$, $g^{\omega}_{G_n}(x) \in \mathcal{X}$, and $T_{G_n}(\omega, x) = T^{\omega}_{G_n}(x) \in GL(d, \mathbb{R})$. Then $G_{n-1} \circ \cdots \circ G_1 \circ G_0(\omega, x, M)$ is a version of the Markov chain $\mathcal{Z}_n$ provided $\mathcal{Z}_0 = (\omega, x, M)$, and so without loss of generality I can assume that both objects coincide.

Define $\rho \in \mathcal{P}(\tilde{\mathcal{X}})$ by $d\rho(\omega, x) = d\rho^{\omega}(x)dP(\omega)$. As explained at the beginning of Section 2 ρ is μ-stationary (i.e. $\mu * \rho = \rho$) and ergodic if and only if ρ^{ω} is a μ^{ω}-stationary ergodic family and the latter holds true by the assumption. Let $\mathcal{T}$ and $\tilde{\mathcal{T}}$ be the spaces of bundle maps $F : E \to E$ and $G : \tilde{E} \to \tilde{E}$ acting by the formulas $F(x, u) = (f_F(x), T_F(x)u)$ and $G(\omega, x, u) = (\theta\omega, g^{\omega}_G(x), T_G(\omega, x)u)$, respectively, where $f_F, g^{\omega}_G : \mathcal{X} \to \mathcal{X}$ are Borel maps and $T_F(x), T_G(\omega, x) \in GL(d, \mathbb{R})$. Let $\pi_\omega : \tilde{\mathcal{T}} \to \mathcal{T}$ be determined by $\pi_\omega G(x, u) = \pi(G(\omega, x, u))$, where $\pi : \Omega \times \mathcal{X} \times \mathbb{R}^d \to \mathcal{X} \times \mathbb{R}^d$ is the natural projection, then by Proposition 2.4, $\mu^{\omega} = \pi_\omega \mu$. Thus if φ is a μ^{ω}-integrable function on $\mathcal{T}$ then

$$\int_{\tilde{\mathcal{T}}} \varphi(\pi_\omega G) d\mu(G) = \int_{\mathcal{T}} \varphi(F) d\mu^{\omega}(F). \tag{3.8}$$

This together with (3.1) and (3.5) give

$$\begin{aligned} &\int (\log^+ \|T_G(\omega, x)\| + \log^+ \|T_G^{-1}(\omega, x)\|) d\mu(G) d\rho(x) \\ &= \int (\log^+ \|T_F(x)\| + \log^+ \|T_F^{-1}(x)\|) d\mu^{\omega}(F) d\rho^{\omega}(x) dP(\omega) < \infty \end{aligned} \tag{3.9}$$

and

$$\gamma(\nu) = \int \log \frac{\|T_G(\omega, x)\hat{u}\|}{\|\hat{u}\|} d\mu(G) d\nu(\omega, x, u) \tag{3.10}$$

where $d\nu(\omega, x, u) = d\nu^{\omega}(x, u)dP(\omega)$ and $d\nu^{\omega}(x, u) = d\nu^{\omega}_x(u)d\rho^{\omega}(x)$. As explained at the beginning of Section 2 ν is μ-stationary, i.e. $\mu * \nu = \nu$, and ergodic if and only if ν^{ω} is a μ^{ω}-stationary ergodic family. Now I apply Theorem III.1.2. from [Ki1] to the sequence of independent random bundle maps G_i, $i = 0, 1, \dots$ having the same distribution μ which in view of the above yields the assertions of Theorem 2.3. $\square$

Consider now a continuous time multiplicative Markov process $Y_t^\omega=(X_t^\omega,M_t^\omega)$, $M_0^\omega=\mathrm{Id}$ described in the end of Introduction and assume

$$\int\int \sup_{0\le t\le 1} E_x^\omega(\log^+\|M_t^\omega\|+\log^+\|(M_t^\omega)^{-1}\|)d\rho^\omega(x)dP(\omega)<\infty \tag{3.11}$$

where E_x^ω is the expectation given that $X_0^\omega=x$. Then in the same way as in Lemma 2.6 from [Bo1] I derive from (3.11) that for any $u\in\mathbb{R}^d$, $P-$a.a.ω, P_x^ω-a.s.,

$$\lim_{n\to\infty} n^{-1}\log\|M_{t+n}^\omega u\|=\lim_{n\to\infty} n^{-1}\log\|M_n^\omega u\| \tag{3.12}$$

where P_x^ω is the path distribution of Y_t^ω given that $X_0^\omega=x$. This leads to the continuous time version of Theorem 3.1 where the corresponding spaces $\mathcal{L}_{x,\omega}^i$ satisfy

$$M_t^\omega\mathcal{L}_{x,\omega}^i=\mathcal{L}^i_{X_t^\omega,\theta^t\omega} \tag{3.13}$$

P_x^ω-a.s. for ρ^ω-a.a x and $P-$a.a.ω.

If $Y_t^\omega=(X_t^\omega,M_t^\omega)$ is given by the stochastic differential equation (1.8) with, say, bounded $\|A_i(x,\omega)\|$, $i=0,1,\dots,m$ then employing standard estimates of moments of stochastic integrals together with Gronwall's inequality one verifies that (3.11) will be satisfied in this case.

4. Largest Lyapunov exponent

The main result of this section is the following

4.1. Theorem. *In the set up of Theorem 3.1 suppose that there exists no μ^ω-stationary family $\nu^\omega\in\mathcal{P}(\mathbb{P}E)$ having marginal ρ^ω on $\mathcal{X}$ such that*

$$T_F(x)\nu_x^\omega=\nu_{f_Fx}^{\theta\omega} \text{ for } \rho^\omega\text{-a.a.}x,\ \mu^\omega\text{-a.a.}F,\quad \text{and } P\text{-a.a.}\omega. \tag{4.1}$$

Then $\lambda_0(\rho)(=\beta_0(\rho))$ satisfies

$$\lambda_0(\rho)>\frac{1}{d}\int\int\log|\det T_F(x)|d\rho^\omega(x)d\mu^\omega(F)dP(\omega). \tag{4.2}$$

In particular, if $\det T_F(x) = 1$ *for* ρ*-a.a.* x, μ^ω*-a.a.* F, *and* $P-$*a.a.*ω *then* $\lambda_0(\rho) > 0$.

Proof. Observe that the right hand side of (4.2) equals d^{-1} times the sum of all Lyapunov exponents of the random matrix product $T(n,\omega,x)$ (see, for instance, [Ar], Section 5.3) and since $\lambda_0(\rho)$ is the biggest such exponent the inequality (4.2) is equivalent to the claim that $\lambda_0(\rho)$ is larger than the minimal Lyapunov exponent $\lambda_{\min}(\rho)$ which for ρ^ω-a.a.x, Q^ω-a.a.ξ, and P-a.a.ω is given by

$$\lambda_{\min}(\rho) = -\lim_{n\to\infty} \frac{1}{n} \log \|T^{-1}(n,\omega,x,\xi)\|. \tag{4.3}$$

The existence of the limit follows from the subadditive ergodic theorem which together with Proposition 2.2 yields that $\lambda_{\min}(\rho)$ is constant.

Set

$$\mathcal{K} = \mathcal{X}^{\mathbb{Z}} \times \Xi = (\mathcal{X} \times \mathcal{T})^{\mathbb{Z}} = \{\kappa = ((x_i,\xi_i), i \in \mathbb{Z}), x_i \in \mathcal{X}, \xi_i \in \mathcal{T}\}$$

and let $\eta : \mathcal{K} \to \mathcal{K}$ be the shift transformation $(\eta\kappa)_i = \kappa_{i+1} = (x_{i+1}, \xi_{i+1})$. Introduce on $\mathcal{K}$ a probability measure R^ω which is the Markov measure corresponding to the process (X_n^ω, F_n^ω) with the initial distribution $\rho^\omega \times \mu^\omega$. Namely, if

$$R^\omega((x,F), U \times \Gamma) = \delta_{f_F x}(U)\mu^{\theta\omega}(\Gamma) \tag{4.4}$$

then for $-\infty < m < n < \infty$,

(4.5)
$$\begin{aligned} &R^\omega\{\kappa = ((x_i,F_i)) : x_j \in U_j,\ F_j \in \Gamma_j,\ j = m, m+1, \dots, n\} \\ &= \int\limits_{U_m \times \Gamma_m} \cdots \int\limits_{U_n \times \Gamma_n} d\rho^{\theta^m\omega}(x_m) d\mu^{\theta^m\omega}(F_m) R^{\theta^m\omega}((x_m,F_m), d(x_{m+1},F_{m+1})) \dots \\ &\dots R^{\theta^{n-1}\omega}((x_{n-1},F_{n-1}), d(x_n,F_n)). \end{aligned}$$

By Proposition 2.3, $\rho^\omega \times \mu^\omega$ is a R^ω-stationary family and it is ergodic if and only if ρ^ω is an ergodic family.

Set $T(\omega,\kappa) = T(\omega,(x_i,\xi_i)) = T^\omega_{\xi_0}(x_0)$. Then one has a stationary sequence of matrices $T(\theta^n\omega, \eta^n\kappa)$, $n \in \mathbb{Z}$ on the space $\Omega \times \mathcal{K}$ with the invariant measure R_P such that $dR_P(\omega,\kappa) = dR^\omega(\kappa)dP(\omega)$. Thus I have the set up of Theorem 1 from

[Le] which implies that if $\lambda_0(\rho)=\lambda_{\min}(\rho)$ then there exists a measurable in ω,κ family of measures $\nu_{\omega,\kappa}\in\mathcal{P}(\mathbb{P}^{d-1})$ such that

$$T(\omega,\kappa)\nu_{\omega,\kappa}=\nu_{\theta\omega,\eta\kappa}\qquad R_P\text{-a.s.}\tag{4.6}$$

and $\nu_{\omega,\kappa}$ is measurable with respect to the σ-algebra $\mathcal{G}_+\cap\mathcal{G}_-$ where $\mathcal{G}_+$ is generated by all $T(\theta^n\omega,\eta^n\kappa)$, $n\geq 0$ and $\mathcal{G}_-$ is generated by all $T(\theta^n\omega,\eta^n\kappa)$, $n<0$.

Let $\mathcal{G}_+^\omega=\{A^\omega=\{\kappa:(\omega,\kappa)\in A\},A\in\mathcal{G}_+\}$ and $\mathcal{G}_-^\omega=\{B^\omega=\{\kappa:(\omega,\kappa)\in B\},B\in\mathcal{G}_-\}$. Since for each $\omega\in\Omega$ the measure R^ω defines a Markov chain on $\mathcal{X}\times\mathcal{T}$ then

$$\mathcal{G}_+^\omega\cap\mathcal{G}_-^\omega=\mathcal{G}_0^\omega\subset\sigma\{X_0^\omega,F_0^\omega\}.$$

Thus $\nu_{\omega,\kappa}$ depends only on ω and κ_0 i.e. $\nu_{\omega,\kappa}=\nu_{\omega,x_0,\xi_0}$. By (4.6),

$$T^\omega_{\xi_0}(x_0)\nu_{\omega,x_0,\xi_0}=\nu_{\theta\omega,x_1,\xi_1}\qquad R_P\text{-a.s.}\tag{4.7}$$

Since $x_1=f_{\xi_0}x_0$ and the left hand side of (4.7) does not depend on ξ_1 then in fact,

$$\nu_{\theta\omega,x_1,\xi_1}=\nu_{\theta\omega,f_{\xi_0}x_0}\qquad R_P-\text{a.s.}$$

and it does not depend on ξ_1. Similarly, $\nu_{\omega,x_0,\xi_0}=\nu_{\theta\omega,f_{\xi_{-1}}x_{-1}}$ does not depend on ξ_0, i.e. $\nu_{\omega,x_0,\xi_0}=\nu_{\omega,x_0}$ R_P-a.s., and I arrive at (4.1). $\square$

Under (3.11) all Lyapunov exponents of the continuous time system are the same as for the same system considered only at integer times $n=0,1,\dots,$ and so Theorem 4.1 can be applied in this case, as well.

5. Simplicity of Lyapunov exponents

Let Γ be a measurable subset of Ω with $P(\Gamma)>0$. Set $n_\Gamma(\omega)=\min\{k>0:\theta^k\omega\in\Gamma\}$ then $\theta_\Gamma:\Gamma\to\Gamma$ acting by the formula $\theta_\Gamma\omega=\theta^{n_\Gamma(\omega)}\omega$ is called an induced transformation. Since P is an ergodic invariant measure of θ then (see, for instance, [Br], p. 30 or [CFS], p. 22) its normalized restriction to Γ, i.e. $P_\Gamma=(P(\Gamma))^{-1}P$, is an ergodic invariant measure of θ_Γ. Set $n_\Gamma^{(1)}=n_\Gamma$ and recursively $n_\Gamma^{(i+1)}=n_\Gamma^{(i)}+n_\Gamma\circ\theta^i_\Gamma$. Let $\varphi_x:\mathcal{T}\to GL(d,\mathbb{R})$ be the map acting by

$\varphi_x(F) = T_F(x)$. This map induces the map of $\mathcal{P}(\mathcal{T})$ into $\mathcal{P}(GL(d,\mathbb{R}))$ which will be denoted again by φ_x. For any $\omega \in \Gamma$ and $x \in \mathcal{X}$ set

$$\eta_\Gamma^\omega = \sum_{k=1}^{\infty} 2^{-k} \mu_\Gamma^{\theta_\Gamma^{k-1}\omega} * \cdots * \mu_\Gamma^{\theta_\Gamma \omega} * \mu_\Gamma^\omega \text{ and } \zeta_\Gamma^{\omega,x} = \varphi_x \eta_\Gamma^\omega \tag{5.1}$$

where $\mu_\Gamma^\omega = \mu^{\theta^{n_\Gamma(\omega)-1}\omega} * \cdots * \mu^{\theta\omega} * \mu^\omega$.

Recall, that a measure $\nu \in \mathcal{P}(\mathbb{P}^{d-1})$ is called proper if $\nu(L) = 0$ for any $L \subset \mathbb{P}^{d-1}$ corresponding to a proper linear subspace of $\mathbb{R}^d$. A subset $S \subset GL(d,\mathbb{R})$ is called $\kappa-$contracting if there exists a sequence $A_n \in S$, $n = 1, 2, ...$ for which $\|A_n\|^{-1}A_n$ converges to a matrix A of rank $\leq \kappa$. If such a sequence $\{A_n\}$ can be found in S for any $u \in \mathbb{R}^d$, $u \neq 0$ with the limiting matrix A satisfying $Au \neq 0$ then I shall call S strongly $\kappa-$contracting. Let ρ^ω be a μ^ω-stationary ergodic family from $\mathcal{P}(\mathcal{X})$.

5.1. Assumption.

(i) *There exists a measurable set $\Gamma \subset \Omega$ with $P(\Gamma) > 0$, a compact subset $\mathcal{N} \subset \mathcal{P}(\mathbb{P}^{d-1})$ (with respect to the weak convergence topology) consisting of proper measures, and a measurable in ω, x family $\nu_x^\omega \in \mathcal{N}$, $x \in \mathcal{X}$, $\omega \in \Gamma$ such that for P_Γ-a.a.ω and ρ^ω-a.a.x,*

$$\int T_F^*(x) \nu_{f_F x}^{\theta_\Gamma \omega} d\mu_\Gamma^\omega(F) = \nu_x^\omega \tag{5.2}$$

where M^* is the conjugate of a matrix $M \in GL(d,\mathbb{R})$;

(ii) *For Γ from (i) there is an integer $\kappa \geq 1$ such that* $\operatorname{supp}\zeta_\Gamma^{\omega,x}$ *is strongly $\kappa-$contracting for P_Γ-a.a.ω and ρ^ω-a.a.x.*

5.2. Theorem. *Let ρ^ω be a μ^ω-stationary ergodic family from $\mathcal{P}(\mathcal{X})$ satisfying (3.1) and suppose that Assumption 5.1 holds true. Then $\lambda_0(\rho) > \lambda_\kappa(\rho)$ where, recall, $\lambda_0(\rho) \geq \lambda_1(\rho) \geq \ldots \geq \lambda_{d-1}(\rho)$ are Lyapunov exponents of the sequence of random bundle maps F_i^ω, $i = 0, 1, \ldots$.*

Proof. Consider the multiplicative Markov process (Z_k, M_k), where $Z_k = (\theta_\Gamma^{k-1}\omega, X^\omega_{n_\Gamma^{(k)}(\omega)})$, and $M_k = T(n_\Gamma^{(k)}(\omega), \omega, x)$, $\omega \in \Gamma$, $x \in \mathcal{X}$, which has the transition probabilities

$$R_\Gamma(((\omega,x),\mathrm{Id}), U \times V \times W) = \delta_{\theta_\Gamma\omega}(U)\mu_\Gamma^\omega\{F : f_F x \in V, T_F(x) \in W\} \tag{5.3}$$

for all measurable $U \subset \Omega$, $V \subset \mathcal{X}$, and $W \subset GL(d,\mathbb{R})$. Since ρ^ω is μ^ω-stationary then $\mu^\omega_\Gamma * \rho^\omega = \rho^{\theta_\Gamma \omega}$, i.e. ρ^ω is a μ^ω_Γ-stationary family. Let $\tau : \Omega \times \mathcal{X} \times \Xi \to \Omega \times \mathcal{X} \times \Xi$ be the skew product transformation acting by $\tau(\omega, x, \xi) = (\theta\omega, f_{\xi_0} x, \sigma\xi)$. Set $\tau_\Gamma(\omega, x, \xi) = \tau^{n_\Gamma(\omega)}(\omega, x, \xi)$. Since ρ^ω is an ergodic family then according to Lemma 2.1 and Proposition 2.2 the measure $\rho \in \mathcal{P}(\Omega \times \mathcal{X} \times \Xi)$ defined by $d\rho(\omega, x, \xi) = d\Pi^\omega(\xi) d\rho^\omega(x) dP(\omega)$ is τ-invariant and ergodic. Then by the general results on induced transformations (see [Br], p. 30 or [CFS], p. 22) the measure ρ_Γ defined by $d\rho_\Gamma(\omega, x, \xi) = d\Pi^\omega(\xi) d\rho^\omega(x) dP_\Gamma(\omega)$ is τ_Γ-invariant and ergodic. This together with Proposition 2.2 yield that $\rho^\omega_\Gamma, \omega \in \Gamma$ is a μ^ω_Γ-stationary ergodic family. Now by (3.1) and the ergodic theorem ρ_Γ-a.s,

(5.4)

$$\int (\log^+ \|M_1\| + \log^+ \|M_1^{-1}\|) d\rho_\Gamma = \lim_{k\to\infty} \frac{1}{k} \sum_{\ell=0}^{k-1} (\log^+ \|M_1\| + \log^+ \|M_1^{-1}\|) \circ \tau^\ell_\Gamma$$

$$\leq \lim_{k\to\infty} \frac{1}{k} \sum_{j=0}^{n_\Gamma^{(k)}(\omega)-1} (\log^+ \|T_{\xi_j}(X_j^\omega(\xi))\| + \log^+ \|T^{-1}_{\xi_j}(X^\omega_j(\xi))\|)$$

$$= (P(\Gamma))^{-1} \int (\log^+ \|T_F(x)\| + \log^+ \|T_F^{-1}(x)\|) d\rho^\omega(x) d\mu^\omega(F) dP(\omega) < \infty.$$

Let $\lambda_0^\Gamma(\rho) \geq \lambda_1^\Gamma(\rho) \geq \ldots \geq \lambda^\Gamma_{d-1}(\rho)$ be the Lyapunov exponents of the multiplicative Markov process (Z_k, M_k) then by the ergodic theorem $\lambda_j(\rho) = \lambda_j^\Gamma(\rho) P(\Gamma)$. The arguments above enable me to apply Proposition 3.3 from [Bo2] to (Z_k, M_k) which yields $\lambda_0^\Gamma(\rho) > \lambda^\Gamma_\kappa(\rho)$ and the assertion of Theorem 5.2. follows. $\square$

In general, it is not possible to verify directly Assumption 5.1 and in the remaining part of this section I shall derive it from more straightforward nondegeneracy conditions. Since the evolution in ω is quite degenerate here I cannot employ the corresponding results from [Bo2] any further and have to proceed in an ω-wise fashion.

Observe, that by Lemma 4.1 from [Bo2] there always exists a measurable in ω, x family $\nu^\omega_x \in \mathcal{P}(\mathbb{P}^{d-1})$ such that for P-a.a.ω and ρ^ω-a.a.x,

$$\int T_F^*(x) \nu^{\theta\omega}_{f_F x} d\mu^\omega(F) = \nu^\omega_x, \tag{5.5}$$

and so (5.2) holds true, as well.

To ensure other requirements of Assumption 5.1 I introduce the following assumption.

5.3. Assumption.

(i) *$\mathcal{X}$ is a compact metric space;*

(ii) *$\rho^\omega \in \mathcal{P}(\mathcal{X})$, $\omega \in \Omega$ is a μ^ω-stationary ergodic family and* $\mathrm{supp}\rho^\omega = \mathcal{X}$ *P-a.s.;*

(iii) *The operator Q^ω, acting by the formula*

$$Q^\omega g(x, M) = \int g(f_F x, T_F(x)M) d\mu^\omega(F), \tag{5.6}$$

maps the space of bounded continuous functions on $\mathcal{X} \times GL(d, \mathbb{R})$ into itself;

(iv) *There exist random variables $\gamma = \gamma_\omega \in (0, 1)$ and $C = C(\omega) \in (0, \infty)$ such that for P-a.a.ω, all $x \in \mathcal{X}$, each $n \in \mathbb{Z}_+$, and any Borel set $U \subset \mathcal{X}$,*

$$|q^{\theta^{-n}\omega}(n, x, U) - \rho^\omega(U)| \le C(\omega)(1 - \gamma_\omega)^n \tag{5.7}$$

where $q^\omega(n, x, U)$ is the n-step transition probability of the Markov chain X_k^ω appearing in (1.6).

It is easy to give simple sufficient conditions which ensure that (ii) and (iii) in Assumption 5.3 are satisfied. For instance, this will be the case when μ^ω-a.s. F is a continuous bundle map and P-a.s. the set $\{f_F x : F \in \mathrm{supp}\mu^\omega\}$ is dense in $\mathcal{X}$ for all x, though, in fact, much less is needed. The property (iv) holds true if the random Doeblin condition introduced in [Ki2] is satisfied. This means that there exist random variable $N = N_\omega \in \mathbb{Z}_+$, $\iota = \iota_\omega > 0$ and a measurable family $m^\omega \in \mathcal{P}(\mathcal{X})$ such that for P-a.a.ω, any $x \in \mathcal{X}$, and each Borel $U \subset \mathcal{X}$,

$$q^{\theta^{-N}\omega}(N, x, U) \ge \iota_\omega m^\omega(U). \tag{5.8}$$

I say that the family $\mu^\omega \in \mathcal{P}(\mathcal{T})$, $\omega \in \Omega$ is strongly irreducible if there exists no finite collection $\{V^{(1)}_{\omega,x}, V^{(2)}_{\omega,x}, \dots, V^{(k)}_{\omega,x}\}$ of proper subspaces of $\mathbb{R}^d$ (with 0 being not a proper subspace) measurably depending on $\omega \in \Omega$, $x \in \mathcal{X}$ and such that for P-a.a.ω and μ^ω-a.a.F,

$$T_F(x)(\bigcup_{i=1}^k V^{(i)}_{\omega,x}) = \bigcup_{i=1}^k V^{(i)}_{\theta\omega, f_F x}. \tag{5.9}$$

If (5.9) can not hold true only for $k = 1$ then I call the family $\mu^\omega \in \mathcal{P}(\mathcal{T})$, $\omega \in \Omega$ irreducible. It is clear that irreducibility and strong irreducibility follow if $\mathrm{supp}\mu^\omega$ is sufficiently large with positive probability. The following result is a generalization of Proposition 4.4 from [Bo2].

5.4. Theorem. *Let $\mu^\omega \in \mathcal{P}(\mathcal{X})$, $\omega \in \Omega$ be a strongly irreducible family and $\rho^\omega \in \mathcal{P}(\mathcal{X})$, $\omega \in \Omega$ be a μ^ω-stationary ergodic family. Suppose that Assumption 5.3 holds true then Assumption 5.1(i) is satisfied and for any $\varepsilon > 0$ the set Γ can be chosen there with $P(\Gamma) > 1 - \varepsilon$.*

Proof. Let $\nu_x^\omega \in \mathcal{P}(\mathbb{P}^{d-1})$, $\omega \in \Omega$, $x \in \mathcal{X}$ be a measurable family satisfying (5.5). Define a measurable map $g : \Omega \times \mathcal{X} \to \mathcal{P}(\mathbb{P}^{d-1})$ by $g_\omega(x) = \nu_x^\omega$. By a version of Lusin's theorem (see [Do], Section V.15) for any integer $n \geq 1$ there exists closed subsets $C_n^\omega \subset \mathcal{X}$ such that $g_\omega(x)$ is continuous in x on C_n^ω and $\rho^\omega(C_n^\omega) \geq 1 - \frac{1}{n}$. Take a continuous function φ on $\mathbb{P}^{d-1}$ and for each $\nu \in \mathcal{P}(\mathbb{P}^{d-1})$ set $r(\nu) = \int \varphi(u) d\nu(u)$ which defines a continuous function on $\mathcal{P}(\mathbb{P}^{d-1})$. Then $\beta_\omega(x, M) = r(M^* \nu_x^\omega) = \int \varphi(M^* u) d\nu_x^\omega(u)$ is a continuous function in $(x, M) \in C_n^\omega \times GL(d, \mathbb{R})$. By the Tietze extension theorem (see [Ku], §14) there exists a function $\beta_\omega^{(n)}(x, M)$ continuous in $(x, M) \in \mathcal{X} \times GL(d, \mathbb{R})$ and such that $\beta_\omega^{(n)}(x, M) = \beta_\omega(x, M)$ for all $(x, M) \in C_n^\omega \times GL(d, \mathbb{R})$ and $|\beta_\omega^{(n)}(x, M)| \leq \alpha = \sup_n |\varphi(u)|$. By (5.5) for $P-$a.a.ω and ρ^ω-a.a. x,

(5.10)
$$
\begin{aligned}
\beta_\omega(x, \mathrm{Id}) &= \int \varphi(u) d\nu_x^\omega(u) = \int\int \varphi(T_F^*(x) u) d\nu_{f_F x}^{\theta\omega}(u) d\mu^\omega(F) = \\
&= \int \beta_{\theta\omega}(f_F x, T_F(x)) d\mu^\omega(F) = Q^\omega \beta_{\theta\omega}(x, \mathrm{Id})
\end{aligned}
$$

with the operator Q^ω defined by (5.6). Set $Q_n^\omega = Q^\omega Q^{\theta\omega} \dots Q^{\theta^{n-1}\omega}$ then (5.10) yields that for $P-$a.a.ω, ρ^ω-a.a.x and all integers $n \geq 1$,

$$
r(g_\omega(x)) = \beta_\omega(x, \mathrm{Id}) = Q_n^\omega \beta_{\theta^n\omega}(x, \mathrm{Id}). \tag{5.11}
$$

Put $h_\omega^{(n)}(x, M) = \mathbb{I}_{C_n^\omega}(x)$ then for ω, x satisfying (5.11) I derive from (5.7) that

(5.12)
$$
\begin{aligned}
&|\int \varphi(u) d\nu_x^\omega(u) - Q_n^\omega \beta_{\theta^n\omega}^{(n)}(x, \mathrm{Id})| = |Q_n^\omega(\beta_{\theta^n\omega} - \beta_{\theta^n\omega}^{(n)})(x, \mathrm{Id})| \\
&\leq 2\alpha Q_n^\omega(1 - h_{\theta^n\omega}^{(n)})(x, \mathrm{Id}) = 2\alpha(1 - Q_n^\omega h_{\theta^n\omega}^{(n)}(x, \mathrm{Id})) \\
&= 2\alpha(1 - q^\omega(n, x, C_n^{\theta^n\omega})) \leq 2\alpha(1 - \rho^{\theta^n\omega}(C_n^{\theta^n\omega}) + C(\theta^n\omega)(1 - \gamma_{\theta^n\omega})^n) \\
&\leq 2\alpha(\frac{1}{n} + C(\theta^n\omega)(1 - \gamma_{\theta^n\omega})^n).
\end{aligned}
$$

Fix $L > 0$ large enough so that $\tilde{\Gamma}_L = \{\omega : C(\omega) \le L \text{ and } \gamma_\omega \ge L^{-1}\}$ satisfies $P(\tilde{\Gamma}_L) > 0$. Then by ergodicity of θ it follows that for $P-$a.a.ω there exists a sequence $n_i = n_i(\omega) \to \infty$ as $i \to \infty$ such that $\theta^{n_i(\omega)}\omega \in \tilde{\Gamma}_L$. Thus for P-a.a.ω and ρ^ω-a.a.x $\int \varphi(u) d\nu_x^\omega(u)$ is the uniform in x limit of the sequence $Q^\omega_{n_i(\omega)}\beta^{(n_i(\omega))}_{\theta^{n_i(\omega)}\omega}(x, \mathrm{Id})$.

In view of Assumption 5.3(iii) the latter sequence consists of continuous in x functions which together with Assumption 5.3(ii) yield that P-a.s. and ρ^ω-a.s., $\int \varphi(u) d\nu_x^\omega(u)$ coincides with a continuous in x function, i.e. it has a continuous in x modification. Applying this to a countable dense set of continuous functions φ on $\mathbb{P}^{d-1}$ I conclude that there exist a measurable in ω and continuous in x family ν_x^ω satisfying (5.5) for P-a.a.ω and ρ^ω-a.a.x. By Assumption 5.3(i) this implies, in particular, that for P-a.a.ω (5.5) holds true for all $x \in \mathcal{X}$.

Consider a specific representation of $(\Omega, \mathcal{A}, P)$ where Ω is an interval $[a, b)$ together with countably many points A_i, $i = 1, 2, \ldots$, P is the Lebesgue measure on $[a, b)$ and has atoms at A_i's, and $\mathcal{A}$ is the completion of the Borel σ-algebra on Ω. Then by a version of Lusin's theorem (see [Do], V.15) there exists a sequence of compact sets $\Gamma_n \subset \Omega$ such that $P(\Omega \setminus \Gamma_n) \le \frac{1}{n}$ and ν_x^ω is continuous in $(\omega, x) \in \Gamma_n \times \mathcal{X}$. Assumption 5.1(i) would follow if I show that for $P-$a.a.ω the measures ν_x^ω are proper for all $x \in \mathcal{X}$. Indeed, if this is true I can choose a sequence of closed sets $\tilde{\Gamma}_n \subset \Omega$ such that $P(\Omega \setminus \tilde{\Gamma}_n) \le \frac{1}{n}$ and ν_x^ω is proper when $\omega \in \tilde{\Gamma}_n$. Now, given $\varepsilon > 0$ take $\Gamma = \Gamma_n \cap \tilde{\Gamma}_n$ for some $n \ge 2\varepsilon^{-1}$. Then $\mathcal{N} = \{\nu_x^\omega, \omega \in \Gamma, x \in \mathcal{X}\}$ will be a compact set of proper measures.

In order to show that ν_x^ω are proper denote by $\Pi(\ell)$ the set of projective subspaces of $\mathbb{P}^{d-1}$ having the dimension ℓ and set $\ell^\omega(x) = \min\{\ell \in \{0, \ldots, d-1\} : \exists H \in \Pi(\ell), \nu_x^\omega(H) \ne 0\}$. Clearly, $\Pi(\ell^\omega(x))$ may contain at most countably many subspaces, and so I can define $r^\omega(x) = \max\{\nu_x^\omega(H) : H \in \Pi(\ell^\omega(x))\}$ and $L^\omega(x) = \{H \in \Pi(\ell^\omega(x)) : \nu_x^\omega(H) = r^\omega(x)\}$. As in Proposition 4.4 of [Bo2] I see that $\ell^\omega(x)$ and $r^\omega(x)$ are measurable on each set $\Gamma_n \times \mathcal{X}$ (where ν_x^ω is continuous), and so these functions are measurable on the whole $\Omega \times \mathcal{X}$.

Put $m^\omega = \operatorname*{essinf}_{x \in \mathcal{X}} \ell^\omega(x)$. I have to show that $m^\omega = d-1$ P-a.s. Set $\varphi^\omega(x) = r^\omega(x)\mathbb{I}_{\{x : \ell^\omega(x) = m^\omega\}}$. If $H \in \Pi(m^\omega)$ then $\nu_x^\omega(H) \le \varphi^\omega(x)$ for any $x \in \mathcal{X}$. If $\ell^\omega(x) = m^\omega$ and $H \in L^\omega(x)$ then by (5.5) (which is true now for all $x \in \mathcal{X}$) P-a.s. for all

$x \in \mathcal{X}$,

$$\varphi^\omega(x) = \nu_x^\omega(H) = \int \nu^{\theta\omega}_{f_F x}((T_F^*(x))^{-1}H)d\mu^\omega(F). \tag{5.13}$$

If $m^\omega = \dim H < m^{\theta\omega}$ this would imply that $\nu_x^\omega(H) = 0$, and so $m^\omega \geq m^{\theta\omega}$ P-a.s. By ergodicity of θ I conclude that $m^\omega = m = \text{const}$ P-a.s. But then P-a.s. the right hand side of (5.13) does not exceed $q^\omega \varphi^{\theta\omega}(x) \overset{\text{def}}{=} \int \varphi^{\theta\omega}(f_F x)d\mu^\omega(F)$. Hence

$$\varphi^\omega(x) \leq q^\omega \varphi^{\theta\omega}(x) \tag{5.14}$$

and since $\varphi^\omega(x) = 0$ if $\ell^\omega(x) \neq m^\omega$ then, in fact, (5.14) holds true for all $x \in \mathcal{X}$ and P-a.a.ω.

Set $q_n^\omega = q^\omega q^{\theta\omega} \dots q^{\theta^{n-1}\omega}$. Then (5.7) and (5.14) give that P-a.s. for all $x \in \mathcal{X}$,

$$\limsup_{n\to\infty} \varphi^{\theta^{-n}\omega}(x) \leq \lim_{n\to\infty} q_n^{\theta^{-(n-1)}\omega}\varphi^\omega(x) = \int \varphi^\omega(x)d\rho^\omega(x) \overset{\text{def}}{=} \alpha_\omega. \tag{5.15}$$

It follows by ergodicity of θ that P-a.s. $\varphi^\omega(x) \leq \alpha^\omega$ for all $x \in \mathcal{X}$ which together with (5.14) yield $\varphi^\omega(x) \leq \alpha^{\theta\omega}$ for P-a.a.ω and all $x \in \mathcal{X}$. The right hand side of (5.15) and $\varphi^\omega \leq \alpha^\omega$ imply that

$$\varphi^\omega(x) = \alpha^\omega > 0 \qquad \rho^\omega\text{-a.s.},\ P\text{-a.s.} \tag{5.16}$$

which together with $\varphi^\omega \leq \alpha^{\theta\omega}$ yield $\alpha^\omega \leq \alpha^{\theta\omega}$ P-a.s. and by ergodicity of θ I derive that $\alpha^\omega = \alpha = \text{const}$ P-a.s.

Set $U^\omega = \{x \in \mathcal{X} : q^\omega r^{\theta\omega}(x) = q^\omega\varphi^{\theta\omega}(x) = r^\omega(x) = \varphi^\omega(x) = \alpha,\ \ell^\omega(x) = m,\ \mu^\omega\{F : \ell^{\theta\omega}(f_F x) = m\} = 1\}$. Since $\mu^\omega * \rho^\omega = \rho^{\theta\omega}$ P-a.s. it follows from above that $\rho^\omega(U^\omega) = 1$ P-a.s. Let $x \in U^\omega$ and $H \in L^\omega(x)$. Then $H \in \Pi(m)$, and so $(T_F^*(x))^{-1}H \in \Pi(\ell^{\theta\omega}(f_F x))$ for μ^ω-a.s.F, which implies that $\nu^{\theta\omega}_{f_F x}((T_F^*(x))^{-1}H) \leq r^{\theta\omega}(f_F x)$. Therefore by (5.5) for P-a.a.ω and ρ^ω-a.a.x,

$$\begin{aligned} 0 &\geq \int (\nu^{\theta\omega}_{f_F x}((T_F^*(x))^{-1}H) - r^{\theta\omega}(f_F x))d\mu^\omega(F) \\ &= \nu_x^\omega(H) - q^\omega r^{\theta\omega}(x) = \varphi^\omega(x) - \alpha = 0. \end{aligned} \tag{5.17}$$

It follows that $\nu^{\theta\omega}_{f_F x}((T_F^*(x))^{-1}H) = r^{\theta\omega}(f_F x)$ for P-a.a.ω, ρ^ω-a.a.x, μ^ω-a.a.F, and so $H \in T_F^*(x)L^{\theta\omega}(f_F x)$. Since this is true for all $H \in L^\omega(x)$ I conclude that

$L^\omega(x) \subset T_F^*(x)L^{\theta\omega}(f_Fx)$ for P-a.a.ω, ρ^ω-a.a.x and μ^ω-a.a.F. Thus for such ω, x, F the number $n^\omega(x)$ of subspaces in $L^\omega(x)$ (which is clearly finite and whose measurability follows in the same way as in [Bo2]) satisfies $n^\omega(x) \leq n^{\theta\omega}(f_Fx)$. Since ρ^ω is an ergodic family, this together with Proposition 2.2 yield that $n^\omega(x) = n = \text{const}$ for P-a.a.ω and ρ^ω-a.a.x. It follows that $L^\omega(x) = T_F^*(x)L^{\theta\omega}(f_Fx)$ for P-a.a.ω, ρ^ω-a.a.x and μ^ω-a.a.F. This means that if $W^\omega(x)$ is the union of subspaces orthogonal to subspaces which form $L^\omega(x)$ then $T_F(x)W^\omega(x) = W^{\theta\omega}(f_Fx)$ for P-a.a.ω, ρ^ω-a.a.x, μ^ω-a.a.F which contradicts the strong irreducibility assumption unless $m = d-1$. $\square$

Next, I shall discuss sufficient conditions for Assumption 5.1(ii).

5.5. Theorem. *Let Γ be a measurable set with $P(\Gamma) > 0$ so that $\mu_\Gamma^\omega, \omega \in \Gamma$ is an irreducible family and either* $\text{supp}\zeta_\Gamma^{\omega,x}$ *is $\kappa-$contracting for $P_\Gamma-$a.a.ω and $\rho^\omega-$a.a.x or Assumption 5.3 holds true and for $P_\Gamma-$a.a.ω,*

$$\rho^\omega\{x : \text{supp}\zeta_\Gamma^{\omega,x} \textit{ is } \kappa\textit{-contracting}\} = \delta_\omega > 0 \tag{5.18}$$

where δ is a random variable. Then Assumption 5.1(ii) is satisfied.

Proof. I shall show first that Assumption 5.3 together with (5.18) yield that $\text{supp}\zeta_\Gamma^{\omega,x}$ is $\kappa-$contracting for $P_\Gamma-$ a.a.ω and for $\rho^\omega-$a.a.x (even for all x) and then I shall obtain that this together with the irreducibility of the family μ_Γ^ω imply Assumption 5.1(ii).

As before, denote by $Q^\omega(n,(x,M),\cdot)$ the $n-$step transition probability of the multiplicative Markov process in random environments Y_n^ω where $Q^\omega(1,(x,M),U) = Q^\omega((x,M),U) = \mu^\omega\{F : (f_Fx, T_F(x)M) \in U\}$. Let $n_\Gamma^{(i)} = n_\Gamma^{(i)}(\omega)$ be the arrival times at Γ defined at the beginning of Section 5, $\theta_\Gamma = \theta^{n_\Gamma^{(1)}}$, $D_{\omega,x}^{(k)}$ be the support of $Q^\omega(n_\Gamma^{(k)}(\omega),(x,Id),\cdot)$, and $S_{\omega,x}^{(k)}$ be the minimal closed subset of $GL(d,\mathbb{R})$ such that $Q^\omega(n_\Gamma^{(k)}(\omega),(x,Id),\mathcal{X}\times S_{\omega,x}^{(k)}) = 1$. Set $S_{\omega,x}^{(k)}(y) = \{M \in GL(d,\mathbb{R}) : (y,M) \in D_{\omega,x}^{(k)}\}$ and $A_{\omega,x}^{(k)} = \{y \in \mathcal{X} : S_{\omega,x}^{(k)}(y) \neq \emptyset\}$. It is not difficult to see that these are measurable sets (cf. [Bo2], Section 4). By the definition, $q^\omega(n_\Gamma^{(k)}(\omega),x,A_{\omega,x}^{(k)}) = 1$, and so by Assumption 5.3(iv),

$$\sup_{x\in\mathcal{X}} |1 - \rho^{\theta_\Gamma^k\omega}(A_{\omega,x}^{(k)})| \leq C(\theta_\Gamma^k\omega)(1-\gamma_{\theta_\Gamma^k\omega})^{n_\Gamma^{(k)}(\omega)} \tag{5.19}$$

Set $S_{\omega,x}=\cup_{k=0}^\infty S^{(k)}_{\omega,x}$ and $V_\omega=\{y\in\mathcal{X}:\ S_{\omega,y}$ is $\kappa-$contracting$\}$. It is clear that $S_{\omega,y}$ is $\kappa-$contracting if and only if $\mathrm{supp}\zeta_\Gamma^{\omega,y}$ is $\kappa-$contracting, and so by (5.18),

$$\rho^{\theta_\Gamma^k\omega}(V_{\theta_\Gamma^k\omega})=\delta_{\theta_\Gamma^k\omega}. \tag{5.20}$$

Observe that $S_{\theta^k_\Gamma\omega,y}S^{(k)}_{\omega,x}(y)\subset S_{\omega,x}$ for any $x,y\in\mathcal{X}$, $\omega\in\Omega$, and $n=0,1,...$ where the product of sets of matrices is understood as the set of corresponding products and the latter set is considered empty if one of the sets in the product is empty. Hence, if $S_{\theta^k\omega,y}$ is $\kappa-$contracting and $S^{(k)}_{\omega,x}(y)\neq\emptyset$ then $S_{\omega,x}$ is $\kappa-$ contracting. By (5.19) and (5.20) for $P_\Gamma-$a.a.ω I can choose $k=k(\omega)$ so that $\rho^{\theta_\Gamma^k\omega}(A^{(k)}_{\omega,x}\cap V_\omega)>0$, i.e. the set in brackets is not empty, and so it contains a point y. Thus for $P_\Gamma-$a.a.ω and all $x\in\mathcal{X}$ the set $S_{\omega,x}$ is $\kappa-$contracting, and so $\mathrm{supp}\zeta_\Gamma^{\omega,x}$ is $\kappa-$contracting, as well.

Assuming that $\mathrm{supp}\zeta_\Gamma^{\omega,x}$ is $\kappa-$contracting the set

$$L_{\omega,x}=\bigcap\{\mathrm{Ker}A:\ A=\lim_{n\to\infty}\|A_n\|^{-1}A_n,\ \mathrm{rank}A\leq\kappa\ \text{for some}\ A_n\in\mathrm{supp}\zeta_\Gamma^{\omega,x}\}$$

is well defined (where $\mathrm{Ker}A$ denotes the kernel of a matrix A) and it is either a proper subspace of $\mathbb{R}^d$ or it contains just 0. Then for any $g\in GL(d,\mathbb{R})$,

$$\begin{aligned}g^{-1}L_{\theta_\Gamma\omega,y}=&\bigcap\{\mathrm{Ker}(Ag):\ A=\lim_{n\to\infty}\|A_n\|^{-1}A_n,\ \mathrm{rank}A\leq\kappa\ \text{for some}\\ &A_n\in\mathrm{supp}\zeta_\Gamma^{\theta_\Gamma\omega,y}\}=\bigcap\{\mathrm{Ker}B:\ B=\lim_{n\to\infty}\|B_n\|^{-1}B_n,\ \mathrm{rank}B\leq\kappa\\ &\text{for some}\ B_n\in(\mathrm{supp}\zeta_\Gamma^{\theta_\Gamma\omega,y})g\}\end{aligned} \tag{5.21}$$

since $\mathrm{Ker}(\lim_{n\to\infty}\|B_ng^{-1}\|^{-1}B_n)=\mathrm{Ker}(\lim_{n\to\infty}\|B_n\|^{-1}B_n)$ (provided both limits exist) and $\mathrm{rank}B=\mathrm{rank}(Bg^{-1})$. Now if $g\in S^{(1)}_{\omega,x}(y)$ then $(\mathrm{supp}\zeta_\Gamma^{\theta_\Gamma\omega,y})g\subset\mathrm{supp}\zeta_\Gamma^{\omega,x}$, and so $g^{-1}L_{\theta_\Gamma\omega,y}\supset L_{\omega,x}$. It follows that for $P_\Gamma-$a.a.ω and $\mu_\Gamma^\omega-$a.a.F,

$$\dim L_{\theta_\Gamma\omega,f_Fx}\geq\dim L_{\omega,x}. \tag{5.22}$$

Since $\rho^\omega-$is an ergodic family, this together with Proposition 2.2 yield that (5.22) is, in fact, an equality for $P_\Gamma-$a.a.ω, $\mu_\Gamma^\omega-$a.a.F, and $\rho^\omega-$a.a.x, and so for such ω, F, and x,

$$L_{\theta_\Gamma\omega,f_Fx}=T_F(x)L_{\omega,x}. \tag{5.23}$$

But if μ_Γ^ω is an irreducible family then this is only possible if $L_{\omega,x} = 0$ for $P_\Gamma-$a.a.ω and $\rho^\omega-$a.a.x which implies Assumption 5.1(ii). $\square$

Under additional continuity assumptions it is possible to replace (5.18) by the condition that for $P_\Gamma-$a.a.ω there exists just one point x depending on ω such that supp$\zeta_\Gamma^{\omega,x}$ is $\kappa-$contracting (cf. [Bo2], Section 5). Since it is usually difficult to verify the above $\kappa-$contraction condition I shall give also a simpler sufficient condition based on richness of supports of μ_Γ^ω's.

5.6. Corollary. *Suppose that there exists a closed $V \subset GL(d, \mathbb{R})$ such that supp$\varphi_x\mu_\Gamma^\omega \supset V$ for P_Γ-a.a.ω and $\rho^\omega-$a.a.x and the minimal semigroup S generated by V is strongly irreducible (i.e. it does not leave invariant a finite union of proper subspaces of $\mathbb{R}^d$) and κ-contracting. Then Assumption 5.1(ii) holds true.*

Proof. Since supp$\zeta_\Gamma^{\omega,x}$ contains the semigroup S for $P_\Gamma-$a.a.ω and $\rho^\omega-$a.a.x then Assumption 5.1(ii) is satisfied in view of Theorem 5.5. $\square$

Note that if supp$\varphi_x\mu^\omega$ both contains V and the identity matrix Id for all x and $P-$a.a.ω then supp$\varphi_x(\mu^{\theta^k\omega} * \cdots * \mu^{\theta\omega} * \mu^\omega) \supset V$ for all $k = 0, 1, ...$ and, in particular, supp$\varphi_x\mu_\Gamma^\omega \supset V$. Recall, that if the algebraic (Zariski) closure of a semigroup S coincides with $GL(d, \mathbb{R})$ then S is 1-contracting, as well, as all its actions on exterior products (see [GM]). As usual, in order to derive that all Lyapunov exponents are different one has to ensure 1-contraction of the actions $F^{\wedge^k}(x, u_1 \wedge u_2 \wedge \dots u_k) = (f_F x, T_F(x)u_1 \wedge \dots \wedge T_F(x)u_k)$ on exterior products $u_1 \wedge u_2 \wedge \dots \wedge u_k$, $k = 1, 2, ..., d-1$.

Observe that (5.7) holds true for the continuous time case given by (1.8) under a version of Hörmander's hypoellipticity conditions which ensures that a Doeblin type condition from [Ki2] is satisfied.

6. Limit theorems

In this section I shall extend the machinery of [Bo1] to derive an ω-wise central limit theorem for "random" random bundle maps under the following condition

6.1. Assumption.

(i) *There exist random variables $\gamma = \gamma_\omega \in (0, 1)$ and $C = C_\omega \in (0, \infty)$ such that for*

P-a.a.ω, all $x\in\mathcal{X}$, each $n\in\mathbb{Z}_+$, and any bounded Borel function φ on $\mathcal{X}$,

$$|\int q^{\theta^{-n}\omega}(n,x,dy)\varphi(y)-\int\varphi(y)d\rho^\omega(y)|\le C_\omega(1-\gamma_\omega)^n\|\varphi(x)\| \tag{6.1}$$

where, as before, $\rho^\omega\in\mathcal{P}(\mathcal{X})$ is a μ^ω-stationary ergodic family and $\|\cdot\|$ is the supremum norm;

(ii) *There exists $a>0$ such that*

$$\int\left(\sup_{x\in\mathcal{X}}\int\exp(a\chi(T_F(x)))d\mu^\omega(F)\right)dP(\omega)<\infty \tag{6.2}$$

where $\chi(M)=\max(\log\|M\|,\log\|M^{-1}\|)$ for any $M\in GL(d,\mathbb{R})$;

(iii) *$\lambda_0(\rho)>\lambda_1(\rho)$, where, recall, $\lambda_0(\rho)\ge\lambda_1(\rho)\ge\ldots\ge\lambda_{d-1}(\rho)$ are the Lyapunov exponents.*

(iv) *The filtration (3.3) of Theorem 3.1 is trivial, i.e. the number $r(\rho)$ appearing there is 0.*

I note that Assumption 6.1(i) holds true under the random Doeblin condition (5.8) which can be shown exactly in the same way as in [Ki2]. Assumption 6.1(iii) is satisfied under conditions discussed in Section 5. Observe that when $d=2$ Assumption 6.1(iii) is satisfied under the conditions of Theorem 4.1. Assumption 6.1(iv) holds true under an irreducibility condition, i.e. when there are no nontrivial measurable subbundles satisfying (3.6) and the latter follows if the supports of measures $\zeta_\Omega^{\omega,x}$ are sufficiently large, for instance, contain open sets.

Denote by Y_n^ω the Markov chain (in random environment) (X_n^ω,M_n^ω) where $M_n^\omega=T(n,\omega,X_0^\omega)$ and let P_x^ω and E_x^ω be the probability and the expectation for $\{Y_n^\omega,n\ge0\}$ provided $Y_0=(x,\mathrm{Id})$.

6.2. Lemma. *Suppose that Assumption 6.1 holds true. Then uniformly in $x\in\mathcal{X}$ and u belonging to the unit sphere S^{d-1},*

$$\lim_{n\to\infty}n^{-1}E_x^\omega\log\|M_n^\omega u\|=\lambda_0(\rho)\ P\text{-a.s.} \tag{6.3}$$

and

$$\lim_{n\to\infty}n^{-1}\int E_x^\omega\log\|M_n^\omega u\|dP(\omega)=\lambda_0(\rho). \tag{6.4}$$

Proof. Observe that

$$\chi(T(n,\omega,x,\xi)) \le \sum_{k=0}^{n-1} \chi(T_{\xi_k}(X_k^\omega(\xi))), \tag{6.5}$$

and

$$E_x^\omega(\chi(T_{\xi_k}(X_k^\omega(\xi))))^2 = E_x^\omega E^{\theta^k\omega}_{X_k^\omega(\xi)}(T_{\xi_k}(X_0^{\theta^k\omega}(\xi)))^2 \le \sup_{x\in\mathcal{X}}\int(\chi(T_F(x)))^2 d\mu^{\theta^k\omega}(F).$$

Thus setting $\Xi_{n,k} = \{\xi : \frac{1}{n}\chi(T(n,\omega,x,\xi)) > K\}$ I have

(6.6)

$$\begin{aligned} &n^{-1}E_x^\omega \mathbb{I}_{\Xi_{n,K}}\chi(M_n^\omega) \le K^{-1}E_x^\omega \mathbb{I}_{\Xi_{n,K}}(\frac{1}{n}\chi(M_n^\omega))^2 \\ &\le K^{-1}n^{-1}\sum_{k=0}^{n-1} E_x^\omega(\chi(T_{\xi_k}(X_k^\omega(\xi))))^2 \le K^{-1}n^{-1}\sum_{k=0}^{n-1}\sup_{x\in\mathcal{X}}\int(\chi(T_F(x)))^2 d\mu^{\theta^k\omega}(F). \end{aligned}$$

By the ergodic theorem the right hand side of (6.6) converges P-a.s. to

$$K^{-1}\int\sup_{x\in\mathcal{X}}\int(\chi(T_F(x)))^2 d\mu^\omega(F)dP(\omega) < \infty \tag{6.7}$$

and the latter integral exists in view of Assumption 6.1(ii) and Jensen's inequality. It follows that the sequence $\{\frac{1}{n}\chi(T(n,\omega,x,\xi)), n \ge 0\}$ is uniformly integrable in ξ for all $x \in \mathcal{X}$ and P-a.a.ω.

Set $g_n^\omega(x) = \sup_{u\in S^{d-1}} |n^{-1}E_x^\omega \log\|M_n^\omega u\| - \lambda_0(\rho)|$. Then by (6.5) in the same way as in (6.6),

$$\begin{aligned} |g_n^\omega(x)| &\le |\lambda_0(\rho)| + n^{-1}E_x^\omega\chi(M_n^\omega) \\ &\le |\lambda_0(\rho)| + n^{-1}\sum_{k=0}^{n-1}\sup_{x\in\mathcal{X}}\int\chi(T_F(x))d\mu^{\theta^k\omega}(F). \end{aligned} \tag{6.8}$$

This together with the ergodic theorem yield that the sequence $g_n^\omega(x)$ is uniformly in x bounded for P-a.a.ω. It follows from Proposition 2.8 in [Bo1] that if $u_n \to u$ on S^{d-1} then for ρ^ω-a.a.x, Q^ω-a.a.ξ, P-a.a.ω the sequence $n^{-1}\log\|T(n,\omega,x,\xi)u_n\|$ converges to $\lambda_0(\rho)$. Since this sequence is bounded by $n^{-1}\chi(T(n,\omega,x,\xi))$, and so it is uniformly integrable, it follows that $g_n^\omega(x) \to 0$ as $n \to \infty$ for ρ^ω-a.a.x and P-a.a.ω. I conclude from above that

$$\lim_{n\to\infty}\int g_n^\omega(x)d\rho^\omega(x) = 0 \quad P\text{-a.s.} \tag{6.9}$$

Set $v_k = v_k^\omega = \|T(k,\omega,x)u\|^{-1}T(k,\omega,x)u$ then

$$|n^{-1}E_x^\omega \log\|M_n^\omega u\| - \lambda_0(\rho)| \le |n^{-1}E_x^\omega\|M_k^\omega u\||$$
$$+ |n^{-1}E_x^\omega \log\|T(n-k,\theta^k\omega,X_k^\omega)v_k\| - \lambda_0(\rho)| \le |n^{-1}E_x^\omega\chi(M_k^\omega)|$$
$$+ |\frac{1}{n-k}E_x^\omega E_{X_k^\omega}\log\|T(n-k,\theta^k\omega,X_k^\omega)v_k\| - \lambda_0(\rho)| + \frac{k}{n}|\lambda_0(\rho)|$$
$$\le n^{-1}\sum_{i=0}^{k-1}A^{\theta^i\omega} + E_x^\omega g_{n-k}^{\theta^k\omega}(X_k^\omega) + \frac{k}{n}|\lambda_0(\rho)|$$

where $A^\omega = \sup_{x\in\mathcal{X}}\int\chi(T_F(x))d\mu^\omega(F)$. Hence

$$g_n^\omega(x) \le E_x^\omega g_{n-k}^{\theta^k\omega}(X_k^\omega) + n^{-1}k(|\lambda_0(\rho)| + k^{-1}\sum_{i=0}^{k-1}A^{\theta^i\omega}). \tag{6.10}$$

By (6.8) and Assumption 6.1(i),

$$\sup_{x\in\mathcal{X}}|E_x^\omega g_{n-k}^{\theta^k\omega}(X_k^\varepsilon) - \int g_{n-k}^{\theta^k\omega}(y)d\rho^{\theta^k\omega}(y)| \tag{6.11}$$
$$\le C_{\theta^k\omega}(1-\gamma_{\theta^k\omega})^k(|\lambda_0(\rho)| + (n-k)^{-1}\sum_{i=k}^{n-1}A^{\theta^i\omega}).$$

By the ergodic theorem $L_\omega = \sup_{\ell\ge1}(\ell^{-1}\sum_{i=0}^{\ell-1}A^{\theta^i\omega}) < \infty$ P-a.s. Set $\Gamma_M = \{\omega : \max(C_\omega,\gamma_\omega^{-1},|\lambda_0(\rho)| + L_\omega) \le M\}$, $\Gamma_{\varepsilon,N} = \{\omega : |\int g_\ell^\omega(\omega)d\rho^\omega(x)| \le \varepsilon\ \forall\ell\ge N\}$, and $\Gamma_{\varepsilon,N,M} = \Gamma_M\cap\Gamma_{\varepsilon,N}$. Then $\Gamma_{\varepsilon,N,M}\uparrow$ as $N\uparrow$ and $M\uparrow$ and by (6.9), $P(\bigcup_{N\ge1}\bigcup_{M\ge1}\Gamma_{\varepsilon,N,M}) = 1$ for any $\varepsilon > 0$. Given $\varepsilon > 0$ choose M and N so that $P(\Gamma_{\varepsilon,N,M}) > 0$ and set $\Gamma = \Gamma_{\varepsilon,N,M}$. Let $n_\Gamma^{(i)} = n_\Gamma^{(i)}(\omega)$ be arrival times to Γ defined in the beginning of Section 5. Then by (6.10) and (6.11) for any $n_\Gamma^{(i)}(\omega) < n-N$,

$$\sup_{x\in\mathcal{X}}|g_n^\omega(x)| \le \varepsilon + M^2(1-M^{-1})^{n_\Gamma^{(i)}(\omega)} + n^{-1}n_\Gamma^{(i)}(\omega)M. \tag{6.12}$$

Passing in (6.12) to $\limsup_{n\to\infty}$ and taking into account that $n_\Gamma^{(i)}(\omega)\to\infty$ as $i\to\infty$ I obtain $\limsup_{n\to\infty}\sup_{x\in\mathcal{X}}|g_n^\omega(x)| \le \varepsilon$. Since $\varepsilon > 0$ is arbitrary, the uniform in x and u limit (6.3) follows.

For $\varepsilon > 0$ set $\Psi_{\varepsilon,n} = \{\omega : \sup_{x\in\mathcal{X}}|g_k^\omega(x)| \le \varepsilon\,\forall k\ge n\}$. I know now that $\Psi_{\varepsilon,n}\uparrow\Psi_\varepsilon$ as $n\uparrow\infty$ and $P(\Psi_\varepsilon) = 1$. Then by (6.8) for any $n\ge m$,

$$\int\sup_{x\in\mathcal{X}}|g_n^\omega(x)|dP(\omega) \le \varepsilon + |\lambda_0(\rho)|P(\Omega\setminus\Psi_{\varepsilon,m})$$
$$+ \int_{\Omega\setminus\Psi_{\varepsilon,m}} n^{-1}\sum_{k=0}^{n-1}\sup_{x\in\mathcal{X}}\int\chi(T_F(x))d\mu^{\theta^k\omega}(F)dP(\omega).$$

Employing the $\mathbb{L}^1$ convergence in the ergodic theorem I obtain that for any $m \in \mathbb{Z}_+$,

$$\limsup_{n\to\infty} \int \sup_{x\in\mathcal{X}} |g_n^\omega(x)| dP(\omega) \leq \varepsilon$$
$$+ (|\lambda_0(\rho)| + \int \sup_{x\in\mathcal{X}} \int \chi(T_F(x)) d\mu^\omega(F) dP(\omega)) P(\Omega \setminus \Psi_{\varepsilon,m}).$$

Letting, first, $m \to \infty$ and then $\varepsilon \to 0$ I obtain the uniform in x and u limit (6.4) via the Jensen inequality. □

For each $u \in S^{d-1}$ denote by $\bar{u}$ the corresponding element of $\mathbb{P}^{d-1}$ and define

$$\delta(\bar{u}, \bar{v}) = |\sin \angle(u, v)| = \|u \wedge v\| \tag{6.13}$$

where $u, v \in S^{d-1}$ and $\angle$ and $\wedge$ denote the angle and the exterior product, respectively. Set

$$D_n^\omega = \sup_{x,u,v} E_x^\omega \log(\delta(M_n^\omega \bar{u}, M_n^\omega \bar{v})/\delta(\bar{u}, \bar{v})).$$

6.3. Corollary. *P-a.s.,*

$$\limsup_{n\to\infty} n^{-1} D_n^\omega \leq \lambda_1(\rho) - \lambda_0(\rho) \tag{6.14}$$

and

$$\limsup_{n\to\infty} n^{-1} \int D_n^\omega dP(\omega) \leq \lambda_1(\rho) - \lambda_0(\rho). \tag{6.15}$$

Proof. Let $\Lambda^2 M$, $M \in GL(d, \mathbb{R})$ denotes the exterior product action, i.e. $\Lambda^2 M(u \wedge v) = Mu \wedge Mv$. By the same argument as in Lemma 6.2 I obtain that P-a.s., $n^{-1} E_x^\omega \log \|\Lambda^2 M_n^\omega\|$ converges uniformly in x to $\lambda_0(\rho) + \lambda_1(\rho)$ as $n \to \infty$. Since $\delta(\bar{u}, \bar{v}) = \|u \wedge v\|(\|u\|\|v\|)^{-1}$ for any $u, v, \in \mathbb{R}^d \setminus \{0\}$ it follows that

$$\begin{aligned} & n^{-1} E_x^\omega \log(\delta(M_n^\omega \bar{u}, M_n^\omega \bar{v})/\delta(\bar{u}, \bar{v})) \\ & \leq n^{-1} E_x^\omega \log \|\Lambda^2 M_n^\omega\| - n^{-1} E_x^\omega \log(\|M_n^\omega u\|/\|u\|) \\ & - n^{-1} E_x^\omega \log(\|M_n^\omega v\|/\|v\|). \end{aligned} \tag{6.16}$$

This together with the first part of Lemma 6.2 give (6.14). Taking in (6.16) the supremum in x, u, v and then integrating the inequality against P I derive (6.15) from the second part of Lemma 6.2. □

Observe that in Lemma 6.2 and Corollary 6.3 I used only (6.7) in place of (6.2).

6.4. Proposition. *Suppose that Assumption 6.1 holds true and $\alpha > 0$ is small enough. Then there exists a random variable $K = K_\omega$ and a number $\beta \in (0,1)$ such that for all $n \geq 1$, $x \in \mathcal{X}$, and $\bar{u}, \bar{v} \in \mathbb{P}^{d-1}$,*

$$E_x^\omega(\delta(M_n^\omega \bar{u}, M_n^\omega \bar{v})/\delta(\bar{u},\bar{v}))^\alpha \leq K_\omega \beta^n. \tag{6.17}$$

Proof. By Corollary 6.3 there exists k such that

$$\int D_k^\omega dP(\omega) \leq -1. \tag{6.18}$$

For any $n \in \mathbb{Z}_+$ and $x \in \mathcal{X}$ set

$$c_n^\omega(x) = \sup_{\bar{u},\bar{v}\in\mathbb{P}^{d-1}} E_x^\omega(\delta(M_n^\omega \bar{u}, M_n^\omega \bar{v})/\delta(\bar{u},\bar{v}))^\alpha$$

and $r_n^\omega = \sup_{x\in\mathcal{X}} c_n^\omega(x)$. Observe that for any $M \in GL(d,\mathbb{R})$ and $\bar{u}, \bar{v} \in \mathbb{P}^{d-1}$,

$$-4\chi(M) \leq \log(\delta(M\bar{u}, M\bar{v})/\delta(\bar{u},\bar{v})) \leq 4\chi(M), \tag{6.19}$$

and so by (6.2) and (6.5) it follows that P-a.s.,

$$\begin{aligned} r_n^\omega &\leq \sup_{x\in\mathcal{X}} E_x^\omega \exp(4\alpha\chi(M_n^\omega))) \\ &\leq \prod_{i=0}^{n-1} \sup_{y\in\mathcal{X}} \int \exp(4\alpha\chi(T_F(y)))d\mu^{\theta^i\omega}(F) < \infty \end{aligned} \tag{6.20}$$

provided $\alpha \leq a/4$. If $n, m \in \mathbb{Z}_+$ set $\bar{u}_m = T(m,\omega,x)\bar{u}$ and $\bar{v}_m = T(m,\omega,x)\bar{v}$. Let $\mathcal{F}_m^\omega$ be the σ-algebra on Ξ generated by the Markov chain $Y_i^\omega = (X_i^\omega, M_i^\omega)$ for all $i \leq m$. Then by the Markov property

$$\begin{aligned} &E_x^\omega((\delta(M_{n+m}^\omega \bar{u}, M_{n+m}^\omega \bar{v})/\delta(\bar{u},\bar{v}))^\alpha | \mathcal{F}_m^\omega) \\ &= E_x^\omega((\delta(T(n,\theta^n\omega, X_m^\omega)\bar{u}_m, T(n,\theta^m\omega, X_m^\omega)\bar{v}_m)/\delta(\bar{u},\bar{v}))^\alpha | \mathcal{F}_m) \\ &\leq r_n^{\theta^m\omega}(\delta(T(m,\omega,x)\bar{u}, T(m,\omega,x)\bar{v})/\delta(\bar{u},\bar{v}))^\alpha. \end{aligned} \tag{6.21}$$

Taking E_x^ω and $\sup_{x,\bar{u},\bar{v}}$ in both parts of (6.21) I derive that P-a.s. for all $m, n \in \mathbb{Z}_+$,

$$r_{n+m}^\omega \leq r_n^{\theta^m\omega} r_m^\omega. \tag{6.22}$$

By (6.2) and (6.18), $\int \log^+ r_1^\omega dP(\omega) < \infty$, and so I can apply the subadditive ergodic theorem which yields that P-a.s.,

$$\lim_{n\to\infty} \frac{1}{n} \log r_n^\omega = \inf_n \frac{1}{n} \int \log r_n^\omega dP(\omega). \tag{6.23}$$

Since $e^s \le 1 + s + \frac{1}{2}s^2 e^{|s|}$ then by (6.19),

$$E_x^\omega(\delta(M_k^\omega \bar{u}, M_k^\omega \bar{v})/\delta(\bar{u},\bar{v}))^\alpha \le 1 + \alpha D_k^\omega + 8\alpha^2 B_k^\omega \tag{6.24}$$

where $B_k^\omega = \sup_{x\in\mathcal{X}} E_x^\omega((\chi(M_k^\omega))^2 \exp(\alpha\chi(M_k^\omega)))$. By (6.21), (6.23), and (6.24) $P-$a.s.,

(6.25)
$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \log r_n^\omega \le \frac{1}{k} \int_\Omega \log r_k^\omega dP(\omega) \le \frac{1}{k} \int_\Omega \log(1 + \alpha D_k^\omega + 8\alpha^2 B_k^\omega) dP(\omega) \\
&\le \frac{1}{k} \int (\alpha D_k^\omega + 8\alpha^2 B_k^\omega) dP(\omega) \le -\frac{\alpha}{k} + \frac{8\alpha^2}{k} \int B_k^\omega dP(\omega).
\end{aligned}$$

Choose α sufficiently small so that $\alpha \int_\Omega B_k^\omega dP(\omega) \le \frac{1}{16}$ then $\lim_{n\to\infty} \frac{1}{n} \log r_n^\omega \le -\frac{\alpha}{2k}$ and (6.17) follows with $\beta = e^{-\frac{\alpha}{3k}}$. $\square$

For $\alpha > 0$ denote by $\mathbb{L}_\alpha$ the space of Borel functions $\varphi : \mathcal{X} \times \mathbb{P}^{d-1} \to \mathbb{R}$ such that $\|\varphi\|_\alpha = |\varphi|_\alpha + \|\varphi\| < \infty$ where $|\varphi|_\alpha = \sup\{|\varphi(x,\bar{u}) - \varphi(x,\bar{v})|/(\delta(\bar{u},\bar{v}))^\alpha : x \in \mathcal{X}, \bar{u},\bar{v} \in \mathbb{P}^{d-1}\}$ and $\|\varphi\| = \sup\{|\varphi(x,\bar{u})| : x \in \mathcal{X}, \bar{u} \in \mathbb{P}^{d-1}\}$. Set $R^\omega\varphi(x,\bar{u}) = E_x^\omega \varphi(X_1^\omega, M_1^\omega \bar{u})$, $R_0^\omega = Id$, $R_1^\omega = R^\omega$ and $R_n^\omega = R^\omega \circ R^{\theta\omega} \circ \cdots \circ R^{\theta^{n-1}\omega}$.

Applying Lemma 3.5 from [Bo1] to the Markov multiplicative system $(\theta^n\omega, X_n^\omega, T(n,\omega,x))$ I obtain that there exists $\nu \in \mathcal{P}(\Omega \times \mathcal{X} \times \mathbb{P}^{d-1})$ such that

$$d\nu(\omega,x,\bar{u}) = d\nu^\omega(x,\bar{u})dP(\omega) = d\nu_x^\omega(\bar{u})d\rho^\omega(x)dP(\omega) \text{ and } \nu^\omega R^\omega = \nu^{\theta\omega}, \tag{6.26}$$

i.e. ν^ω is a $\mu-$stationary family and ν is an invariant measure of the Markov multiplicative system above. Set $N^\omega\varphi(x,\bar{u}) = \int \varphi d\nu^\omega$, $\varphi \in \mathbb{L}_\alpha$.

6.5. Proposition. *Suppose that Assumption 6.1 holds true and $\alpha > 0$ is small enough. Then there exists a number ι such that for $P-a.a.\omega$,*

$$\iota = \lim_{n\to\infty} \|R_n^{\theta^{-n}\omega} - N^\omega\|_\alpha^{1/n} < 1. \tag{6.27}$$

Proof. Let $\varphi\in\mathbb{L}_\alpha$, $x\in\mathcal{X}$, and $\bar u,\bar v\in\mathbb{P}^{d-1}$ then for each $n>m$,

(6.28)
$$
\begin{aligned}
&|R_n^{\theta^{-n}\omega}\varphi(x,\bar u)-E_x^{\theta^{-n}\omega}\varphi(X_n^{\theta^{-n}\omega},T(m,\theta^{-m}\omega,X_{n-m}^{\theta^{-n}\omega})\bar v)|\\
&=|E_x^{\theta^{-n}\omega}\varphi(X_n^{\theta^{-n}\omega},T(n,\theta^{-n}\omega,x)\bar u)-E_x^{\theta^{-n}\omega}\varphi(X_n^{\theta^{-n}\omega},T(m,\theta^{-m}\omega,X_{n-m}^{\theta^{-n}\omega})\bar v)|\\
&\le\|\varphi\|_\alpha E_x^{\theta^{-n}\omega}(\delta(T(n,\theta^{-n}\omega,x)\bar u,T(m,\theta^{-m}\omega,X_{n-m}^{\theta^{-n}\omega})\bar v))^\alpha\\
&=\|\varphi\|_\alpha E_x^{\theta^{-n}\omega}(E_x^{\theta^{-n}\omega}(\delta(T(m,\theta^{-m}\omega,X_{n-m}^{\theta^{-n}\omega})T(n-m,\theta^{-n}\omega,x)\bar u,\\
&T(m,\theta^{-m}\omega,X_{n-m}^{\theta^{-n}\omega})\bar v)^\alpha|\mathcal{F}_{n-m}^{\theta^{-n}\omega}))\\
&\le\|\varphi\|_\alpha E_x^{\theta^{-n}\omega}\sup_{u,v}E^{\theta^{-m}\omega}_{X_{n-m}^{\theta^{-n}\omega}}(\delta(M_m^{\theta^{-m}\omega}\bar u,M_m^{\theta^{-m}\omega}\bar v))^\alpha\\
&\le\|\varphi\|_\alpha K_{\theta^{-m}\omega}\beta^m
\end{aligned}
$$

where I employed the Markov property and the last inequality follows from Proposition 6.4.

Now let $\nu\in\mathcal{P}(\Omega\times\mathcal{X}\times\mathbb{P}^{d-1})$ satifies (6.26). Set $\psi_m^\omega(x)=\int E_x^\omega\varphi(X_m^\omega,M_m^\omega\bar v)d\nu_x^\omega(\bar v)$, then $\sup_{x,\omega}|\psi_m^\omega(x)|=\|\psi\|\le\|\varphi\|_\alpha$. Since $X_n^{\theta^{-n}\omega}(\xi)=X_m^{\theta^{-m}\omega}(\sigma^{n-m}\xi)$ then

$$
\begin{aligned}
&|\int E_x^{\theta^{-n}\omega}\varphi(X_n^{\theta^{-n}\omega},T(m,\theta^{-m}\omega,X_{n-m}^{\theta^{-n}\omega})\bar v)d\nu^{\theta^{-m}\omega}_{X_{n-m}^{\theta^{-n}\omega}}(\bar v)\\
&-\int E_x^{\theta^{-m}\omega}\varphi(X_m^{\theta^{-m}\omega},T(m,\theta^{-m}\omega,x)\bar v)d\nu^{\theta^{-m}\omega}(x,\bar v)|\\
&\le|E_x^{\theta^{-n}\omega}\psi_m^{\theta^{-m}\omega}(X_{n-m}^{\theta^{-n}\omega})-\int\psi_m^{\theta^{-m}\omega}(x)d\rho^{\theta^{-m}\omega}(x)|\\
&\le\|\varphi\|_\alpha C_{\theta^{-m}\omega}(1-\gamma_{\theta^{-m}\omega})^{n-m}.
\end{aligned}\tag{6.29}
$$

where I employed Assumption 6.1(i).

Set $\bar R_n^\omega=R_n^{\theta^{-n}\omega}-N^\omega$. Since

$$
\begin{aligned}
&\int E_x^{\theta^{-m}\omega}\varphi(X_m^{\theta^{-m}\omega},T(m,\theta^{-m}\omega,\bar v)d\nu^{\theta^{-m}\omega}(x,\bar v)=\int R_m^{\theta^{-m}\omega}\varphi(x,\bar v)d\nu^{\theta^{-m}\omega}(x,\bar v)\\
&=\int\varphi(x,\bar v)d\nu^\omega(x,\bar v)=N^\omega\varphi,
\end{aligned}
$$

I obtain from (6.28) and (6.29) that for any $m,n\in\mathbb{Z}_+$, $m<n$,

$$
\|\bar R_n^\omega\|_\alpha\le K_{\theta^{-m}\omega}\beta^m+C_{\theta^{-m}\omega}(1-\gamma_{\theta^{-m}\omega})^{n-m}.\tag{6.30}
$$

Set $\Gamma_M=\{\omega:\max(K_\omega,C_\omega,\gamma_\omega^{-1})\le M\}$ and choose M large enough so that $P(\Gamma_M)>0$. Let $m_\Gamma^{(0)}(\omega)=0$ and recursively $m_\Gamma^{(i+1)}(\omega)=\min\{m>m_\Gamma^{(i)}(\omega):$

$\theta^{-m}\omega \in \Gamma_M\}$, $i = 0, 1, ...$ Set $i_\Gamma(\omega, n) = \max\{i : m_\Gamma^{(i)}(\omega) \le \frac{n}{2}\}$ and $m_\Gamma(\omega, n) = m_\Gamma^{(i(\omega,n))}(\omega)$. Then by (6.30),

$$\|\bar{R}_n^\omega\|_\alpha \le 2M \max(\beta^{m_\Gamma(\omega,n)}, (1 - M^{-1})^{\frac{n}{2}}).$$

By the ergodic theorem $P-$a.s., $\lim_{n\to\infty} n^{-1} i_\Gamma(\omega, n) = 2P(\Gamma_M)$ and since $m_\Gamma(\omega, n) \ge i_\Gamma(\omega, n)$ I obtain that $P-$a.s.,

$$\iota \stackrel{\text{def}}{=} \limsup_{n\to\infty} \|\bar{R}_n^\omega\|_\alpha^{1/n} < 1. \tag{6.31}$$

Observe that $\bar{R}_n^\omega = \bar{R}_{n-m}^{\theta^{-m}\omega} \bar{R}_m^\omega$ since by (6.26), $N^{\theta^{-m}\omega} R_m^{\theta^{-m}\omega} = N^\omega$, and so $\|\bar{R}_n^\omega\|_\alpha \le \|\bar{R}_{n-m}^{\theta^{-m}\omega}\|_\alpha \|\bar{R}_m^\omega\|_\alpha$. By the subadditive ergodic theorem it follows that, in fact, the limit in (6.31) exists and $P-$a.s. it is constant, concluding the proof of Proposition 6.5. $\square$

Set $L_\varepsilon(\omega) = \min\{L \in \mathbb{Z}_+ : \|R_n^{\theta^{-n}\omega} - N^\omega\|_\alpha \le (1 - \varepsilon)^n \ \forall n \ge L\}$. By (6.27), $L_\varepsilon(\omega) < \infty$ P-a.s. provided $\varepsilon \in (0, 1 - \rho)$. Let $\Gamma = \Gamma_\varepsilon = \{\omega : L_\varepsilon(\omega) \le \varepsilon^{-1}\}$ and $n_\varepsilon^{(i)} = n_\varepsilon^{(i)}(\omega) = n_{\Gamma_\varepsilon}^{(i)}(\omega)$ be the arrival times at Γ defined at the beginning of Section 5. For any $x \in X$, $\bar{u} \in \mathbb{P}^{d-1}$ and $F \in \mathcal{F}$ set

$$\eta(x, \bar{u}, F) = \log \frac{\|T_F(x)u\|}{\|u\|}, \quad u \ne 0$$

and denote

$$\lambda_0^\omega(\rho) = \int \eta(x, \bar{u}, F) d\nu^\omega(x, \bar{u}) d\mu^\omega(F), \quad t_n^\omega = \sum_{i=0}^{n-1} \lambda_0^{\theta^i\omega}(\rho).$$

Observe that since ν^ω is the unique μ^ω-stationary family it must be ergodic and so by Theorem 3.1 $\int \lambda_0^\omega(\rho) dP(\omega) = \lambda_0(\rho)$. Next, I can derive the following limit theorem.

6.6. Theorem. *Suppose that Assumption 6.1 holds true and, in addition, for some $\varepsilon \in (0, 1 - \iota)$ with $P(\Gamma_\varepsilon) > 0$ and for some $a > 0$ one has*

$$\int n_\varepsilon^{(1)}(\omega) \prod_{i=0}^{n_\varepsilon^{(1)}(\omega)-1} \sup_{x\in X} \int \exp(a\chi(T_F(x))) d\mu^{\theta^i\omega}(F) dP_\varepsilon(\omega) < \infty \tag{6.32}$$

where P_ε is the normalized restriction of P to Γ_ε. Then *(i) For P-a.s. $\omega\in\Omega$ and all $x\in X$, $u\in S^{d-1}$ the limit*

$$\sigma^2=\lim_{n\to\infty}n^{-1}E_x^\omega(\log\|M_n^\omega u\|-t_n^\omega)^2$$

exists. Moreover, there exists a measurable in ω family of functions $\varphi_\omega\in\mathbb{L}_\alpha$ with $\|\varphi_\omega\|_\alpha\in L^2(\Gamma_\varepsilon,P_\varepsilon)$ such that

$$\sigma^2=P(\Gamma_\varepsilon)\iint g_\omega^\varphi(x,\bar u)d\nu^\omega(x,\bar u)dP_\varepsilon(\omega) \tag{6.33}$$

where $g_\omega^\varphi(x,\bar u)=E_x^\omega\big(\log\|M^\omega_{n_\varepsilon^{(1)}(\omega)}u\|-t^\omega_{n_\varepsilon^{(1)}(\omega)}+\varphi_{\theta^{n_\varepsilon^{(1)}(\omega)}\omega}(X^\omega_{n_\varepsilon^{(1)}(\omega)},M^\omega_{n_\varepsilon^{(1)}(\omega)}\bar u)-\varphi_\omega(x,u)\big)^2$. Furthermore, $\sigma=0$ if and only if for some family of functions $\varphi_\omega\in\mathbb{L}_\alpha$ with $\|\varphi_\omega\|_\alpha\in L^2(\Gamma_\varepsilon,P_\varepsilon)$ the corresponding $g^\varphi=0$ ν^ω-a.s., P_ε-a.s.

(ii) For each $u\in S^{d-1}$ and $\omega\in\Omega$ define the sequence of continuous random processes

$$\begin{aligned}S_n^\omega(t,\xi)&=(n\sigma^2)^{-1/2}\big(\log\|M^\omega_{[nt]}(\xi)u\|-t^\omega_{[nt]}\\&+(nt-[nt])(\log\|M^\omega_{[nt]+1}(\xi)u\|+\lambda_0^{\theta^{[nt]}\omega}(\rho)-\log\|M^\omega_{[nt]}(\xi)u\|)\big),\end{aligned} \tag{6.34}$$

$t\in[0,1]$, distributed according to P_x^ω. Then for P-a.a.ω the processes $\{S_n^\omega(t,\cdot),t\in[0,1]\}$ converge in distribution as $n\to\infty$ to the one dimensional Brownian motion on the time interval $[0,1]$. The same remains true if in the definition of S_n^ω the expressions $\|M_\cdot^\omega(\xi)u\|$ are replaced by $\|M_\cdot^\omega(\xi)\|$.

(iii) For each x,u and for $P-$a.a.ω, P_x^ω-a.a.ξ the set of limit points in $C[0,1]$ of the sequence $\{(2\log\log n)^{-1/2}S_n^\omega(t,\xi),t\in[0,1]\}$, $n=1,2,\dots$ coincides with the compact set of functions q absolutely continuous on $[0,1]$ such that $q(0)=0$ and $\int_0^1(q'(s))^2ds\le 1$.

Proof. Introduce Markov chains in random environments defined by

$$Z_n^\omega=(X_n^\omega,\bar U_n^\omega,F_n^\omega)\ \text{ where }\ U_n^\omega=T(n,\omega,x)u\ \text{ with }\ x\in X\ \text{ and }\ u\in S^{d-1}.$$

Observe that

$$\log\|U_n^\omega\|=\sum_{k=1}^{n-1}\eta(X_k^\omega,\bar U_k^\omega,F_k^\omega)=\sum_{k=1}^{n-1}\eta(Z_k^\omega). \tag{6.35}$$

Let $\tilde R^\omega$ be the transition operator of the Markov chain Z_n^ω, i.e.

$$\tilde R^\omega\varphi(x,\bar u,F)=\int\varphi(f_Fx,T_F(x)\bar u,G)d\mu^{\theta\omega}(G)$$

for any bounded Borel function φ on $X \times \mathbb{P}^{d-1} \times \mathcal{T}$ and set $\tilde{R}^\omega_n = \tilde{R}^\omega \circ \tilde{R}^{\theta\omega} \circ \cdots \circ \tilde{R}^{\theta^{n-1}\omega}$. Let $\psi_\omega(x,\bar{u}) = \int \eta(x,\bar{u},F)d\mu^\omega(F)$ then

(6.36)
$$\tilde{R}^\omega_n \eta(x,\bar{u},F) = \int \eta(X^\omega_{n-1}(\sigma\xi, f_F x), T(n-1,\theta\omega, f_F x, \sigma\xi)T_F(x)\bar{u}, F^\omega_n(\xi))$$
$$d\Pi^\omega(\xi) = R^{\theta\omega}_{n-1}\psi_{\theta^n\omega}(f_F x, T_F(x)\bar{u})$$

where y in $X^\omega_k(\tau,y)$ indicates that $X^\omega_0 = y$.

Set $\Psi_\omega(x,\bar{u}) = \sum_{k=0}^{n^{(1)}_\varepsilon(\omega)-1} R^\omega_k \psi_{\theta^k\omega}(x,\bar{u})$. It follows from Lemma V.4.2 in [BL] that for any $\alpha \in (0,1]$ and some constant $C > 0$,

$$\|\Psi_\omega\|_\alpha \leq C\alpha^{-1} \sum_{j=0}^{n^{(1)}_\varepsilon(\omega)-1} \prod_{i=0}^{j} \sup_x\Big(\int e^{3\alpha\chi(T_F(x))} d\mu^{\theta^i\omega}(F)\Big). \tag{6.37}$$

This together with (6.32) and the Hölder inequality yield that if $6\alpha \leq a \leq 1$ then

$$\int \|\Psi_\omega\|^2_\alpha dP_\varepsilon(\omega) < \infty. \tag{6.38}$$

In view of Proposition 6.5 and the definitions of Γ_ε and $n^{(i)}_\varepsilon$ it follows that for any $\omega \in \Gamma_\varepsilon$,

$$\|\hat{R}^\omega_i \Psi_{\theta^i_\Gamma\omega} - N^{\theta^i_\Gamma\omega}\Psi_{\theta^i_\Gamma\omega}\|_\alpha \leq (1-\varepsilon)^{n^{(i)}_\varepsilon(\omega)} \|\Psi_{\theta^i_\Gamma\omega}\|_\alpha, \tag{6.39}$$

provided $n^{(i)}_\varepsilon(\omega) \geq \varepsilon^{-1}$, where $\hat{R}^\omega_i = R^\omega_{n^{(i)}_\varepsilon(\omega)}$ and $\theta_\Gamma = \theta^{n^{(1)}_\varepsilon(\omega)}$ is the P_ε-preserving ergodic transformation of Γ_ε (see [Br], p. 30).

It follows that for $P_\varepsilon - a.a.\omega$ the series

$$\varphi_\omega = \sum_{i=0}^{\infty} (\hat{R}^\omega_i - N^{\theta^i_\Gamma\omega})\Psi_{\theta^i_\Gamma\omega}$$

converges in $\mathbb{L}_\alpha$ and

$$\int \|\varphi_\omega\|^2_\alpha dP_\varepsilon(w) < \infty. \tag{6.40}$$

Since $N^{\theta^m\omega}R^{\theta^m\omega}_k = N^{\theta^{m+k}\omega}$ by (6.26) then $N^\omega\varphi_\omega = 0$, and so $P_\varepsilon - a.s.$,

$$\Psi_\omega(x,\bar{u}) - \hat{t}^\omega_1 = \varphi_\omega(x,\bar{u}) - \hat{R}^\omega_1\varphi_{\theta_\Gamma\omega}(x,\bar{u}) \tag{6.41}$$

where I set $\hat{t}_i^\omega = t^\omega_{n_\varepsilon^{(i)}(\omega)}$. Observe also that

$$\sum_{i=0}^{\ell-1} \hat{R}_i^\omega \Psi_{\theta_\Gamma^i\omega} = \sum_{k=0}^{n_\varepsilon^{(\ell)}(\omega)-1} R_k^\omega \psi_{\theta^k\omega}$$

and

$$\sum_{i=0}^{\ell-1} N^{\theta_\Gamma^i\omega} \Psi_{\theta_\Gamma^i\omega} = \sum_{k=0}^{n_\varepsilon^{(\ell)}(\omega)-1} N^{\theta^k\omega}\psi_{\theta^k\omega} = \sum_{k=0}^{n_\varepsilon^{(\ell)}(\omega)-1} \lambda_0^{\theta^k\omega}(\rho).$$

Define $Y_0^\omega(x,\bar{u}) = 0$ and recursively

(6.42)

$$\begin{aligned} Y^\omega_{n+1}(x,\bar{u}) &= Y_n^\omega(x,\bar{u}) + \log(\|\hat{M}^\omega_{n+1}u\|\,\|\hat{M}_n^\omega u\|^{-1}) \\ &- (\hat{t}^\omega_{n+1} - \hat{t}^\omega_n) + \varphi_{\theta_\Gamma^{n+1}\omega}(\hat{X}^\omega_{n+1}, \hat{M}^\omega_{n+1}\bar{u}) - \varphi_{\theta_\Gamma^n\omega}(\hat{X}_n^\omega, \hat{M}_n^\omega\bar{u}) \\ &\text{where } \hat{M}_i^\omega = M^\omega_{n_\varepsilon^{(i)}(\omega)} = T(n_\varepsilon^{(i)}(\omega),\omega,x) \text{ and } \hat{X}_i^\omega = X^\omega_{n_\varepsilon^{(i)}(\omega)}. \end{aligned}$$

Let $\mathcal{F}_n^\omega$ be the σ-algebra generated by $\{(X_\ell^\omega, M_\ell^\omega), \ell = 0,1,\dots,n\}$, $n = 0,1,\dots$ and $\hat{\mathcal{F}}_i^\omega = \mathcal{F}^\omega_{n_\varepsilon^{(i)}(\omega)}$. Since $\hat{M}^\omega_{n+1} = T(n_\varepsilon^{(1)}(\theta_\Gamma^n\omega), \theta_\Gamma^n\omega, \hat{X}_n^\omega)\hat{M}_n^\omega$ I derive from (6.41), (6.42) and the Markov property that

$$\begin{aligned} E_x^\omega(Y^\omega_{n+1}(x,\bar{u}) - Y_n^\omega(x,\bar{u})|\hat{\mathcal{F}}_n^\omega) &= \Psi_{\theta_\Gamma^n\omega}(\hat{X}_n^\omega, \hat{M}_n^\omega\bar{u}) - \hat{t}_1^{\theta_\Gamma^n\omega} \\ &+ \hat{R}_1^{\theta_\Gamma^n\omega}\varphi_{\theta_\Gamma^{n+1}\omega}(\hat{X}_n^\omega, \hat{M}_n^\omega\bar{u}) = \varphi_{\theta_\Gamma^n\omega}(\hat{X}_n^\omega, \hat{M}_n^\omega\bar{u}) = 0. \end{aligned} \tag{6.43}$$

Thus $(Y_n^\omega(x,\bar{u}), \hat{\mathcal{F}}_n^\omega), n = 0,1,\dots$ is a martingale for $P_\varepsilon - a.a.\omega$.

Next, I am going to check the conditions of invariance principles in the central limit theorem and the law of iterated logarithm for martingales from Ch. 4 in [HH] (cf. [Ru1] and [Ki3]). First, I claim that $P_\varepsilon - a.s.$,

$$\lim_{n\to\infty} n^{-1}\sum_{\ell=1}^n E_x^\omega((Y^\omega_{\ell+1} - Y^\omega_\ell)^2|\hat{\mathcal{F}}^\omega_\ell) = (P(\Gamma_\varepsilon))^{-1}\sigma^2 \tag{6.44}$$

with σ^2 given by (6.33). Indeed, by (6.42)

$$E_x^\omega((Y^\omega_{n+1}(x,\bar{u}) - Y_n^\omega(x,\bar{u}))^2|\hat{\mathcal{F}}_n^\omega) = g_{\theta_\Gamma^n\omega}(\hat{X}_n^\omega, \hat{M}_n^\omega\bar{u}) \tag{6.45}$$

where $g = g^\varphi$ is the same as in (6.33). Since $\varphi_\omega \in \mathbb{L}_\alpha$ then in view of (6.32), $g_\omega \in \mathbb{L}_\alpha$. Set $b_\omega = \int g_\omega(x,\bar{u})d\nu^\omega(x,\bar{u})$ then by Proposition 6.5 and the definition of Γ_ε for any $\omega \in \Gamma_\varepsilon$, $n_\varepsilon^{(k)}(\omega) \geq \varepsilon^{-1}, y \in X$ and $\bar{v} \in \mathbb{P}^{d-1}$,

$$|E_y^\omega g_{\theta_\Gamma^k\omega}(\hat{X}_k^\omega, \hat{M}_k^\omega\bar{v}) - b_{\theta_\Gamma^k\omega}| \leq (1-\varepsilon)^{n_\varepsilon^{(k)}(\omega)}\|g_{\theta_\Gamma^k\omega}\|_\alpha. \tag{6.46}$$

In view of (6.32), (6.41) and Lemma V.4.2 from [BL] I conclude that

$$\int \|g_\omega\|_\alpha^2 dP_\varepsilon(\omega) < \infty.$$

This together with (6.46) yield that

$$\int E_x^\omega\Big(\sum_{n=1}^{\infty} n^{-1}\big(E_x^\omega((Y_{n+1}^\omega - Y_n^\omega)^2|\hat{\mathcal{F}}_n^\omega) - b_{\theta_\Gamma^n\omega}\big)\Big)^2 dP_\varepsilon(\omega) < \infty. \tag{6.47}$$

Since by the ergodic theorem

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} b_{\theta_\Gamma^i\omega} = (P(\Gamma_\varepsilon))^{-1}\sigma^2$$

both $P_\varepsilon - a.s.$ and in $L^2(\Gamma_\varepsilon, P_\varepsilon)$ then (6.44) follows by the Kronecker lemma and the convergence in (6.44) is both $P_\varepsilon - a.s.$ and in $L^2(\Gamma_\varepsilon, P_\varepsilon)$.

Next, I have to check the Lindenberg condition saying that for any $k > 0, P_\varepsilon - a.s.$,

$$\lim_{n\to\infty} n^{-1}\sum_{j=1}^{n} E_x^\omega((Y_{j+1}^\omega - Y_j^\omega)^2 \mathbb{I}_{\{|Y_{j+1}^\omega - Y_j^\omega| > \kappa\sqrt{n}\}}) = 0. \tag{6.48}$$

By (6.42),

$$|Y_{n+1}^\omega - Y_n^\omega| \le A_n^\omega \overset{\mathrm{def}}{=} \chi(T(n_\varepsilon^{(1)}(\theta_\Gamma^n\omega), \theta_\Gamma^n\omega, \hat{X}_n^\omega)) + \hat{t}_1^{\theta_\Gamma^n\omega} + \|\varphi_{\theta_\Gamma^{n+1}\omega}\| + \|\varphi_{\theta_\Gamma^n\omega}\| \tag{6.49}$$

and it follows by the Markov property that

$$E_x^\omega(A_n^\omega)^2 \mathbb{I}_{\{A_n^\omega > L\}} \le B_L(\theta_\Gamma^n\omega) \tag{6.50}$$

where

$$B_L(\omega) = 4\sup_{x\in\mathcal{X}} E_x^\omega\big((\chi^2(\hat{M}_1^\omega) + (\hat{t}_1^\omega) + (\hat{t}_1^\omega)^2 + \|\varphi_{\theta_\Gamma\omega}\|^2 + \|\varphi_\omega\|^2)$$
$$\times\, \mathbb{I}_{\{\chi(\hat{M}_1^\omega) + \hat{t}_1^\omega + \|\varphi_{\theta_\Gamma\omega}\| + \|\varphi_\omega\| > L\}}\big).$$

By the ergodic theorem $P_\varepsilon - a.s.$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1} B_L(\theta_\Gamma^i\omega) = \int B_L dP_\varepsilon \tag{6.51}$$

and it is easy to see from (6.32) and (6.40) that the right hand side of (6.51) tends to zero as $L \to \infty$, which implies (6.48).

Consider random processes on the probability space (Ξ, P_x^ω) given by

$$\text{(6.52)}\quad \hat{S}_n^\omega(t,\cdot) = (n\sigma^2 P(\Gamma_\varepsilon)^{-1})^{\frac{1}{2}}(Y_{[nt]}^\omega(x,\bar{u}) + (nt-[nt])(Y_{[nt]+1}(x,\bar{u}) - Y_{[nt]}^\omega(x,\bar{u}))),$$

$n = 0, 1, \dots, t \in [0,1]$. Then (6.43), (6.44) and (6.48) together with Ch. 4 in [HH] yield that $P_\varepsilon - a.s.$ as $n \to \infty$ the processes $\hat{S}_n^\omega(t,\cdot)$ satisfy invariance principles in the central limit theorem and in the law of iterated logarithm as described in assertions (ii)–(iii) of Theorem 6.6.

Set $D_x^\omega(\xi) = \sum_{i=0}^{n_\varepsilon^{(1)}(\omega)-1} \chi(T_{\xi_i}(X_i^\omega))$, where $X_0^\omega = x$, and $\ell_n(\omega) = \max\{\ell \colon n_\varepsilon^{(\ell)}(\omega) \le n\}$. Then

$$L_n^\omega \stackrel{\text{def}}{=} |\log \|M_n^\omega u\| - t_n^\omega - Y_{\ell_n(\omega)-1}^{\theta_\Gamma \omega}| \le \int D_x^\omega d\rho^\omega(x) d\Pi^\omega$$
$$+ \int D_x^{\theta_\Gamma^{\ell_n(\omega)}\omega} d\rho^{\theta_\Gamma^{\ell_n(\omega)}\omega}(x) d\Pi^{\theta_\Gamma^{\ell_n(\omega)}\omega} + D_x^\omega + D_{\hat{X}_{\ell_n(\omega)}^\omega}^{\theta_\Gamma^{\ell_n(\omega)}\omega} + \|\varphi_{\theta_\Gamma^{\ell_n(\omega)}\omega}\|_\alpha + \|\varphi_\omega\|_\alpha.$$

By the ergodic theorem $\lim_{n\to\infty} n^{-1}\ell_n = P(\Gamma_\varepsilon)$ $P - a.s.$, and so, by (6.32) and (6.40) I derive that for $P - a.a.\omega, P_x^\omega - a.s.$, $\lim_{n\to\infty} n^{-\frac{1}{2}} L_n^\omega = 0$ which yields the assertions (i)–(iii) of Theorem 6.6 for $S_n^\omega(t,\cdot)$ defined by (6.34). In view of Proposition 2.8 from [Bo1] the same result follows if $S_n^\omega(t,\cdot)$ is defined with $\|M_\cdot^\omega(\xi)\|$ in place of $\|M_\cdot^\omega(\xi)u\|$. □

If (6.27) holds true in the supremum norm in place of $\|\cdot\|_\alpha$ then Theorem 6.6 can be proved under weaker than (6.32) integrability conditions. According to [Ki2] the former takes place if the following random Doeblin condition is satisfied.

6.7. Assumption. *There exist random variables $N = N_\omega \in \mathbb{Z}_+$ and $\gamma = \gamma_\omega > 0$ and a measurable in ω family $m^\omega \in \mathcal{P}(\mathcal{X} \times \mathbb{P}^{d-1})$ such that for any $x \in \mathcal{X}$ and a Borel $U \subset \mathcal{X} \times \mathbb{P}^{d-1}$ one has $R_N^{\theta^{-N}\omega} \mathbb{I}_U(x) \ge \gamma_\omega m^\omega(U)$.*

Under Assumption 6.7 it follows from [Ki2] that there exists a measurable in ω family $\nu^\omega \in \mathcal{P}(\mathcal{X} \times \mathbb{P}^{d-1})$ such that for any bounded Borel function q on $\mathcal{X} \times \mathbb{P}^{d-1}$

$$\text{(6.55)}\qquad \|R_n^{\theta^{-n}\omega} q - \int q d\nu^\omega\| \le C_\omega (1-\kappa_\omega)^n \|q\|$$

for some random variables $C_\omega > 0$ and $\kappa_\omega \in (0,1)$. Let $\Gamma_\varepsilon = \{\omega\colon \max(N_\omega, \gamma_\omega^{-1}) \leq \varepsilon^{-1}\}$ and define the sequence $n_\varepsilon^{(i)}(\omega)$ in the same way as before Theorem 6.6. Set

$$c(\omega) = (\int (\log \|T_F(x)u\|)^2 d\mu^\omega(F) d\nu^\omega(x,\omega))^{\frac{1}{2}}.$$

In view of (6.35), the following result follows from [Ki4] (see also [Ru2]).

6.8. Theorem. *The assertions of Theorem 6.6 hold true if (6.32) and Assumption 6.1 are replaced by Assumption 6.7 together with the condition that for some $\varepsilon > 0$, $P(\Gamma_\varepsilon) > 0$ and*

$$\int (\sum_{i=0}^{n_\varepsilon^{(1)}(\omega)-1} c \circ \theta^i)^2 dP_\varepsilon < \infty, \tag{6.56}$$

where, again, P_ε is the normalized restriction of P to Γ_ε.

In the same way as in Corollary 4.6 of [Bo1] the continuous time versions of Theorems 6.6 and 6.8 follow if, in addition, one assumes that

$$\int \sup_{x\in\mathcal{X}} E_x^\omega (\sup_{0\leq t\leq 1} \chi(M_t^\omega)^2) dP(\omega) < \infty.$$

7. Random harmonic functions and measures

Let Z_n^ω be a Markov chain in random environments on a Borel subset of a Polish space $\mathcal{V}$ with transition probabilities $R^\omega(v,\cdot)$ as in the beginning of Section 2. I denote by P_v^ω and E_v^ω the corresponding path distribution and the expectation provided $Z_0^\omega = v$. Let also $\mathcal{F}_{m,n}^\omega$, $0 \leq m \leq n \leq \infty$ be the σ-algebra on the path space $\Xi = \mathcal{V}^{\mathbb{Z}_+}$ generated by all Z_j^ω, $m \leq j < n+1$ and set $\mathcal{F}_{\infty,\infty}^\omega = \bigcap_{k\geq 0} \mathcal{F}_{k,\infty}$ which is called the tail $\sigma-$algebra. A measurable in ω family of functions $h = h_\omega(v)$ is called (random) harmonic if (2.2) holds true for all $v \in \mathcal{V}$ and P-a.a.ω (cf. [Ru1]). The following simple result is a basis for the boundary theory of random harmonic functions.

7.1. Proposition. *Let $h = h_\omega(v)$ be a harmonic family and*

$$r_\omega = \sup_{v\in\mathcal{V}} |h_\omega(v)| < \infty \quad P\text{-}a.s. \tag{7.1}$$

Then $h_{\theta^n\omega}(Z_n^\omega)$ is a bounded martingale with respect to $\mathcal{F}_n^\omega$, $n=0,1,\dots$ P-a.s. Hence for P-a.a ω the limit

$$\lim_{n\to\infty} h_{\theta^n\omega}(Z_n^\omega)=\varphi_\omega \tag{7.2}$$

exists P_v^ω-a.s. (and in any $L^k(\Xi,P_v^\omega)$) where φ_ω is a random variable on the probability space $(\Xi,\mathcal{F}_{\infty,\infty}^\omega,P_v^\omega)$.

Proof. By (2.2),

$$r_\omega\le\int R^\omega(v,d\omega)r_{\theta\omega}=r_{\theta\omega}$$

and by ergodicity of θ with respect to P I conclude that $r_\omega\equiv r$ is a constant P-a.s. Hence $h_\omega(v)$ is a bounded measurable function on $\Omega\times\mathcal{V}$. By the Markov property

$$E_v^\omega(h_{\theta^{n+1}\omega}(Z_{n+1}^\omega)|\mathcal{F}_n^\omega)=\int R^{\theta^n\omega}(Z_n^\omega,dw)h_{\theta^{n+1}\omega}(\omega)=h_{\theta^n\omega}(Z_n^\omega), \tag{7.3}$$

and so $h_{\theta^n\omega}(Z_n^\omega)$ is a bounded martingale. Now the result follows via the martingale convergence theorem. $\square$

By (2.2) and (7.2) I can also write

$$h_\omega(v)=E_v^\omega h_{\theta^n\omega}(Z_n^\omega)=E_v^\omega\varphi_\omega \tag{7.4}$$

which is a general form of the Poisson formula and one of the main problems of the boundary theory is a detailed description of such representations for specific models.

Let now G be a locally compact semigroup, μ^ω be a measurable in $\omega\in\Omega$ family of probability measures on G, $\Xi=G^{\mathbb{Z}_+}$, $\Pi^\omega=\prod_{i\in\mathbb{Z}_+}\mu^{\theta^i\omega}$, and $g_i^\omega(\xi)=g_0^{\theta^i\omega}(\sigma^i\xi)=\xi_i$ for $\xi=\{(\xi_i),i\in\mathbb{Z}_+\}$ where σ is the left shift on Ξ. Then g_i^ω are independent random elements of G with distributions $\mu^{\theta^i\omega}$, $i\in\mathbb{Z}_+$. Set $L_{-1}^\omega(\xi)=Id$, $L_n^\omega=L_n^\omega(\xi)=g_0^\omega(\xi)g_1^\omega(\xi)\cdots g_{n-1}^\omega(\xi)$ and $Z_n^\omega=gL_n^\omega$ for $g\in G$ which defines a Markov chain in random environments on G starting at g. The n-step transition probabilities of Z_n^ω can be expressed in the form

$$R^\omega(n,g,\Gamma)=\delta_g*\mu^\omega*\mu^{\theta\omega}*\cdots\mu^{\theta^{n-1}\omega}(\Gamma). \tag{7.5}$$

Let B be a compact space on which G acts minimally, i.e. for any $u\in B$ the set Gu is dense in B. Then G acts also on the space $\mathcal{P}(B)$ of probability measures on B, and so for any $\nu\in\mathcal{P}(B)$ the convolution $\mu^\omega*\nu$ is defined by (2.5). Suppose that $\nu^\omega\in\mathcal{P}(B)$ is a measurable in ω family satisfying

$$\mu^\omega*\nu^{\theta\omega}=\nu^\omega \tag{7.6}$$

which amounts to (2.7) with θ replaced by θ^{-1} and then, $\nu^{\theta^{-1}\omega}$ replaced by ν^ω. Then for any bounded Borel function φ on B the function

$$h_\omega(g)=\int\varphi(gu)d\nu^\omega(u)=\int\varphi(v)dg\nu^\omega(v) \tag{7.7}$$

satisfies

$$\int R^\omega(g,d\gamma)h_{\theta\omega}(\gamma)=\int h_{\theta\omega}(g\gamma)d\mu^\omega(\gamma)=\int\varphi(gv)d\mu^\omega*\nu^{\theta\omega}(v)=h_\omega(g), \tag{7.8}$$

i.e. h_ω is a random harmonic function for the Markov chain Z_n^ω according to the definition (2.2). Thus, the study of families of measures satisfying (7.6), which are naturally to call random harmonic measures, is important in the description of random harmonic functions on G. By analogy with the deterministic case one may call the pair (B,ν) a random μ-boundary. One can consider dual to h_ω and ν^ω objects replacing in (7.6) and (7.8) θ by θ^{-1}.

I shall not enter here into an extensive study of random μ-boundaries (for some results in this direction see [KKR]) but, instead, restrict myself to the case when $G=SL(d,\mathbb{R})$ and $B=\mathbb{P}^{d-1}$ (which is, essentially, the set up of previous sections with $\mathcal{X}$ being a point) though in order to describe the random Poisson boundary here one has to deal with B being the space of flags. I assume that $\int\chi(g)d\mu^\omega(g)dP(\omega)<\infty$. Recall, that $\mu^\omega\in\mathcal{P}(G)$, $\omega\in\Omega$ is a strongly irreducible family if there exist no finite collection $\{V_\omega^{(1)},V_\omega^{(2)},\dots,V_\omega^{(k)}\}$ of proper subspaces of $\mathbb{R}^d$ measurably depending on ω such that $g(\bigcup_{i=1}^k V_\omega^{(i)})=\bigcup_{i=1}^k V_{\theta\omega}^{(i)}$ for μ^ω-a.a. g and P-a.a. ω. The corresponding notion defined with θ^{-1} in place of θ will be called the reverse strong irreducibility. Let $\hat\mu^\omega$ denotes the distribution of g^* provided g has the distribution μ^ω then I conclude in the same way as at the end of proof of Theorem 5.4 that μ^ω is strongly irreducible if and only if $\hat\mu^\omega$ is reverse strongly irreducible.

7.2. Proposition. *(i) Measurable families $\nu^\omega \in \mathcal{P}(\mathbb{P}^{d-1})$ and $\hat\nu^\omega \in \mathcal{P}(\mathbb{P}^{d-1})$ satisfying*

$$\mu^\omega * \nu^\omega = \nu^{\theta\omega} \text{ and } \hat\mu^\omega * \hat\nu^{\theta\omega} = \hat\nu^\omega \tag{7.9}$$

always exist. If μ^ω is a strongly irreducible family or, equivalently, $\hat\mu^\omega$ is a reverse strongly irreducible family, then both ν^ω and $\hat\nu^\omega$ are proper $P-$a.s. If, in addition, the two largest Lyapunov exponents λ_0, λ_1 of the product $M_n^\omega(\xi) = g_{n-1}^\omega(\xi)\cdots g_1^\omega(\xi)g_0^\omega(\xi)$ are different, i.e. $\lambda_0 > \lambda_1$, then for each proper measure $m \in \mathcal{P}(\mathbb{P}^{d-1})$, in particular, for the normalized Lebesgue measure on $\mathbb{P}^{d-1}$, for $P-$a.a.ω and $\Pi^\omega-$a.a.ξ,

$$w\text{-}\lim_{n\to\infty}(M_n^{\theta^{-n}\omega}\circ\sigma^{-n})m = w\text{-}\lim_{n\to\infty}(M_n^{\theta^{-n}\omega}\circ\sigma^{-n})\nu^{\theta^{-n}\omega} = \delta_{V_\infty^\omega} \tag{7.10}$$

and

$$w\text{-}\lim_{n\to\infty}(M_n^\omega)^* m = w\text{-}\lim_{n\to\infty}(M_n^\omega)^*\hat\nu^{\theta^n\omega} = \delta_{\hat V_\infty^\omega}, \tag{7.11}$$

where w-$\lim$ denotes the weak limit, δ_u denotes the Dirac measure at u, and $V_\infty^\omega = V_\infty^\omega(\xi)$ and $\hat V_\infty^\omega = \hat V_\infty^\omega(\xi)$ are random points having the distributions ν^ω and $\hat\nu^\omega$, respectively, i.e. $\int \delta_{V_\infty^\omega(\xi)}d\Pi^\omega(\xi) = \nu^\omega$ and $\int \delta_{\hat V_\infty^\omega(\xi)}d\Pi^\omega(\xi) = \hat\nu^\omega$, and V_∞^ω and $\hat V_\infty^\omega$ are directions of the ranges of any limit point of the sequences $\|M_n^{\theta^{-n}\omega}\circ\sigma^{-n}\|^{-1}M_n^{\theta^{-n}\omega}\circ\sigma^{-n}$ and $\|(M_n^\omega)^*\|^{-1}(M_n^\omega)^*$, respectively. Hence, under the conditions above, the measurable families ν^ω, $\hat\nu^\omega \in \mathcal{P}(\mathbb{P}^{d-1})$ satisfying (7.9) are unique.

(ii) Similarly, replacing θ by θ^{-1}, if $\hat\mu^\omega$ is a strongly irreducible family or, equivalently, μ^ω is reverse strongly irreducible then ν^ω satisfying (7.6) and $\hat\nu^\omega$ satisfying $\hat\mu^\omega * \hat\nu^\omega = \hat\nu^{\theta\omega}$ are proper. If, in addition, the two largest Lyapunov exponents of the product $L_n^\omega = L_n^\omega(\xi) = g_0(\xi)\cdots g_{n-1}^\omega(\xi)$ are different then such families ν^ω and $\hat\nu^\omega$ are unique $P-$a.s. and for any proper $m \in \mathcal{P}(\mathbb{P}^{d-1})$,

$$w\text{-}\lim_{n\to\infty} L_n^\omega m = w\text{-}\lim_{n\to\infty} L_n^\omega \nu^{\theta^n\omega} = \delta_{W_\infty^\omega} \tag{7.12}$$

and

$$w\text{-}\lim_{n\to\infty}(L_n^{\theta^{-n}\omega}\circ\sigma^{-n})^* m = w\text{-}\lim_{n\to\infty}(L_n^{\theta^{-n}\omega}\circ\sigma^{-n})^*\hat\nu^{\theta^{-n}\omega} = \delta_{\hat W_\infty^\omega} \tag{7.13}$$

where the random directions $W^\omega_\infty = W^\omega_\infty(\xi)$ and $\hat{W}^\omega_\infty = \hat{W}^\omega_\infty(\xi)$ are the ranges of limit points of the sequences $\|L^\omega_n\|^{-1}L^\omega_n$ and $\|(L^{\theta^{-n}\omega}_n \circ \sigma^{-n})^*\|^{-1}(L^{\theta^{-n}\omega}_n \circ \sigma^{-n})^*$, respectively.

Proof. The existence of families ν^ω and $\hat{\nu}^\omega$ satisfying (7.9) follows from Kakutani's fixed point theorem (see Lemma 3.5 in [Bo1] and Lemma 4.1 in [Bo2]). Under the corresponding strong irreducibility condition I derive in the same way as in the proof of Theorem 5.4 that such measures ν^ω and $\hat{\nu}^\omega$ are proper $P-$a.s.

Next, assume that $\lambda_0 > \lambda_1$. Consider a polar decomposition $M^{\theta^{-n}\omega}_n(\sigma^{-n}\xi) = K^\omega_n(\xi)A^\omega_n(\xi)U^\omega_n(\xi)$ where $K^\omega_n(\xi)$ and $U^\omega_n(\xi)$ are orthogonal matrices and $A^\omega_n(\xi) =$diag$(a^\omega_{0,n}(\xi), ..., a^\omega_{d-1,n}(\xi))$ is a diagonal matrix with $a^\omega_{0,n} \geq a^\omega_{1,n} \geq \cdots \geq a^\omega_{d-1,n}$. The measure Π such that $d\Pi(\omega,\xi) = d\Pi^\omega(\xi)dP(\omega)$ is $\theta\times\sigma-$invariant and ergodic (by a trivial partial case of Proposition 2.2). It follows from the Oseledec "multiplicative ergodic theorem" (see, for instance, [Ar], Ch.4.) that $\lim_{n\to\infty}\frac{1}{n}\log a^\omega_{i,n} = \lambda_i$ $\Pi-$a.s. where λ_i is the $i-$th Lyapunov exponent and, in particular, I conclude that $\Pi-$a.s.,

$$\lim_{n\to\infty}\frac{a^\omega_{1,n}(\xi)}{a^\omega_{0,n}(\xi)} = 0. \tag{7.14}$$

It follows that $\Pi-$a.s. all limit points as $n\to\infty$ of the sequence $\|M^{\theta^{-n}\omega}_n \circ \sigma^{-n}\|^{-1}(M^{\theta^{-n}\omega}_n \circ \sigma^{-n})$ are rank one matrices and for any proper measure $m \in \mathcal{P}(\mathbb{P}^{d-1})$, $\Pi-$a.s. all weak limit points of $(M^{\theta^{-n}\omega}_n \circ \sigma^{-n})m$ are Dirac measures (see [BL], Ch.III and [GR]).

Call a family $\mathcal{N}$ of measures $\nu\in\mathcal{P}(\mathbb{P}^{d-1})$ equi proper if for any $\varepsilon > 0$ there is $\gamma(\varepsilon) > 0$ such that for any proper subspace V one has $\sup_{\nu\in\mathcal{N}}\nu(\bar{V}_{\gamma(\varepsilon)}) < \varepsilon$, where $\bar{V}_\gamma$ denotes the $\gamma-$neghborhood of the projective subspace corresponding to V. By a compactness argument the family containing a single proper measure is, of course, equi proper. If $M_n \in GL(d,\mathbb{R})$ is a sequence of matrices such that $\|M_n\|^{-1}M_n$ converges to a rank one matrix M and $\nu_n \in \mathcal{P}(\mathbb{P}^{d-1})$, $n = 1, 2, ...$ is an equi proper sequence then an easy compactness argument yields that $w-\lim_{n\to\infty}M_n\nu_n = \delta_{\bar{z}}$ where $\bar{z}\in\mathbb{P}^{d-1}$ is the direction of the range of M.

Now, consider a measurable family $\nu^\omega\in\mathcal{P}(\mathbb{P}^{d-1})$ satisfying (7.9). It is easy to check directly that $(M^{\theta^{-n}\omega}_n \circ \sigma^{-n})\nu^{\theta^{-n}}$, $n = 1, 2, ...$ is a martingale with respect to the $\sigma-$algebras $\mathcal{F}^\omega_n$ generated by $g_0^\omega, g_0^{\theta^{-1}\omega}\circ\sigma^{-1},...,g_0^{\theta^{-(n-1)}\omega}\circ\sigma^{-(n-1)}$ (cf. [Bo2], Lemma 3.6). Thus $\Pi-$a.s. the limit

$$w-\lim_{n\to\infty}(M_n^{\theta^{-n}\omega}\circ\sigma^{-n})\nu^{\theta^{-n}\omega}=\nu^{\omega,\xi}\in\mathcal{P}(\mathbb{P}^{d-1})$$

exists. Furthermore, set $\Gamma_{n,l}=\{\omega:\nu^\omega(\bar V_{\frac{1}{n}})<\frac{1}{l}$ for any proper subspace $V\subset\mathbb{R}^d\}$, $N(l)=\min\{n:P(\Gamma_{n,l})>1-3^{-l}\}$, $\Gamma(l)=\Gamma_{N(l),l}$, and $\Gamma=\cap_{l=1}^\infty\Gamma(l)$. Since $\Gamma_{n,l}\uparrow\Gamma_l$ as $n\uparrow\infty$ and $P(\Gamma_l)=1$ then $N(l)<\infty$ for any l and I conclude that $P(\Gamma)\geq\frac{1}{2}$. Clearly, $\{\nu^\omega,\omega\in\Gamma\}$ is an equi proper family. Define P_Γ and the arrival times $n_\Gamma^{(i)}=n_\Gamma^{(i)}(\omega)$ to Γ as in the beginning of Section 5, but for θ^{-1} in place of θ so that $\theta^{-n_\Gamma^{(i)}(\omega)}\omega\in\Gamma$. Then for $P_\Gamma-$a.a.ω and $\Pi^\omega-$a.a.ξ all weak limit points as $i\to\infty$ of the sequence $(M_{n_\Gamma^{(i)}(\omega)}^{\theta^{-n_\Gamma^{(i)}(\omega)}\omega}\circ\sigma^{-n_\Gamma^{(i)}(\omega)})\nu^{\theta^{-n_\Gamma^{(i)}(\omega)}\omega}$ are Dirac measures. Since, on the other hand, the sequence $(M_n^{\theta^{-n}\omega}\circ\sigma^{-n})\nu^{\theta^{-n}\omega}$ converges $\Pi-$a.s., I conclude from above that $\Pi-$a.s. all limit points of the sequence $\|M_n^{\theta^{-n}\omega}(\sigma^{-n}\xi)\|^{-1}M_n^{\theta^{-n}\omega}(\sigma^{-n}\xi)$ have the same one dimensional range with a random direction $V_\infty^\omega\in\mathbb{P}^{d-1}$ and (7.10) holds true. Since $\int(M_n^{\theta^{-n}\omega}\circ\sigma^{-n})\nu^{\theta^{-n}\omega}d\Pi^\omega=\nu^\omega$ the distribution of the random point V_∞^ω, which depends only on the sequence $M_n^{\theta^{-n}\omega}\circ\sigma^{-n}$, is ν^ω and it follows that ν^ω satisfying (7.9) is unique. Other assertions of Proposition 7.2 hold true, as well, in view of relations explained above. $\square$

7.3. Remark. Observe that Proposition 7.2 remains true in the more general case of independent random bundle maps considered in previous sections. Indeed, if $\rho^\omega\in\mathcal{P}(\mathcal{X})$ is a $\mu^\omega-$stationary ergodic family, $\lambda_0(\rho)>\lambda_1(\rho)$, and ν^ω, say, satisfies (5.5) then I can consider the stationary ergodic process $(\theta^n\omega,X_n^\omega)$ with X_0^ω having the distribution ρ^ω and then the same proof as above yield that $T^*(n,\omega,X_0^\omega)\nu^{\theta^n\omega}_{X_n^\omega}$ weakly converges $\Pi-$a.s. as $n\to\infty$ to $\delta_{V_\infty^\omega}$ where V_∞^ω is a random point in $\mathcal{P}(\mathbb{P}^{d-1})$ (cf. the proof of Proposition 3.3 in [Bo2]). This implies also that for $P-$a.a.ω, $\Pi^\omega-$a.a.ξ, and $\rho^\omega-$a.a.x the sequence $T^*(n,\omega,x,\xi)\nu^{\theta^n\omega}_{X_n^\omega}$, $X_0^\omega=x$ weakly converges to $\delta_{V_\infty^\omega(\xi,x)}$ where $V_\infty^\omega(\xi,x)$ is the direction of the range of any limit point of the sequence $\|T^*(n,\omega,x,\xi)\|^{-1}T^*(n,\omega,x,\xi)$. It follows that the family ν_x^ω satisfying (5.5) is unique $\rho^\omega-$a.s., $P-$a.s.

Modifying arguments of Theorem VI.2.1 from [BL] in the spirit of the first half of Section 6 above one can show proceeding similarly to Section VI.4 in [BL] that under Assumption 6.1 the Hausdorff dimension of measures ν^ω is a positive constant P-a.s. Next, I consider a specific example of random continued fractions where this dimension can be computed explicitly. In the case when Ω is a point this example was considered in [KP] and its connection to products of random i.i.d. matrices was discussed in Section VI.5 of [BL].

Let $A_0^\omega, A_1^\omega, A_2^\omega, \dots$ be independent positive integer valued variables with distributions $\mu^\omega, \mu^{\theta\omega}, \mu^{\theta^2\omega}, \dots \in \mathcal{P}(\mathbb{Z}_+)$ and set $p_i^\omega = \mu^\omega(\{i\})$, $\bar{p}_i = \int p_i^\omega dP(\omega)$. Assume that

$$0 < \sum_{i=1}^{\infty} \bar{p}_i \log i < \infty. \tag{7.15}$$

Suppose that μ^ω is not a Dirac measure with positive probability. Denote by $\Xi = \mathbb{Z}_+^{\mathbb{Z}_+}$ the sequence space and set $\Pi^\omega = \prod_{i=0}^{\infty} \mu^{\theta^i\omega} \in \mathcal{P}(\Xi)$. Now I can write $A_n^\omega = A_n^\omega(\xi) = \xi_n$ where $\xi = (\xi_0, \xi_1, \dots) \in \Xi$. Consider independent random matrices $g_n^\omega(\xi) = \begin{pmatrix} 0 & 1 \\ 1 & A_n^\omega(\xi) \end{pmatrix}$, $n = 0, 1, \dots$ and denote, again, by μ^ω the distribution of g_0^ω in $GL(2, \mathbb{Z})$ so that g_n^ω is distributed according to $\mu^{\theta^n\omega}$. For any vector $(a, b) \in \mathbb{R}^2$ represent the corresponding point of the projective space $\mathbb{P}^1$ by the number $\frac{a}{b}$ which is the cotangent of the appropriate angle. Since $\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}\begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ a + x \end{pmatrix}$ is represented by $\frac{1}{a+x}$ then $g = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$ acts on $x \in \mathbb{P}^1$ by the formula $g \cdot x = \frac{1}{a+x}$, and so $L^\omega(n) = L^\omega(n, \xi) = g_0^\omega(\xi) g_1^\omega(\xi) \cdots g_{n-1}^\omega(\xi)$ acts by

$$L^\omega(n) \cdot x = \cfrac{1}{A_0^\omega + \cfrac{1}{A_1^\omega + \cdots + \cfrac{1}{A_{n-1}^\omega + x}}}. \tag{7.16}$$

It follows that $L^\omega(n) \cdot 0$ converges Π^ω-a.s. to a real random variable V_∞^ω with values in $[0, 1]$ and a distribution $\nu^\omega \in \mathcal{P}(\mathbb{P}^1)$ satisfying (7.6).

Let σ be the left shift on Ξ, $\tau(\omega, \xi) = (\theta\omega, \sigma\xi)$, $\xi \in \Xi$ and define $\Pi \in \mathcal{P}(\Omega \times \Xi)$ by $d\Pi(\omega, \xi) = d\Pi^\omega(\xi) dP(\omega)$. It follows trivially from Lemma 2.1 and Proposition 2.2 (with the space $\mathcal{V}$ there being a point) that Π is τ-invariant and ergodic.

Observe that

$$|L^\omega(n) \cdot x - L^\omega(n) \cdot y| \le |x - y| (\prod_{i=0}^{n-1} A_i^\omega)^{-2}. \tag{7.17}$$

By (7.15), ergodicity of Π and the ergodic theorem for P-a.a.ω, Π^ω-a.s.,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{n-1}\log A_j^\omega=\sum_{i=1}^{\infty}\bar p_i\log i>0. \tag{7.18}$$

Thus Π-a.s. the right hand side of (7.17) tends to zero exponentially fast.

The contraction property above yields also that the family μ^ω is reverse strongly irreducible. Indeed, if $V_\omega^{(i)}$, $i=1,\dots,k$ are one dimensional subspaces of $\mathbb{R}^2$ satisfying $g(\bigcup_{i=1}^k V_{\theta\omega}^{(i)})=\bigcup_{i=1}^k V_\omega^{(i)}$ for $\mu^\omega-$a.a.g and $P-$a.a.ω then

$$L^\omega(n)(\bigcup_{i=1}^k V_{\theta^n\omega}^{(i)})=\bigcup_{i=1}^k V_\omega^{(i)}\quad \Pi-\text{a.s.} \tag{7.19}$$

Let $\bar V_\omega^{(i)}$ denotes the representation in $\mathbb{P}^1$ of the line $V_\omega^{(i)}$ as a point in $\mathbb{R}$ and set $\Gamma_K=\{\omega:|\bar V_\omega^{(i)}|\le K\ \forall i=1,\dots,k\}$. Choose K large enough so that $P(\Gamma_K)>0$. If $\theta^n\omega\in\Gamma_K$ then by (7.17),

$$\max_{i\ne j}|L^\omega(n)\cdot\bar V_{\theta^n\omega}^{(i)}-L^\omega(n)\cdot\bar V_{\theta^n\omega}^{(j)}|\le 2K(\prod_{i=0}^{n-1}A_i^\omega)^{-2}.$$

Taking a subsequence $n_\ell=n_\ell(\omega)\to\infty$ such that $\theta^{n_\ell}\omega\in\Gamma_K$ I conclude from here and (7.18) that (7.19) is only possible if $k=1$. But since μ^ω is not a Dirac measure with positive probability and $\begin{pmatrix}0&1\\1&a\end{pmatrix}V\ne\begin{pmatrix}0&1\\1&b\end{pmatrix}V$ for any line V if $a\ne b$ then (7.19) cannot hold true also for $k=1$.

It follows by Proposition 7.2 that ν^ω is proper, i.e. it has no atoms P-a.s. Recall, that any real number $t\in(0,1)$ can be expanded in a continued fraction

$$t=\lim_{n\to\infty}(\xi_0(t);\xi_1(t);\dots;\xi_n(t))$$

where $(\xi_0;\xi_1;\dots,\xi_{n-1})=\frac{1}{\xi_0+\frac{1}{\xi_2+\dots+\frac{1}{\xi_{n-1}}}}$. When t is irrational this expansion is unique and since ν^ω has no atoms P-a.s. then this expansion is ν^ω-a.s. unique, i.e. the map $\pi:\Xi\to(0,1)$ given by

$$\pi(\xi)=(\xi_0;\xi_1;\dots)=\frac{1}{\xi_0+\frac{1}{\xi_1+\cdots}}$$

has the unique inverse $\pi^{-1}(t)$ for ν^ω-a.a. t. It follows that the law of $A_0^\omega,A_1^\omega,\dots,A_n^\omega$ under Π^ω is the same as of $\xi_0(t),\xi_1(t),\dots,\xi_n(t)$ under ν^ω and $\pi\Pi^\omega=\nu^\omega$. Consider the map $T:(0,1)\to(0,1)$ given by $T(t)=\frac{1}{t}-[\frac{1}{t}]$. Then $T\pi=\pi\sigma$, and so

$$T\nu^\omega=\nu^{\theta\omega}\ P\text{-a.s.} \tag{7.20}$$

If $\hat\tau(\omega,t)=(\theta\omega,T(t))$ then the measure ν defined by $d\nu(\omega,t)=d\nu^\omega(t)dP(\omega)$ is $\hat\tau$-invariant and I conclude from ergodicity of Π that ν is ergodic, as well.

Next, I claim that ν^ω is singular with respect to the Lebesgue measure on $(0,1)$ for $P-$a.a.ω. Indeed, let γ be the Gauss measure, i.e. $\gamma(U)=\frac{1}{\log 2}\int_U\frac{dt}{1+t}$ for any Borel $U\subset(0,1)$. It is known that γ is $T-$invariant and mixing (see [CFS], p.174) and since P is $\theta-$invariant and ergodic it follows that the product measure $\gamma\times P$ is ergodic with respect to the product transformation $T\times\theta$ (see [CFS], p. 229). Since ν is ergodic with respect to $T\times\theta$ and have the same marginal P on Ω as $\gamma\times P$ then either ν^ω coincide with γ for $P-$a.a.ω or ν^ω is singular with γ for $P-$a.a.ω. The first case is impossible since by elementary computation $\gamma\{t:\xi_0(t)=1,\xi_1(t)=1\}\neq\gamma\{t:\xi_0(t)=1\}\gamma\{t:\xi_1(t)=1\}$, and so the claim is proved.

Since the family μ^ω is reverse strongly irreducible then by Theorem 3.1 the largest Lyapunov exponent λ_0 satisfies

$$\lambda_0=\lim_{n\to\infty}\frac{1}{n}\log\|L^\omega(n)\begin{pmatrix}0\\1\end{pmatrix}\|\quad\Pi\text{-a.s.}\tag{7.21}$$

It is easy to check that

$$\begin{aligned}L^\omega(n)\begin{pmatrix}0\\1\end{pmatrix}&=\prod_{i=0}^{n-1}(A_i^\omega+\frac{1}{A_{i+1}^\omega+\cdots\frac{1}{A_{n-1}^\omega}})\begin{pmatrix}L^\omega(n)\cdot 0\\1\end{pmatrix}\\&=(\prod_{i=0}^{n-1}(L^{\theta^i\omega}(n-i)\circ\sigma^i)\cdot 0)^{-1}\begin{pmatrix}L^\omega(n)\cdot 0\\1\end{pmatrix}.\end{aligned}\tag{7.22}$$

This together with (7.21) and ergodicity of Π yield by the left hand side of (7.15) that

(7.23)

$$\begin{aligned}\lambda_0&=-\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\iint(\log((L^{\theta^i\omega}(n-i)\circ\sigma^i)\cdot 0))d\Pi^\omega dP(\omega)=\\&=-\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\iint(\log(L^\omega(n-i)\cdot 0)d\Pi^\omega dP(\omega)=-\iint\log V_\infty^\omega d\Pi^\omega dP(\omega)\\&=-\iint\log t d\nu^\omega(t)dP(\omega)>\sum_{i=1}^\infty\bar p_i\log i>0.\end{aligned}$$

On the other hand, by the right hand side of (7.15),

$$\lambda_0 \le \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \log \|g_i^\omega\| = \int \log(1+A_0^\omega) d\Pi^\omega dP(\omega) \tag{7.24}$$
$$= \sum_{i=1}^{\infty} \bar p_i \log(1+i) < \infty.$$

Since $\det g_i^\omega = 1$ then the other Lyapunov exponent λ_1 must be negative. Observe that g_i^ω's are self adjoint, and so $\mu^\omega = \hat\mu^\omega$. Since μ^ω is reverse strongly irreducible I conclude from Proposition 7.2 that ν^ω satisfying (7.6) is unique and it must be the distribution of V_∞^ω.

Observe that by Jensen's inequality

$$-\sum_{i=2}^{\infty} \frac{p_i^\omega}{1-p_1^\omega} \log \frac{i^2 p_i^\omega}{1-p_1^\omega} \le \log(\sum_{i=2}^{\infty} i^{-2}) = \log(\frac{\pi^2}{6} - 1). \tag{7.25}$$

Set $h = -\sum_{i=1}^{\infty} \int p_i^\omega \log p_i^\omega dP(\omega)$ which is the relativized entropy of T with respect to the measure ν (see [Ki1]). Integrating in (7.25) in ω and applying Jensen's inequality to the function $-x\log x$ I obtain

$$h \le 2\sum_{i=1}^{\infty} \bar p_i \log i + (1-\bar p_1)\log(\frac{\pi^2}{6} - 1) - \bar p_1 \log \bar p_1 - (1-\bar p_1)\log(1-\bar p_1). \tag{7.26}$$

This together with (7.15) imply, in particular, that $h < \infty$.

7.4. Proposition. *P-a.s.,*

$$\dim_H \nu^\omega = \frac{h}{2\lambda_0} \tag{7.27}$$

where $\dim_H$ *denotes the Hausdorff dimension of a measure, i.e. the infimum of Hausdorff dimensions of sets of full measure.*

Proof. Set $J_n^\omega(\xi) = L^\omega(n,\xi)\cdot[0,1]$. Then in the same way as in Section VI.5 from [BL] I derive that for $P-$a.a.ω and $\Pi-$a.a.ξ,

$$\lim_{n\to\infty} \frac{1}{n} \log |J_n^\omega(\xi)| = -2\lambda_0 \tag{7.28}$$

where $|I|$ denotes the length of an interval I. On the other hand,

$$\nu^\omega(J_n^\omega(\xi)) = \nu^\omega\{t \in (0,1) : \xi_i(t) = A_i^\omega(\xi)\ \forall i = 0,1,\dots,n-1\} \tag{7.29}$$
$$= \prod_{i=0}^{n-1} \mu^{\theta^i\omega}(\{A_i^\omega(\xi)\}).$$

This together with ergodicitiy of Π imply that for P-a.a.ω and $\Pi-$a.a.ξ,

$$\lim_{n\to\infty}\frac{1}{n}\log\nu^\omega(J_n^\omega(\xi))=h. \tag{7.30}$$

Thus $\lim_{n\to\infty}\frac{\log\nu^\omega(J_n^\omega(\xi))}{\log|J_n^\omega(\xi)|}$ equals the right hand side of (7.27) for P-a.s.ω and $\Pi-$a.a.ξ and Proposition 7.4 follows by an easy "random" modification of Lemma 3.1 in [KP] (cf. [Ki3]). $\square$

Since $2\lambda_0$ is the Lyapunov exponent of the map T corresponding to the measure ν then (7.27) has the usual in the one dimensional situation form: dimension= $\frac{\text{entropy}}{\text{exponent}}$. Several arguments were suggested which should lead to the proof that $\dim_H\nu^\omega<1$ for $P-$a.a.ω but they are outside of the scope of this paper. This should follow also from the explicit formula (7.27) but, as far as I know, even in the case of [KP] when Ω is just one point, no good estimates of the right hand side in (7.27) for the general case appeared in the literature though it is easy to show that this expression is strictly less than one for some partial cases, for example, when $\bar p_1$ is close to 0 (which follows from (7.23) and (7.26)) and when $\bar p_1$ is close to 1 since then h is close to 0 and $-\lambda_0$ is close to the logarithm of the golden mean $\frac{1}{2}(\sqrt{5}-1)$.

7.5. Remark. The example above can be generalized in the spirit of [KP] considering random $f-$expansions, namely, representing a number $x\in(0,1)$ in the form

$$x=\lim_{n\to\infty}f_\omega(A_0^\omega+f_{\theta\omega}(A_1^\omega+\cdots+f_{\theta^{n-1}\omega}(A_{n-1}^\omega)...))$$

where f_ω is a random decreasing (or increasing) function satisfying some properties which ensure convergence of such expansions and $A_i^\omega=A_i^\omega(x)$ are positive integer coefficients of the expansion so that $A_0^\omega(x)=[f_\omega^{-1}(x)]$ and $A_n^\omega(x)=A_{n-1}^{\theta\omega}(T_\omega x)$, $n=1,2,...$ with the random transformation $T_\omega x=f_\omega^{-1}(x)-[f_\omega^{-1}(x)]$. Recall, that continued fraction expansions correspond to the particular (nonrandom) decreasing function $f(t)=\frac{1}{t}$. The case of the increasing function $f_\omega(t)=\ell(\omega)t \pmod 1$ with a positive integer valued random variable ℓ leads to random base expansions considered in [Ki3]. If one chooses the coefficients A_i^ω independently with distributions changing stationarily as above or having Markov dependence (with stationarily changing transition probabilities) then modifying arguments from [KP] it is possible to estimate the Hausdorff dimension of the distribution of the corresponding random point on (0,1) similarly to Proposition 7.4.

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