METRIC PAIRS AND THE FUTAKI CHARACTER

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Abstract. The Futaki character gives an obstruction to the existence of Kähler metrics of constant scalar curvature in a fixed Kähler class [F2, C2]. We show that in combination with the resolution of the Calabi conjecture [Yu], one has an analogous obstruction on pairs of metrics in different Kähler classes. If the difference of the Futaki characters on two classes of fixed total volume does not vanish identically, there cannot exist a pair of metrics in these classes which have the same Ricci form *and* the same harmonic Ricci form. When the obstruction vanishes, results in [H] are used to construct non-trivial examples of such pairs which are also extremal [C1].

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1. Introduction

In [F1, F2, C2], Futaki defined a character on the Lie algebra of holomorphic vector fields on a compact Kähler manifold, attached to a fixed Kähler class, whose non-vanishing provides an obstruction to the existence of Kähler metrics of constant scalar curvature in the given class. The purpose of this note is to show that if ones combines the invariant with Yau's solution to the Calabi conjecture, Futaki's character also gives information about pairs of metrics in different Kähler classes. We call a pair of Kähler metrics a *harmonic (Calabi-Yau) pair*, if the two metrics share the same Ricci form *and* the same harmonic Ricci form. Such a condition is, of course, overdetermined. It is of interest, however, that there exists a topological obstruction to its fullfilment. Our result is a slight generalization of the following.

Theorem 1.1. *Let M be a compact Kähler manifold, and $\Omega, \tilde{\Omega}$ a pair of Kähler classes of fixed total volume. If the difference of the Futaki characters of the two classes does not vanish identically, then there does not exist a harmonic pair of Kähler metrics with Kähler forms $(\omega, \tilde{\omega}) \in \Omega \times \tilde{\Omega}$. When this obstruction vanishes, there are examples of harmonic pairs which are also extremal.*

Here an extremal Kähler metric is one which minimizes the L^2-norm of the scalar curvature in its Kähler class [C1]. Note that one can easily find examples of harmonic pairs, each of which is a product of metrics of constant scalar curvature, or indeed a product of extremal metrics. The merit of the examples we give is that they are not product metrics. To construct them we rely heavily on results of Hwang [H].

After gathering the necessary preliminaries in Section 2, we show the existence of the obstruction in Section 3, relate it to Mabuchi's K-energy map [Mb] and to extremal metrics. Section 4 is devoted to giving extremal harmonic examples.

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2. Preliminaries

2.1. **The Futaki Invariant.** Let $M := M_n$ be a compact Kähler manifold of complex dimension n. Given a Kähler metric g on M, with Kähler form ω and Ricci form ρ, denote by ρ_H the harmonic part of ρ. Since ρ and ρ_H belong to the same cohomology class, there exists (cf. [GH, Chapter 1, Section 2]) a smooth real valued function F, called the **Ricci potential**, such that

$$\rho - \rho_H = i\partial\bar{\partial}F. \tag{1}$$

Unless otherwise stated, we normalize F to be L^2-perpendicular to the constants.
We recall the definition of the Futaki invariant.

Definition. Let (M_n, ω) be a compact Kähler manifold with Ricci potential F. The **Futaki character** is the map $\mathcal{F}_{[\omega]} : h(M) \to \mathbb{C}$, where $h(M)$ denotes the Lie algebra of holomorphic vector fields on M, given by

$$\mathcal{F}_{[\omega]}(\Xi) = \int_M \Xi F \frac{\omega^{\wedge n}}{n!}. \tag{2}$$

Remark. The values of this character do not depend on the choice of metric in the Kähler class $[\omega]$ (see [Ba, C2, F2]), i.e., it is a Kähler class invariant.

Remark. This invariance is what in fact implies that $\mathcal{F}_{[\omega]}$ is a Lie algebra character (cf. [C2]).

Remark. $\mathcal{F}_{[\omega]}$ is completely determined by its values on a subalgebra of $h(M)$: by the Hodge decomposition, the $(0,1)$-form α which is metrically dual to a given holomorphic vector field Ξ decomposes as $\alpha = \alpha_H + \bar{\partial} f$, with α_H harmonic and f a complex valued function (the term involving $\bar{\partial}^*$ vanishes by the holomorphicity of Ξ and the local implications of g being Kähler). Then

$$\mathcal{F}_{[\omega]}(\Xi) = \int_M \Xi F \frac{\omega^{\wedge n}}{n!} = \int_M (\alpha_H, \bar{\partial} F) \frac{\omega^{\wedge n}}{n!} + \int_M (\bar{\partial} f, \bar{\partial} F) \frac{\omega^{\wedge n}}{n!},$$

where $(\cdot,\cdot)$ denotes the pointwise inner product induced on $(0,1)$-forms by the Kähler metric. But the first term on the right hand side vanishes since after integrating by parts we see that it equals $\int_M \bar{\partial}^* \alpha_H F \frac{\omega^{\wedge n}}{n!}$ and α_H, being harmonic, is co-closed.

Therefore, it suffices to consider holomorphic vector fields $\Xi = \Xi_f$ whose dual $(0,1)$-form is of the form $\bar{\partial} f$. We call Ξ_f a **gradient vector field**, and $f = f_\Xi$ a **holomorphy potential**. On compact Kähler manifolds the gradient vector fields form a Lie subalgebra. Another way to characterize them is as those holomorphic vector fields having a non-empty zero set (cf. [Kb, Part II, Corollary 4.6]).

2.2. **Holomorphy Potentials and the Ricci Form.** A holomorphy potential also satisfies

$$\imath_\Xi \omega = \bar{\partial} f, \tag{3}$$

and satisfies the following analogous relation with the Ricci form:

Proposition 2.1. *Let M be a complex manifold of Kähler type and Ξ a gradient holomorphic vector field on it. Suppose g is a Kähler metric on M with Kähler form ω and Ricci form ρ. Then if f is a smooth complex valued function on M satisfying*

$$\imath_\Xi \omega = \bar{\partial} f, \tag{4}$$

we have

$$\imath_\Xi \rho = \bar{\partial}(\Delta f). \tag{5}$$

Moreover, if M is compact, the second equation implies the first.

Remark. Equivalently, using the $\bar{\partial}$-Laplacian on 1-forms, one can write:

$$\imath_\Xi \rho = \Delta(\imath_\Xi \omega).$$

Proof of Proposition 2.1 This proposition is well known and goes back to Bochner [Bo] and Yano [Yn]. For a convenient proof of a more general statement involving *any* holomorphic vector field, see [Kb, Theorem 4.2]. □

For an interpretation of these relations in the terms of holomorphic equivariant cohomology [L], see [Ms1, Ms2].

2.3. **Scalar Curvature, Extremal Kähler metrics.** The relation between the scalar curvature s, and the Ricci potential F, is gotten by taking traces with respect to ω in equation (1), and using the Hard Lefschetz Theorem. One has [1]:

$$\Delta F = s - s_0, \tag{6}$$

where $\Delta := \Delta_{\bar\partial}$ denotes the $\bar\partial$-Laplacian: $\Delta := \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$, and s_0 is average value of the scalar curvature over M,

$$s_0 = \frac{\int_M s\frac{\omega^{\wedge n}}{n!}}{\int_M \frac{\omega^{\wedge n}}{n!}}.$$

This quantity depends only on the Kähler class.
Note in particular that a Kähler metric has constant scalar curvature if and only if its Ricci potential is harmonic.

The L^2 norm of the scalar curvature figures in Calabi's notion of an extremal Kähler metric [C1].

Definition. A Kähler metric g with Kähler form ω on a complex manifold M_n will be called an **extremal** Kähler metric if it is critical point of the functional

$$g \longrightarrow \int_M s_g^2 \frac{\omega_g^{\wedge n}}{n!} \tag{7}$$

among Kähler metrics in the class $[\omega]$. Here s_g denotes the scalar curvature of g.

Specializing to compact manifolds, one has:

Proposition 2.2 (Calabi [C1])**.** *For M compact a Kähler metric is extremal if and only if its scalar curvature is a holomorphy potential, i.e. $s^{,\alpha}_{\ ,\bar\beta} = 0$.*

In particular, if g is a Kähler-Einstein metric, or, more generally, any Kähler metric of constant scalar curvature, it is extremal.

2.4. **The K-Energy Map.** Given a Kähler class Ω, fix a Kähler form $\omega_0 \in \Omega$. Let ϕ_t be a one-parameter family of smooth real valued functions such that $\phi_0 = 0$, and $\omega_t := \omega_0 + i\partial\bar\partial\phi_t$ is a Kähler form. Denoting $\omega = \omega_1$, Mabuchi's K-energy map [Mb] is the functional on Kähler forms in Ω, given by

$$\mu(\omega) := \mu_{\Omega,\omega_0}(\omega) := M(\omega_0, \omega) := -\int_0^1 \Big(\int_M \dot\phi_t (s_{\phi_t} - s_0)\frac{\omega_t^{\wedge n}}{n!}\Big)\mathrm{dt}.$$

Here $\dot\phi_t$ deontes differentiation with respect to t, and s_{ϕ_t} is the scalar curvature of ω_t. μ is independent of the path ϕ_t between ω_0 and ω_1, and changes by a constant upon changing the basepoint ω_0. $M(.,.)$ satisifies a cocycle condition. The K-energy map has Kähler forms of metrics of constant scalar curvature as critical points. Moreover, given the real part of a holomorphic vector field, and exponentiating it to get a one-parameter group of diffeomorphisms acting by pull-back on ω, the derivative of $\mu(\omega)$ along this orbit is exactly the real part of the Futaki invariant, evaluated on the original vector field.

[1] For convenience, we have absorbed into s a factor of $-1/2$, as compared with the standard Riemannian definition.

2.5. **The Calabi Conjecture.** Recall that on a compact Kähler manifold M, the Ricci form of any Kähler metric is closed, and its corresponding second cohomology class in $H^2(M, \mathbb{C})$ is the first Chern class $c_1(M)$. Also, the Ricci form is completely determined by the complex structure and the volume form, via the formula

$$\rho = -i\partial\bar{\partial} \log \det g. \tag{8}$$

Given this, one has two equivalent formulations of the Calabi Conjecture, which was resolved by Yau [Yu].

Theorem 2.3 (Calabi-Yau [Yu]). *Let M_n be a compact Kähler manifold. If $\bar{\rho}$ is a real closed $(1,1)$-form representing $c_1(M)$ (if Ψ is a real non-degenerate (n,n)-form), then in every Kähler class there exists a unique Kähler form ω, whose Ricci form equals $\bar{\rho}$ (whose volume form equals a positive multiple of Ψ).*

Let g be a Kähler metric with Kähler form ω and Ricci form ρ. Given a Kähler class $\tilde{\Omega}$, we will call the unique Kähler form $\tilde{\omega} \in \tilde{\Omega}$, the **Calabi-Yau Representative** of ω in $\tilde{\Omega}$.

3. The Reflection Character

3.1. **Definition, Invariance.** We define a new character attached to pairs of Kähler classes, and proceed to relate it to the Futaki character.

Definition. Let M_n be a Kähler manifold, Ω and $\tilde{\Omega}$ two Kähler classes. Let ω be a Kähler form in Ω with Calabi-Yau representative $\tilde{\omega}$ in $\tilde{\Omega}$. Define the **reflection potential** $\Phi := \Phi_{\omega,\tilde{\Omega}}$ of the pair $(\omega, \tilde{\Omega})$ to be the smooth real valued function given up to an additive constant by

$$\rho_H - \tilde{\rho_H} = i\partial\bar{\partial}\Phi,$$

where ρ_H, $\tilde{\rho_H}$ are the ω-harmonic and $\tilde{\omega}$-harmonic representatives, respectively, in the class c_1. The **reflection character** is defined to be the Lie algebra character $\mathcal{R}_{\Omega}^{\tilde{\Omega}} : h(M) \to \mathbb{C}$, given by

$$\mathcal{R}_{\Omega}^{\tilde{\Omega}}(\Xi) = \int_M \Xi(\Phi) \frac{\omega^{\wedge n}}{n!}. \tag{9}$$

We call $(\omega, \tilde{\omega})$ a **Calabi-Yau (metric) pair**, and say they form a **harmonic Calabi-Yau pair**, or simply a harmonic pair, if $\rho_H = \tilde{\rho_H}$.

We at once to show that this invariant is well defined, and indeed is a character.

Proposition 3.1. *Keeping notations as in the definition, $\mathcal{R}_{\Omega}^{\tilde{\Omega}}$ does not depend on the choice of ω in Ω. Furthermore, we have $\mathcal{R}_{\Omega}^{\tilde{\Omega}} \equiv -A\mathcal{R}_{\tilde{\Omega}}^{\Omega}$, where $A := A_{\tilde{\Omega}}^{\Omega} = \frac{\Omega^{\wedge n}}{\tilde{\Omega}^{\wedge n}}$ (the volume ratio of the classes).*

Proof. Since ω, $\tilde{\omega}$ are Calabi-Yau related, one has $A\frac{\tilde{\omega}^{\wedge n}}{n!} = \frac{\omega^{\wedge n}}{n!}$ $(A > 0)$. Also, if F and $\tilde{F}$ are the Ricci potentials of ω and $\tilde{\omega}$, respectively, we have, $i\partial\bar{\partial}\Phi = \rho_H - \tilde{\rho_H} = \rho_H - \rho + \rho - \tilde{\rho_H} = \rho_H - \rho + \tilde{\rho} - \tilde{\rho_H} = -i\partial\bar{\partial}F + i\partial\bar{\partial}\tilde{F}$. Choosing $\Phi = -F + \tilde{F}$, and using the volume form proportionality of the pair, we get

$$\mathcal{R}_{\Omega}^{\tilde{\Omega}}(\Xi) = \int_M \Xi(\Phi)\tfrac{\omega^{\wedge n}}{n!} = -\int_M \Xi(F)\tfrac{\omega^{\wedge n}}{n!} + \int_M \Xi(\tilde{F})\tfrac{\omega^{\wedge n}}{n!} =$$

$$-\int_M \Xi(F)\tfrac{\omega^{\wedge n}}{n!} + A\int_M \Xi(\tilde{F})\tfrac{\tilde{\omega}^{\wedge n}}{n!} = -\mathcal{F}_{[\omega]}(\Xi) + A\mathcal{F}_{[\tilde{\omega}]}(\Xi),$$

which is an expression depending only on Kähler classes. The last statement follows again from the relation between the volume forms, together with the relation $\Phi_{\omega,\tilde{\Omega}} = -\Phi_{\tilde{\omega},\Omega}$. $\square$

3.2. **Relation To The K-Energy Map.** Note that on Kähler classes of metrics of fixed total volume, the reflection character satisfies cocycle condition:

$$\begin{aligned} \mathcal{R}_{\Omega}^{\tilde{\Omega}} &\equiv -\mathcal{R}_{\tilde{\Omega}}^{\Omega} \\ \mathcal{R}_{\Omega}^{\tilde{\Omega}} + \mathcal{R}_{\tilde{\Omega}}^{\tilde{\tilde{\Omega}}} &\equiv \mathcal{R}_{\Omega}^{\tilde{\tilde{\Omega}}}. \end{aligned}$$

Working for simplicity with such classes in what follows, the reflection character can be related to the K-energy map. Fixing $\omega_0 \in \Omega$, if $\tilde{\omega_0} \in \tilde{\Omega}$ is the Calabi-Yau representative of ω_0 in $\tilde{\Omega}$, define

$$\nu_{\Omega,\omega_0}^{\tilde{\Omega}}(\omega) = \mu_{\Omega,\omega_0}(\omega) - \mu_{\tilde{\Omega},\tilde{\omega_0}}(\tilde{\omega}).$$

Then on orbits of the one parameter group constructed from a holomorphic vector field as in subsection 2.4, the functional ν has derivative equal to the real part of the reflection character of the two classes, evaluated on the vector field.

3.3. **Relation To Extremal Metrics.**

Proposition 3.2. *Let M be a compact Kähler manifold and Ω, $\tilde{\Omega}$ two Kähler classes (having the same total volume). Suppose $\omega \in \Omega$ is the Kähler form of a Kähler metric g, $\tilde{\omega} \in \tilde{\Omega}$ its Calabi-Yau representative with corresponding metric $\tilde{g}$, $\rho = \tilde{\rho}$ their shared Ricci form, F, $\tilde{F}$ the corresponding Ricci potentials and Φ the reflection potential of $(\omega, \tilde{\Omega})$. We will assume as usual that these potentials are normalized to be L^2-orthogonal to the constants. Denote also by $\Delta, \tilde{\Delta}$ the respective $\bar{\partial}$-Laplacians of the two metrics. Then we have the following:*

- **A:** *If $(\omega, \tilde{\omega})$ form a harmonic pair then $\mathcal{R}_{\Omega}^{\tilde{\Omega}} \equiv 0$.*
- **B:** *If $\Delta\Phi$ is a holomorphy potential and $\mathcal{R}_{\Omega}^{\tilde{\Omega}}(\Xi_{\Delta\Phi}) = 0$ then $(\omega, \tilde{\omega})$ form a harmonic pair.*
- **C:** *If $(\omega, \tilde{\omega})$ form a harmonic pair then g has constant scalar curvature if and only if $\tilde{g}$ has constant scalar curvature.*
- **D:** *If $(\omega, \tilde{\omega})$ form a harmonic pair, and g is extremal, then $\tilde{g}$ is extremal with respect to the same holomorphic vector field, if and only if $\Delta^2 F = \tilde{\Delta}^2 F$.*
- **E:** *If both g and $\tilde{g}$ are extremal with respect to the same holomorphic vector field, then $(\omega, \tilde{\omega})$ form a harmonic pair if and only if $\Delta^2 F = \Delta^2 \tilde{F}$.*

Proof. **A** and **B** are proved as for the Futaki character, with the reflection potential taking the place of the Ricci potential. **C** follows since the assumptions imply $\tilde{\rho_H} = \rho_H = \rho = \tilde{\rho}$. For **D**, the pair being harmonic means $\Phi = 0$, which implies $F = \tilde{F}$. Since ΔF differs by a constant from the scalar curvature of the extremal metric g, by Proposition 2.1,

$$\bar{\partial}\Delta^2 F = \imath_{\Xi_{\Delta F}}\rho = \imath_{\Xi_{\Delta F}}\tilde{\rho}. \tag{10}$$

Now if $\tilde{g}$ is extremal (with respect to the same vector field $\Xi_{\Delta F}$), then $\imath_{\Xi_{\Delta F}}\tilde{\rho} = \bar{\partial}\tilde{\Delta}^2\tilde{F} = \bar{\partial}\tilde{\Delta}^2 F$, so combining with (10), $\Delta^2 F$ and $\tilde{\Delta}^2 F = \tilde{\Delta}^2\tilde{F}$ differ by a constant. But now the volume forms of the two metrics are equal, so integration of either of these double Laplacians together with the divergence theorem eliminates the constant. Assuming now $\Delta^2 F = \tilde{\Delta}^2 F$, and again combining with equation (10), results in $\imath_{\Xi_{\Delta F}}\tilde{\rho} = \bar{\partial}\tilde{\Delta}^2\tilde{F}$, which, via a second use of Proposition 2.1, implies that $\tilde{g}$ is extremal with respect to $\Xi_{\Delta F}$. Finally, to show **E**, we note in one direction again that $\Phi = 0$ implies $F = \tilde{F}$. The assumption in the other direction means that $\Delta(\Delta\tilde{F} - \Delta F) = 0$, so the difference inside the brackets must be a constant. Integrating with respect to $\omega^{\wedge n}$ and using the Divergence Theorem eliminates the constant. Repeating this argument eliminates the remaining Laplacian, and so we see that $F = \tilde{F}$, $\Phi = 0$, and the pair is harmonic. □

In the next section we give examples of harmonic Calabi-Yau pairs of extremal metrics.

4. Extremal Harmonic Pairs

In [H], families of extremal Kähler metrics are constructed in a manner analogous to the one given in [S, KS1, KS2] for the Kähler-Einstein case. We demonstrate the existence of non-trivial harmonic pairs of extremal metrics using only special cases of these constructions. Recall that we are looking for examples which are not product metrics. Instead, the metrics live on projectivized vector bundles over products.

4.1. **The Construction.** We begin with a summarized description of the construction. We refer the reader to the papers above, as well as to [HS], for further details. Let $M \times M$ be a product of two copies of a Kähler-Einstein manifold M of positive Ricci curvature, second Betti number equal to one and dimension l. Let ω be an indivisible integral Kähler-Einstein form on M with Ricci form $c_1(M,\omega) = k\omega$, $k > 0$. Let $p : (L, h) \to N = M \times M$ be a holomorphic Hermitian line bundle having first Chern form $c_1(L, h) = n\omega_1 + n\omega_2$, with h the Hermitian metric, n a positive integer, and $\omega_i = \pi_i^*\omega$ with π_i the corresponding projections on the factors of N. Also, take B to be the symmetric two-tensor associated with $2\pi c_1(L, h)$ via the complex structure on N. Fix positive real numbers a_1, a_2 and b with $a_i \pm bn > 0, a_1 \neq a_2$. Let g_N be the Kähler metric on N with Kähler form $a_1\omega_1 + a_2\omega_2$. The Ricci tensor r_N of g_N has constant eigenvalues $\frac{k}{a_1}, \frac{k}{a_2}$, each of multiplicity l with respect to g_N, and B has eigenvalues $\frac{n}{a_1}, \frac{n}{a_2}$, also of multiplicity l.

Define two functions

$$Q(x) := \det(I - xg_N^{-1}B) = (1 - \frac{n}{a_1}x)^l(1 - \frac{n}{a_2}x)^l,$$

$$T(x) := \mathrm{tr}_{(g_N - xB)}\, r_N = \frac{kl}{a_1 - nx} + \frac{kl}{a_2 - nx}.$$

Q and $T \cdot Q$ are everywhere defined and positive on $(-b, b)$. To emphasize the dependence on the Kähler class (of the base), we will sometimes write Q_{a_1,a_2}, T_{a_1,a_2}.

Now use Q and T to define $\phi : [-b, b] \to \mathbb{R}$ by

$$(\phi Q)(x) = 2(x + b)Q(-b) - 2\int_{-b}^{x}(\sigma_0 + \lambda y - T(y))(x - y)Q(y)\, dy,$$

where the constants σ_0 and λ can be written in terms of b, a_i, and n, by solving the equations

$$\sigma_0\alpha_0 + \lambda\alpha_1 = Q(b) + Q(-b) + \int_{-b}^{b} T(x)Q(x)\, dx,$$

$$\sigma_0\alpha_1 + \lambda\alpha_2 = b(Q(b) + Q(-b)) + \int_{-b}^{b} xT(x)Q(x)\, dx.$$

Here $\alpha_i = \int_{-b}^{b} x^i Q(x)\,dx, i = 0,1,2$. ϕ is smooth on [-b,b], non-negative, zero exactly at the endpoints, and satisfies $\phi'(\pm b) = \mp 2$.

The above data determine a metric

$$g = dt^2 + (dt\cdot J)^2 + p^* g_N - up^*B$$

on the complement L_0 of the zero section in L, which extends to an extremal Kähler metric on the compactification $\mathbb{P}(L\oplus\mathbb{C})$ of L_0. Here J is the complex structure on L_0, and the two functions $u : L_0 \to [-b,b]$ and $t : L_0 \to (0, \int_{-b}^{b} \frac{dx}{\sqrt{\phi(x)}})$ are obtained from ϕ by precomposing the hermitian norm on L_0 with a respective function (denoted by the same letter) having the same range, but with domain is $(0,\infty)$, determined by the relations

$$\log r = \int_{-b}^{u(r)} \frac{dx}{\phi(x)}, \quad t(r) = \int_{-b}^{u(r)} \frac{dx}{\sqrt{\phi(x)}}.$$

Here t is thought of as determined by u. Viewing u as a function of t, the metric can be written $dt^2 + (dt\cdot J)^2 + g_t$, with g_t a metric on the base N. With this perspective, t measures the distance from a fixed section of $\mathbb{P}(L\oplus\mathbb{C})$ (the one that corresponds to zero hermitian norm in L), u is the moment map for the S^1-part of the natural $\mathbb{C}^*$-action on L_0, and if H is the real gradient vector field generating the $\mathbb{R}^+$-part of this action, $\phi(u) = g(H,H)$. The map $(L_0, g) \to (L_0/S^1 \approx N\times(0,\infty), dt^2 + g_t)$ is a Riemannian submersion.

The explicitness of the description of g allows one to give local coordinate expressions for the various quantities of interest. If z_0 is a fiber coordinate such that $\frac{\partial}{\partial z_0}$ is the generator of the C^*-action, and $z_1,\ldots,z_{2l}$ are coordinates on N, then on a *fiber where* $\frac{\partial u}{\partial z_i} = 0,\ \ i = 1,..2l$, we get

$$g_{0\bar{0}} = 2\phi(u), \quad g_{0\bar{\beta}} = 0, \quad g_{\alpha\bar{\beta}} = (g_N)_{\alpha\bar{\beta}} - uB_{\alpha\bar{\beta}}.$$

For the Ricci tensor one has

$$r_{0\bar{0}} = -\phi(\phi' + \frac{Q'}{Q}\phi)'(u), \quad r_{0\bar{\beta}} = 0, \quad r_{\alpha\bar{\beta}} = (r_N)_{\alpha\bar{\beta}} + \frac{1}{2}(\phi(\log(\phi Q))')(u)B_{\alpha\bar{\beta}},$$

where the prime denotes differentiation with respect to u. The scalar curvature is now

$$s(u) = T(u) - \frac{1}{2Q(u)}(\phi Q)''(u). \tag{11}$$

Finally we record the following expressions that hold for any smooth function $f : [-b,b] \to \mathbb{R}$:

$$\int_{\mathbb{P}(L\oplus\mathbb{C})} f(u)\,\mathrm{dvol}(g) = 2\pi \mathrm{Vol}(N, g_N)\int_{-b}^{b} f(x)Q(x)\,dx,$$

$$\mathrm{grad}\ f(u) = f'(u)H, \quad \Delta f(u) = \frac{(\phi Q f')'}{2Q}(u). \tag{12}$$

The condition for such a function to be a holomorphy potential is simply that it be affine in u, a condition that can be verified directly for s from the expression (11). Thus one obtains extremal Kähler metrics in every Kähler class of the Kähler cone of $\mathbb{P}(L\oplus\mathbb{C})$. These will not in general be of constant scalar curvature, since the required extra condition $\lambda = 0$ is only obtained in (at most) a real-algebraic hypersurface of the

Kähler cone. The metrics all share the same extremal vector field up to a constant multiple, which can be fixed upon normalizing the metrics in their homothety classes.

4.2. **Harmonic Pairs.** The Kähler cone is parametrized by a_1, a_2, b. We normalize by regarding b as fixed, and consider different values of a_1, a_2. To obtain extremal Calabi-Yau pairs from this family, we make the simple observation that

$$Q_{a_1,a_2} = Q_{a_2,a_1}, \quad T_{a_1,a_2} = T_{a_2,a_1}.$$

Going through the above expressions in succession we see that σ_0, λ, then ϕ, u, t and finally the Ricci tensor, s and Δs all remain invariant under this permutation of the a_i's. So g_{a_1,a_2} and g_{a_2,a_1}, for every allowable value of the a_i's, form an extremal Calabi-Yau pair, usually of non-constant scalar curvature (so usually, the individual Futaki invariants are non-zero). To show that this pair is harmonic, we note that $s_{a_1,a_2}(u) = s_{a_2,a_1}(u)$ implies $\Delta_{a_1,a_2} F_{a_1,a_2}(u) = \Delta_{a_2,a_1} F_{a_2,a_1}(u)$. Now since by expression (12) and the above observations, the Laplacians of the two metrics coincide, as operators on functions of u, the right hand side of the last equality equals $\Delta_{a_1,a_2} F_{a_2,a_1}(u)$. Combining these equalities, and operating on the result again via Δ_{a_1,a_2}, one gets

$$\Delta^2_{a_1,a_2} F_{a_1,a_2}(u) = \Delta^2_{a_1,a_2} F_{a_2,a_1}(u).$$

Thus condition **E** in Proposition 3.2 holds, the pair $(\omega_{a_1,a_2}, \omega_{a_2,a_1})$ is harmonic, and by condition **A** of the same proposition,

$$\mathcal{R}^{[\omega_{a_2,a_1}]}_{[\omega_{a_1,a_2}]} \equiv 0.$$

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