VANISHING THEOREMS FOR THE KERNEL OF A DIRAC OPERATOR

MAXIM BRAVERMAN

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Abstract. We obtain a vanishing theorem for the kernel of a Dirac operator on a Clifford module twisted by a sufficiently large power of a line bundle, whose curvature is non-degenerate at any point of the base manifold. In particular, if the base manifold is almost complex, we prove a vanishing theorem for the kernel of a spin^c Dirac operator twisted by a line bundle with curvature of a mixed sign. In this case we also relax the assumption of non-degeneracy of the curvature. These results are generalization of a vanishing theorem of Borthwick and Uribe. As an application we obtain a new proof of the classical Andreotti-Grauert vanishing theorem for the cohomology of a compact complex manifold with values in the sheaf of holomorphic sections of a holomorphic vector bundle, twisted by a large power of a holomorphic line bundle with curvature of a mixed sign.

As another application we calculate the sign of the index of a signature operator twisted by a large power of a line bundle.

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Contents

1. Introduction

One of the most fundamental facts of complex geometry is the Kodaira vanishing theorem for the cohomology of the sheaf of sections of a holomorphic vector bundle twisted by a large power of a positive line bundle. In 1962, Andreotti and Grauert [AG] obtained the following generalization of this result to the case when the line bundle is not necessarily positive. Let $\mathcal{L}$ be a holomorphic line bundle over a compact complex n-dimensional manifold M. Suppose $\mathcal{L}$ admits a holomorphic connection whose curvature $F^{\mathcal{L}}$ has a least q negative and at least p positive eigenvalues at any point of M. Then the Andreotti-Grauert theorem asserts that, for any holomorphic vector bundle $\mathcal{W}$ over M, the cohomology $H^j(M, \mathcal{O}(\mathcal{W}\otimes\mathcal{L}^k))$ of M with coefficients in the sheaf of holomorphic sections of the tensor product $\mathcal{W}\otimes\mathcal{L}^k$ vanishes for $k\gg 0,\ j\neq q, q+1, \dots, n-p$. In particular, if $F^{\mathcal{L}}$ is non-degenerate at all points of M, then the number q of negative eigenvalues of $F^{\mathcal{L}}$ is independent of $x\in M$, and the Andreotti-Grauert theorem implies that the cohomology $H^j(M, \mathcal{O}(\mathcal{W}\otimes\mathcal{L}^k))$ vanishes for $k\gg 0, j\neq q$.

If $M, \mathcal{W}$ and $\mathcal{L}$ are endowed with metrics, then the cohomology $H^*(M, \mathcal{O}(\mathcal{W}\otimes\mathcal{L}^k))$ is isomorphic to the kernel of the Dolbeault-Dirac operator

$$\sqrt{2}(\bar\partial+\bar\partial^*) : \mathcal{A}^{0,*}(M, \mathcal{W}\otimes\mathcal{L}^k)\to\mathcal{A}^{0,*}(M, \mathcal{W}\otimes\mathcal{L}^k).$$

Here $\mathcal{A}^{0,*}(M, \mathcal{W}\otimes\mathcal{L}^k)$ denotes the space of $(0,*)$-differential forms on M with values in $\mathcal{W}\otimes\mathcal{L}^k$. The Andreotti-Grauert theorem implies, in particular, that the restriction of the kernel of the Dolbeault-Dirac operator on the space $\mathcal{A}^{0,\mathrm{odd}}(M, \mathcal{W}\otimes\mathcal{L}^k)$ (resp. $\mathcal{A}^{0,\mathrm{even}}(M, \mathcal{W}\otimes\mathcal{L}^k)$) vanishes provided the curvature $F^{\mathcal{L}}$ is non-degenerate and has an even (resp. an odd) number of negative eigenvalues at any point of M.

The last statement may be extended to the case when the manifold M is not complex. First step in this direction was done by Borthwick and Uribe [BU], who showed that, if M

is an almost Kähler manifold and $\mathcal{L}$ is a positive line bundle over M, then the restriction of the kernel of the spin^c-Dirac operator $D_k : \mathcal{A}^{0,*}(M,\mathcal{L}^k) \to \mathcal{A}^{0,*}(M,\mathcal{L}^k)$ on the space $\mathcal{A}^{0,\mathrm{odd}}(M,\mathcal{W}\otimes\mathcal{L}^k)$ vanishes for $k \gg 0$. Moreover, they showed that, for any $\alpha \in \operatorname{Ker} D_k$, "most of the norm" of α is concentrated in $\mathcal{A}^{0,0}(M,\mathcal{L}^k)$. This result generalizes the Kodaira vanishing theorem to the case of an almost Kähler manifolds.

One of the results of the present paper is the extension of the Borthwick-Uribe theorem to the case when the curvature $F^{\mathcal{L}}$ of $\mathcal{L}$ is not positive. In other words, we extend the Andreotti-Grauert theorem to almost complex manifolds.

More generally, assume that M is a compact oriented even-dimensional Riemannian manifold and let $C(M)$ denote the Clifford bundle of M, i.e., a vector bundle whose fiber at any point is isomorphic to the Clifford algebra of the cotangent space. Let $\mathcal{E}$ be a self-adjoint Clifford module over M, i.e., a Hermitian vector bundle over M endowed with a fiberwise action of $C(M)$. Then (cf. Subsection 2.3) $\mathcal{E}$ possesses a natural grading $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$. Let $\mathcal{L}$ be a Hermitian line bundle endowed with a Hermitian connection $\nabla^{\mathcal{L}}$ and let $\mathcal{E}$ be a Hermitian vector bundle over M endowed with an Hermitian connection $\nabla^{\mathcal{E}}$. These data defines (cf. Section 2) a self-adjoint Dirac operator $D_k : \Gamma(\mathcal{E}\otimes\mathcal{L}^k) \to \Gamma(\mathcal{E}\otimes\mathcal{L}^k)$. The curvature $F^{\mathcal{L}}$ of $\nabla^{\mathcal{L}}$ is an imaginary valued 2-form on M. If it is non-degenerate at all points of M, then $iF^{\mathcal{L}}$ is a symplectic form on M, and, hence, defines an orientation of M. Our main result (Theorem 3.2) states that *the restriction of the kernel of D_k to $\Gamma(\mathcal{E}^-\otimes\mathcal{L}^k)$ (resp. to $\Gamma(\mathcal{E}^+\otimes\mathcal{L}^k)$) vanishes for large k if this orientation coincides with (resp. is opposite to) the given orientation of M.*

Our result may be considerably refined when M is an almost complex $2n$-dimensional manifold and the curvature $F^{\mathcal{L}}$ is a $(1,1)$-form on M. In this case, $F^{\mathcal{L}}$ may be considered as a sesquilinear form on the holomorphic tangent bundle to M. Let $\mathcal{W}$ be a Hermitian vector bundle over M endowed with an Hermitian connection. Then (cf. Subsection 2.7) there is a canonically defined Dirac operator $D_k : \mathcal{A}^{0,*}(M,\mathcal{W}\otimes\mathcal{L}^k) \to \mathcal{A}^{0,*}(M,\mathcal{W}\otimes\mathcal{L}^k)$. We prove (Theorem 3.13) that *if $F^{\mathcal{L}}$ has at least q positive and at least p negative eigenvalues at every point of M, then, for large k, "most of the norm" of any element $\alpha \in \operatorname{Ker} D_k$ is concentrated in $\bigoplus_{j=q}^{n-p} \mathcal{A}^{0,j}(M,\mathcal{W}\otimes\mathcal{L}^k)$.* In particular, *if the sesquilinear form $F^{\mathcal{L}}$ is non-degenerate and has exactly q negative eigenvalues at any point of M, then "most of the norm" of $\alpha \in \operatorname{Ker} D_k$ is concentrated in $\mathcal{A}^{0,q}(M,\mathcal{W}\otimes\mathcal{L}^k)$, and, depending on the parity of q, the restriction of the kernel of D_k either to $\mathcal{A}^{0,odd}(M,\mathcal{W}\otimes\mathcal{L}^k)$ or to $\mathcal{A}^{0,even}(M,\mathcal{W}\otimes\mathcal{L}^k)$ vanishes.* These results generalize both the Andreotti-Grauert and the Borthwick-Uribe vanishing theorems. In particular, we obtain a new proof of the Andreotti-Grauert theorem.

As another application of Theorem 3.2, we study the index of a signature operator twisted by a line bundle having a non-degenerate curvature. We prove (Corollary 3.7) that, *if the orientation defined by the curvature of $\mathcal{L}$ coincides with (resp. is opposite to) the given orientation of M, then this index is non-negative (resp. non-positive).*

The proof of our main vanishing theorem (Theorem 3.2) is based on an estimate of the square D_k^2 of the twisted Dirac operator for large values of k. This estimate is obtained in two steps. First we use the Lichnerowicz formula to compare D_k^2 with the metric Laplacian $\Delta_k = \nabla^{\mathcal{E}\otimes\mathcal{L}^k}\left(\nabla^{\mathcal{E}\otimes\mathcal{L}^k}\right)^*$. Then we use the method of [GU, BU] to estimate the large k behavior of the metric Laplacian.

Contents. The paper is organized as follows:

In Section 2, we briefly recall some basic facts about Clifford modules and Dirac operators. We also present some examples of Clifford modules which are used in the rest of the paper.

In Section 3, we formulate the main results of the paper and discuss their applications. The rest of the paper is devoted to the proof of these results.

In Section 4, we present the proof of Theorem 3.2 (the vanishing theorem for the kernel of a Dirac operator). The proof is based on two statements (Propositions 4.3 and 4.4) which are proven in the later sections.

In Section 5, we prove an estimate on the Dirac operator on an almost complex manifold (Proposition 3.14) and use it to prove Theorem 3.13 (our analogue of the Andreotti-Grauert vanishing theorem for almost complex manifolds). The proof is based on Propositions 4.4 and 5.1 which are proved in later sections.

In Section 6, we prove the Andreotti-Grauert theorem (Theorem 3.9).

In Section 7, we use the Lichnerowicz formula to prove Propositions 4.3, 5.1 and 5.4. These results establish the connection between the Dirac operator and the metric Laplacian. They are used in the proofs of Theorems 3.2, 3.13 and 3.9.

Finally, in Section 8, we apply the method of [GU, BU] to prove Proposition 4.4 (the estimate on the metric Laplacian).

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2. Clifford modules and Dirac operators

In the first part of this section we briefly recall the definitions and some basic facts about Clifford modules and Dirac operators. We refer the reader to [BGV, Du, LM] for details. In our exposition we adopt the notations of [BGV].

In the second part of the section we present some examples of Clifford modules, which will be used in the subsequent sections.

2.1. **The Clifford bundle.** Suppose (M, g) is an oriented Riemannian manifold of dimension $2n$. For any $x \in M$, we denote by $C(T_x^*M) = C^+(T_x^*M) \oplus C^-(T_x^*M)$ the Clifford algebra of the cotangent space T_x^*M, cf. [BGV, §3.3].

The *Clifford bundle* $C(M)$ of M (cf. [BGV, §3.3]) is the $\mathbb{Z}_2$-graded bundle over M, whose fiber at $x \in M$ is $C(T_x^*M)$.

The Riemannian metric g induces the Levi-Civita connection ∇ on $C(M)$ which is compatible with the multiplication and preserves the $\mathbb{Z}_2$-grading on $C(M)$.

2.2. **Clifford modules.** A *Clifford module* on M is a complex vector bundle $\mathcal{E}$ on M endowed with an action of the bundle $C(M)$. We write this action as

$$(a, s) \ \mapsto \ c(a)s, \quad \text{where} \quad a \in \Gamma(M, C(M)), \ s \in \Gamma(M, \mathcal{E}).$$

A Clifford module $\mathcal{E}$ is called *self-adjoint* if it is endowed with a Hermitian metric such that the operator $c(v) : \mathcal{E}_x \to \mathcal{E}_x$ is skew-adjoint, for any $x \in M$ and any $v \in T_x^*M$.

A connection $\nabla^{\mathcal{E}}$ on a Clifford module $\mathcal{E}$ is called a *Clifford connection* if it is *compatible with the Clifford action*, i.e., if for any $a \in \Gamma(M, C(M))$ and $X \in \Gamma(M, TM)$,

$$[\nabla^{\mathcal{E}}_X, c(a)] \ = \ c(\nabla_X a).$$

In this formula, ∇_X is the Levi-Civita covariant derivative on $C(M)$, and $[\nabla^{\mathcal{E}}_X, c(a)]$ denotes the commutator of the operators $\nabla^{\mathcal{E}}_X$ and $c(a)$.

Suppose $\mathcal{E}$ is a Clifford module and $\mathcal{W}$ is a vector bundle over M. The *twisted Clifford module obtained from $\mathcal{E}$ by twisting with $\mathcal{W}$* is the bundle $\mathcal{E} \otimes \mathcal{W}$ with Clifford action $c(a) \otimes 1$. Note that the twisted Clifford module $\mathcal{E} \otimes \mathcal{W}$ is self-adjoint if and only if so is $\mathcal{E}$.

Let $\nabla^{\mathcal{W}}$ be a connection on $\mathcal{W}$ and let $\nabla^{\mathcal{E}}$ be a Clifford connection on $\mathcal{E}$. Then the *product connection*

$$\nabla^{\mathcal{E}\otimes\mathcal{W}} \ = \ \nabla^{\mathcal{E}} \otimes 1 \ + \ 1 \otimes \nabla^{\mathcal{W}} \tag{2.1}$$

is a Clifford connection on $\mathcal{E} \otimes \mathcal{W}$.

2.3. **The chirality operator. The natural grading.** Let $e_1, \dots, e_{2n}$ be an oriented orthonormal basis of $C(T_x^*M)$ and consider the element

$$\Gamma \ = \ i^n \, e_1 \cdots e_{2n} \ \in \ C(T_x^*M) \otimes \mathbb{C}. \tag{2.2}$$

This element is independent of the choice of the basis, anti-commutes with any $v \in T_x^*M \subset C(T_x^*M)$, and satisfies $\Gamma^2 = -1$, cf. [BGV, §3.2]. This element Γ is called the *chirality operator*. We also denote by Γ the section of $C(M)$ whose restriction to each fiber is equal to the chirality operator.

Let $\mathcal{E}$ be a Clifford module, i.e. (cf. Subsection 2.2), a vector bundle over M endowed with a fiberwise action of $C(M)$. Set

$$\mathcal{E}^{\pm} \;=\; \{v\in\mathcal{E}:\ \Gamma\, v=\pm v\}. \tag{2.3}$$

Then $\mathcal{E}=\mathcal{E}^+\oplus\mathcal{E}^-$ is a *graded module* over $C(M)$ in the sense that $C^+(M)\cdot\mathcal{E}^\pm\subset\mathcal{E}^\pm$ and $C^-(M)\cdot\mathcal{E}^\pm\subset\mathcal{E}^\mp$.

We refer to the grading (2.3) as the *natural grading* on $\mathcal{E}$. Note that this grading is preserved by any Clifford connection on $\mathcal{E}$. Also, if $\mathcal{E}$ is a self-adjoint Clifford module (cf. Subsection 2.2), then the chirality operator $\Gamma:\mathcal{E}\to\mathcal{E}$ is self-adjoint. Hence, the subbundles $\mathcal{E}^\pm$ are orthogonal with respect to the Hermitian metric on $\mathcal{E}$. *In this paper we endow all our Clifford modules with the natural grading.*

2.4. **Dirac operators.** The *Dirac operator* associated to a Clifford connection $\nabla^{\mathcal{E}}$ is defined by the following composition

$$\Gamma(M,\mathcal{E})\ \xrightarrow{\ \nabla^{\mathcal{E}}\ }\ \Gamma(M,T^*M\otimes\mathcal{E})\ \xrightarrow{\ c\ }\ \Gamma(M,\mathcal{E}). \tag{2.4}$$

In local coordinates, this operator may be written as $D=\sum c(dx^i)\,\nabla^{\mathcal{E}}_{\partial_i}$. Note that D sends even sections to odd sections and vice versa: $D:\,\Gamma(M,\mathcal{E}^\pm)\to\Gamma(M,\mathcal{E}^\mp)$.

Suppose that the Clifford module $\mathcal{E}$ is endowed with a Hermitian structure and consider the L_2-scalar product on the space of sections $\Gamma(M,\mathcal{E})$ defined by the Riemannian metric on M and the Hermitian structure on $\mathcal{E}$. By [BGV, Proposition 3.44], *the Dirac operator associated to a Clifford connection $\nabla^{\mathcal{E}}$ is formally self-adjoint with respect to this scalar product if and only if $\mathcal{E}$ is a self-adjoint Clifford module and $\nabla^{\mathcal{E}}$ is a Hermitian connection.*

We finish this section with some examples of Clifford modules, which will be used later.

2.5. **Spinor bundles.** Assume that M is a spin manifold and let $\mathcal{S}=\mathcal{S}^+\oplus\mathcal{S}^-$ be a *spinor bundle* over M (cf. [BGV, §3.3]). It is a minimal Clifford module in the sense that any other Clifford module $\mathcal{E}$ may be decomposed as a tensor product

$$\mathcal{E}\;=\;\mathcal{S}\,\otimes\mathcal{W}, \tag{2.5}$$

where $\mathcal{W}=\operatorname{Hom}_{C(M)}(\mathcal{S},\mathcal{E})$ and the action of the Clifford bundle $C(M)$ is trivial on the second factor of (2.5). In this case, the natural grading $\mathcal{E}=\mathcal{E}^+\oplus\mathcal{E}^-$ is defined by $\mathcal{E}^\pm=\mathcal{S}^\pm\otimes\mathcal{W}$.

The Riemannian metric on M induces the Levi-Civita connection $\nabla^{\mathcal{S}}$ on $\mathcal{S}$, which is compatible with the Clifford action. Moreover, a connection $\nabla^{\mathcal{E}}$ on the twisted Clifford module $\mathcal{E}=\mathcal{S}\otimes\mathcal{W}$ is a Clifford connection if and only if

$$\nabla^{\mathcal{E}}\;=\;\nabla^{\mathcal{S}}\otimes 1\;+\;1\otimes\nabla^{\mathcal{W}} \tag{2.6}$$

for some connection $\nabla^{\mathcal{W}}$ on $\mathcal{W}$.

Note that locally the spinor bundle and, hence, the decompositions (2.5), (2.6) always exist. In particular, suppose that $\widetilde{\mathcal{S}}$ is a Clifford module whose fiber dimension is equal to $\dim \mathcal{S} = 2^n$. Then locally $\widetilde{\mathcal{S}} = \mathcal{S} \otimes \mathcal{W}$ for some locally defined complex line bundle $\mathcal{W}$. In this case $\widetilde{\mathcal{S}}$ is called a spin^c vector bundle over M ([Du, Ch. 5], [LM, Appendix D]).

A Dirac operator on a spin^c vector bundle is called a spin^c *Dirac operator.*

2.6. **The exterior algebra.** Consider the exterior algebra $\Lambda T^*M = \bigoplus_i \Lambda^i T^*M$ of the cotangent bundle T^*M. There is a canonical action of the Clifford bundle $C(M)$ on ΛT^*M such that

$$c(v)\,\alpha \;=\; v \wedge \alpha \;-\; \iota(v)\,\alpha, \qquad v \in \Gamma(M, T^*M),\ \alpha \in \Gamma(M, \Lambda T^*M). \tag{2.7}$$

Here $\iota(v)$ denotes the contraction with the vector $v^* \in T_xM$ dual to v.

The chirality operator (2.2) coincides in this case (cf. [BGV, §3.6]) with the Hodge $*$-operator. Hence, the usual grading $\Lambda T^*M = \Lambda^{\mathrm{even}} T^*M \oplus \Lambda^{\mathrm{odd}} T^*M$ is not the natural grading in the sense of Subsection 2.3. *We will always consider ΛT^*M with the natural grading.* The positive and negative elements of $\Gamma(M, \Lambda T^*M)$, with respect to this grading, are called *self-dual* and *anti-self-dual* differential forms respectively.

The action (2.7) is self-adjoint with respect to the metric on ΛT^*M defined by the Riemannian metric on M. The connection induced on ΛT^*M by the Levi-Civita connection on T^*M is a Clifford connection. The Dirac operator associated with this connection is equal to $d + d^*$ and is called the *signature operator*, [BGV, §3.6]. If the dimension of M is divisible by four, then its index is equal to the signature of the manifold M.

2.7. **Almost complex manifolds.** Assume that M is an almost complex manifold with an almost complex structure $J : TM \to TM$. Then J defines a structure of a complex vector bundle on the tangent bundle TM. Let h^{TM} be a Hermitian metric on $TM \otimes \mathbb{C}$. The real part $g^{TM} = \operatorname{Re} h^{TM}$ of h^{TM} is a Riemannian metric on M. Note also that J defines an orientation on M.

Let $\Lambda^q = \Lambda^q (T^{0,1}M)^*$ denote the bundle of $(0,q)$-forms on M and set

$$\Lambda^+ \;=\; \bigoplus_{q \text{ even}} \Lambda^q, \quad \Lambda^- \;=\; \bigoplus_{q \text{ odd}} \Lambda^q.$$

Let $\lambda^{1/2}$ be the square root of the complex line bundle $\lambda = \det T^{1,0}M$ and let $\mathcal{S}$ be the spinor bundle over M associated to the Riemannian metric g^{TM}. Although $\lambda^{1/2}$ and $\mathcal{S}$ are defined only locally, unless M is a spin manifold, it is well known (cf. [LM, Appendix D]) that the products $\mathcal{S}^\pm \otimes \lambda^{1/2}$ are globally defined and

$$\Lambda^\pm \;=\; \mathcal{S}^\pm \otimes \lambda^{1/2}.$$

It follows that Λ is a spin^c vector bundle over M (cf. Subsection 2.5). In particular, the grading $\Lambda = \Lambda^+ \oplus \Lambda^-$ is natural.

The Clifford action of $C(M)$ on Λ may be described as follows: if $f\in\Gamma(M,T^*M)$ decomposes as $f=f^{1,0}+f^{0,1}$ with $f^{1,0}\in\Gamma(M,(T^{1,0}M)^*)$ and $f^{0,1}\in\Gamma(M,(T^{0,1}M)^*)$, then the Clifford action of f on $\alpha\in\Gamma(M,\Lambda)$ equals

$$c(f)\alpha \;=\; \sqrt{2}\,\left(f^{0,1}\wedge\alpha \;-\; \iota(f^{1,0})\,\alpha\right). \tag{2.8}$$

Here $\iota(f^{1,0})$ denotes the interior multiplication by the vector field $(f^{1,0})^*\in T^{0,1}M$ dual to the 1-form $f^{1,0}$. This action is self-adjoint with respect to the Hermitian structure on Λ defined by the Riemannian metric g^{TM} on M.

The Levi-Civita connection ∇^{TM} of g^{TM} induces a Hermitian connection on $\lambda^{1/2}$ and on $\mathcal{S}$. Let ∇^M be the product connection (cf. Subsection 2.2),

$$\nabla^M \;=\; \nabla^{\mathcal{S}}\otimes 1 \;+\; 1\otimes\nabla^{\lambda^{1/2}}.$$

Then ∇^M is a well-defined Hermitian Clifford connection on the spin^c bundle Λ. Hence, it gives rise to a self-adjoint spin^c Dirac operator.

More generally, assume that $\mathcal{W}$ is a Hermitian vector bundle over M and let $\nabla^{\mathcal{W}}$ be a Hermitian connection on $\mathcal{W}$. Consider the twisted Clifford module $\mathcal{E}=\Lambda\otimes\mathcal{W}$. The product connection $\nabla^{\mathcal{E}}=\nabla^{\Lambda\otimes\mathcal{W}}$ determines a Dirac operator $D:\Gamma(M,\mathcal{E})\to\Gamma(M,\mathcal{E})$.

2.8. **Kähler manifolds.** If (M,J,g^{TM}) is a Kähler manifold, then (cf. [BGV, Proposition 3.67]) the Dirac operator defined in (2.4) coincides with the Dolbeault-Dirac operator

$$D \;=\; \sqrt{2}\,\left(\bar\partial+\bar\partial^*\right). \tag{2.9}$$

Here $\bar\partial^*$ denotes the operator adjoint to $\bar\partial$ with respect to the L_2-scalar product on $\mathcal{A}^{0,*}(M,\mathcal{W})$. Hence, the restriction of the kernel of D to $\mathcal{A}^{0,i}(M,\mathcal{W})$ is isomorphic to the cohomology $H^i(M,\mathcal{O}(\mathcal{W}))$ of M with coefficients in the sheaf of holomorphic section of $\mathcal{W}$.

3. Vanishing theorems and their applications

In this section we state the main theorems of the paper. The section is organized as follows:

In Subsection 3.1, we formulate our main result – the vanishing theorem for the kernel of a Dirac operator (Theorem 3.2).

In Subsection 3.3, we briefly indicate the idea of the proof of Theorem 3.2.

In Subsection 3.5, we apply this theorem to calculate the sign of the signature of a vector bundle twisted by a high power of a line bundle.

In Subsection 3.8, we refine Theorem 3.2 for the case of a complex manifold. In particular, we recover the Andreotti-Grauert vanishing theorem for a line bundle with curvature of a mixed sign, cf. [AG, DPS].

Finally, in Subsection 3.12, we present an analogue of the Andreotti-Grauert theorem for almost complex manifolds. This generalizes a result of Borthwick and Uribe [BU].

3.1. **Twisting by a line bundle. The vanishing theorem.** Suppose $\mathcal{E}$ is a self-adjoint Clifford module over M. Recall from Subsection 2.3 that $\mathcal{E}=\mathcal{E}^+\oplus\mathcal{E}^-$ denotes the *natural* grading on $\mathcal{E}$. Let $\mathcal{L}$ be a Hermitian line bundle over M, and let $\nabla^{\mathcal{L}}$ be a Hermitian connection on $\mathcal{L}$. The connection $\nabla^{\mathcal{E}\otimes\mathcal{L}^k}$ (cf. (2.1)) is a Hermitian Clifford connection on the twisting Clifford module $\mathcal{E}\otimes\mathcal{L}^k$. Consider the self-adjoint Dirac operator

$$D_k:\,\Gamma(M,\mathcal{E}\otimes\mathcal{L}^k)\ \to\ \Gamma(M,\mathcal{E}\otimes\mathcal{L}^k)$$

associated to this connection and let $D_k^\pm$ denote the restriction of D_k to the spaces $\Gamma(M,\mathcal{E}^\pm\otimes\mathcal{L}^k)$.

The curvature $F^{\mathcal{L}}=(\nabla^{\mathcal{L}})^2$ of the connection $\nabla^{\mathcal{L}}$ is an imaginary valued closed 2-form on M. If it is non-degenerate, then $iF^{\mathcal{L}}$ is a symplectic form on M and, hence, defines an orientation of M. Our main result is the following

Theorem 3.2. *Let $\mathcal{E}$ be a self-adjoint Clifford module over a compact oriented even-dimensional Riemannian manifold M. Let $\nabla^{\mathcal{E}},\mathcal{L},\,\nabla^{\mathcal{L}},D_k$ be as above. Assume that the curvature $F^{\mathcal{L}}=(\nabla^{\mathcal{L}})^2$ of the connection $\nabla^{\mathcal{L}}$ is non-degenerate at all points of M. If the orientation defined by the symplectic form $iF^{\mathcal{L}}$ coincides with the original orientation of M, then*

$$\operatorname{Ker} D_k^-\ =\ 0\qquad\textit{for}\qquad k\gg 0. \tag{3.1}$$

Otherwise, $\operatorname{Ker} D_k^+=0$ *for* $k\gg 0$.

The theorem is a generalization of a vanishing theorem of Borthwick and Uribe [BU], who considered the case where M is an almost Kähler manifold, D is a spin^c-Dirac operator and $\mathcal{L}$ is a positive line bundle over M.

The theorem is proven in Section 4. Here we only explain the main ideas of the proof.

3.3. **The scheme of the proof.** Our proof of Theorem 3.2 follows the lines of [BU]. It is based on an estimate from below on the large k behavior of the square D_k^2 of the Dirac operator. Using this estimate we show that, if the orientation defined by $iF^{\mathcal{L}}$ coincides with (resp. is opposite to) the given orientation of M, then, for large k, the restriction of D_k^2 to $\mathcal{E}^-\otimes\mathcal{L}^k$ (resp. to $\mathcal{E}^+\otimes\mathcal{L}^k$) is a strictly positive operator and, hence, has no kernel.

This estimate for D_k^2 is obtained in two steps. First we use the Lichnerowicz formula (cf. Subsection 7.3) to compare D_k^2 with the *metric Laplacian* $\Delta_k=\nabla^{\mathcal{E}\otimes\mathcal{L}^k}(\nabla^{\mathcal{E}\otimes\mathcal{L}^k})^*$.

Then it remains to study the large k behavior of the metric Laplacian Δ_k. This is done in Section 8. In fact, the estimate which we need is essentially obtained in [BU, GU]. Roughly speaking it says that Δ_k grows linearly in k.

The proof of the estimate for Δ_k also consists of two steps. First we consider the principal bundle $\mathcal{Z} \to M$ associated to the vector bundle $\mathcal{E}\otimes\mathcal{L}$, and construct a differential operator $\widetilde{\Delta}$ (*horizontal Laplacian*) on $\mathcal{Z}$, such that the operator Δ_k is "equivalent" to a restriction of $\widetilde{\Delta}$ on a certain subspace of the space of L_2-functions on the total space of $\mathcal{Z}$. Then we apply the *a priory* Melin estimates [Me] (see also [Ho, Theorem 22.3.3]) to the operator $\widetilde{\Delta}$.

Remark 3.4. It would be very interesting to obtain an effective estimates of a minimal value of k which satisfies (3.1) at least for the simplest cases (say, when $\mathcal{E}$ is a spinor bundle over a spin manifold M). Unfortunately, such an estimate can not be obtained using our method. This is because the Melin inequalities [Me], [Ho, Theorem 22.3.3] (see also Subsection 8.3), used in our proof, contain a constant C, which can not be estimated effectively.

We will now discuss applications and refinements of Theorem 3.2. In particular, we will see that Theorem 3.2 may be considered as a generalization of the vanishing theorems of Kodaira, Andreotti-Grauert [AG] and Borthwick-Uribe [BU].

3.5. **The signature operator.** Recall from Subsection 2.6 that, for any oriented even-dimensional Riemannian manifold M, the exterior algebra ΛT^*M of the cotangent bundle is a self-adjoint Clifford module. The connection induced on ΛT^*M by a Levi-Civita connection on T^*M is a Hermitian Clifford connection and the Dirac operator associated to this connection is the signature operator $d + d^*$.

Consider a twisted Clifford module $\mathcal{E} = \Lambda T^*M \otimes \mathcal{W}$, where $\mathcal{W}$ is a Hermitian vector bundle over M endowed with a Hermitian connection $\nabla^{\mathcal{W}}$.

Let $\mathcal{L}$ be a Hermitian line bundle over M and let $\nabla^{\mathcal{L}}$ be an Hermitian connection on $\mathcal{L}$. The space $\Gamma(M, \mathcal{E} \otimes \mathcal{L}^k)$ of sections of the twisted Clifford module $\mathcal{E} \otimes \mathcal{L}^k$ coincides with the space $\mathcal{A}^*(M, \mathcal{W} \otimes \mathcal{L}^k)$ of differential forms on M with values in $\mathcal{W} \otimes \mathcal{L}^k$. The positive and negative elements of $\mathcal{A}^*(M, \mathcal{W} \otimes \mathcal{L}^k)$ with respect to the natural grading are called the *self-dual* and the *anti-self-dual* differential forms respectively.

Let $D_k : \mathcal{A}^*(M, \mathcal{W} \otimes \mathcal{L}^k) \to \mathcal{A}^*(M, \mathcal{W} \otimes \mathcal{L}^k)$ denote the Dirac operator corresponding to the tensor product connection $\nabla^{\mathcal{W}\otimes\mathcal{L}^k}$ on $\mathcal{W} \otimes \mathcal{L}^k$. Then

$$D_k \;=\; \nabla^{\mathcal{W}\otimes\mathcal{L}^k} \;+\; \left(\nabla^{\mathcal{W}\otimes\mathcal{L}^k}\right)^*, \tag{3.2}$$

where $\left(\nabla^{\mathcal{W}\otimes\mathcal{L}^k}\right)^*$ denotes the adjoint of $\nabla^{\mathcal{W}\otimes\mathcal{L}^k}$ with respect to the L_2-scalar product on $\mathcal{W} \otimes \mathcal{L}^k$. The operator (3.2) is called the *signature operator* of the bundle $\mathcal{W} \otimes \mathcal{L}^k$.

Let D_k^+ and D_k^- denote the restrictions of D_k on the spaces of self-dual and anti-self-dual differential forms respectively.

As an immediate consequence of Theorem 3.2, we obtain the following

Theorem 3.6. *Suppose M is a compact oriented even-dimensional Riemannian manifold. Let $\mathcal{W}, \nabla^{\mathcal{W}}, \mathcal{L}, \nabla^{\mathcal{L}}, D_k^{\pm}$ be as above. Assume that the curvature $F^{\mathcal{L}} = (\nabla^{\mathcal{L}})^2$ of the connection $\nabla^{\mathcal{L}}$ is non-degenerate at any point $x \in M$. If the orientation defined by $iF^{\mathcal{L}}$ coincides with the given orientation of M, then $\operatorname{Ker} D_k^- = 0$, for $k \gg 0$.*

Otherwise, $\operatorname{Ker} D_k^+ = 0$, for $k \gg 0$.

The index

$$\operatorname{ind} D_k \;=\; \dim \operatorname{Ker} D_k^+ \;-\; \dim \operatorname{Ker} D_k^-$$

of the Dirac operator D_k is called the *signature* of the bundle $\mathcal{W} \otimes \mathcal{L}^k$ and is denoted by $\operatorname{sign}(\mathcal{W} \otimes \mathcal{L}^k)$. It depends only on the manifold M, its orientation and the bundle $\mathcal{W} \otimes \mathcal{L}^k$ (but not on the choice of Riemannian metric on M and of Hermitian structures and connections on the bundles $\mathcal{W}, \mathcal{L}$). If the bundles $\mathcal{W}$ and $\mathcal{L}$ are trivial, then it coincides with the usual signature of the manifold M.

From Theorem 3.6, we obtain the following

Corollary 3.7. *Let $\mathcal{W}$ and $\mathcal{L}$ be respectively a vector and a line bundles over a compact oriented even-dimensional Riemannian manifold M. Suppose that, for some Hermitian metric on $\mathcal{L}$, there exist a Hermitian connection, whose curvature $F^{\mathcal{L}}$ is non-degenerate at any point of M. If the orientation defined by the symplectic form $iF^{\mathcal{L}}$ coincides with the given orientation of M, then*

$$\operatorname{sign}(\mathcal{W} \otimes \mathcal{L}^k) \;\geq\; 0 \qquad \text{for} \qquad k \gg 0.$$

Otherwise, $\operatorname{sign}(\mathcal{W} \otimes \mathcal{L}^k) \leq 0$ for $k \gg 0$.

3.8. **Complex manifolds. The Andreotti-Grauert theorem.** Suppose M is a compact complex manifold, $\mathcal{W}$ is a holomorphic vector bundle over M and $\mathcal{L}$ is a holomorphic line bundles over M. Fix a Hermitian metric $h^{\mathcal{L}}$ on $\mathcal{L}$ and let $\nabla^{\mathcal{L}}$ be the *Chern connection* on $\mathcal{L}$, i.e., the unique holomorphic connection which preserves the Hermitian metric. The curvature $F^{\mathcal{L}}$ of $\nabla^{\mathcal{L}}$ is a $(1,1)$-form which is called the *curvature form of the Hermitian metric $h^{\mathcal{L}}$*.

The orientation condition of Theorem 3.2 may be reformulated as follows. Let $(z^1, \dots, z^n)$ be complex coordinates in the neighborhood of a point $x \in M$. The curvature $F^{\mathcal{L}}$ may be written as

$$iF^{\mathcal{L}} \;=\; \frac{i}{2} \sum_{i,j} F_{ij} dz^i \wedge d\overline{z}^j.$$

Denote by q the number of negative eigenvalues of the matrix $\{F_{ij}\}$. Clearly, the number q is independent of the choice of the coordinates. We will refer to this number as the *number of negative eigenvalues of the curvature $F^{\mathcal{L}}$ at the point x*. Then the orientation defined by the symplectic form $iF^{\mathcal{L}}$ coincides with the complex orientation of M if and only if q is even.

A small variation of the method used in the proof of Theorem 3.2 allows to get a more precise result which depends not only on the parity of q but on q itself. In this way we obtain a new proof of the following vanishing theorem of Andreotti and Grauert [AG, DPS]

Theorem 3.9 (**Andreotti-Grauert**). *Let M be a compact complex manifold and let $\mathcal{L}$ be a holomorphic line bundle over M. Assume that $\mathcal{L}$ carries a Hermitian metric whose curvature form $F^{\mathcal{L}}$ has at least q negative and at least p positive eigenvalues at any point $x \in M$. Then, for any holomorphic vector bundle $\mathcal{W}$ over M, the cohomology $H^j(M, \mathcal{O}(\mathcal{W}\otimes\mathcal{L}^k))$ with coefficients in the sheaf of holomorphic sections of $\mathcal{W}\otimes\mathcal{L}^k$ vanish for $j \neq q, q+1, \dots, n-p$ and $k \gg 0$.*

The proof is given in Subsection 6.2. In contrary to Theorem 3.2, the curvature $F^{\mathcal{L}}$ in Theorem 3.9 needs not be non-degenerate. If $F^{\mathcal{L}}$ is non-degenerate, then the number q of negative eigenvalues of $F^{\mathcal{L}}$ does not depend on the point $x \in M$. Then we obtain the following

Corollary 3.10. *If, in the conditions of Theorem 3.9, the curvature $F^{\mathcal{L}}$ is non-degenerate and has exactly q negative eigenvalues at any point $x \in M$, then $H^j(M, \mathcal{O}(\mathcal{W} \otimes \mathcal{L}^k))$ vanishes for any $j \neq q$ and $k \gg 0$.*

Note that, if $\mathcal{L}$ is a positive line bundle, Corollary 3.10 reduces to the classical Kodaira vanishing theorem (cf., for example, [BGV, Theorem 3.72(2)]).

Remark 3.11. a. It is interesting to compare Corollary 3.10 with Theorem 3.2 for the case when M is a Kähler manifold. In this case the Dirac operator D_k is equal to the Dolbeault-Dirac operator (2.9). Hence (cf. Subsection 2.8), Theorem 3.2 implies that $H^j(M, \mathcal{O}(\mathcal{W}\otimes \mathcal{L}^k))$ vanishes when the parity of j is not equal to the parity of q. Corollary 3.10 refines this result.

b. If M is not a Kähler manifold, then the Dirac operator D_k defined by (2.4) is not equal to the Dolbeault-Dirac operator, and the kernel of D_k is not isomorphic to the cohomology $H^*(M, \mathcal{O}(\mathcal{W} \otimes \mathcal{L}^k))$. However, we show in Section 6 that the operators D_k and $\sqrt{(}\bar{\partial} + \bar{\partial}^*)$ have the same asymptotic as $k \to \infty$. Then the vanishing of the kernel of D_k implies the vanishing of the cohomology $H^j(M, \mathcal{O}(\mathcal{W} \otimes \mathcal{L}^k))$.

c. In Theorem 3.9 the bundle $\mathcal{W}$ can be replaced by an arbitrary coherent sheaf $\mathcal{F}$. This follows from Theorem 3.9 by a standard technique using a resolution of $\mathcal{F}$ by locally free sheaves (see, for example, [SS, Ch. 5] for similar arguments).

3.12. **Andreotti-Grauert-type theorem for almost complex manifolds.** In this section we refine Theorem 3.2 assuming that M is endowed with an almost complex structure J such that the curvature $F^{\mathcal{L}}$ of $\mathcal{L}$ is a $(1,1)$ form on M with respect to J. In other words, we assume that, for any $x \in M$ and any basis $(e^1, \dots, e^n)$ of the holomorphic cotangent space $(T^{1,0}M)^*$, one has

$$iF^{\mathcal{L}} \;=\; \frac{i}{2}\sum_{i,j} F_{ij}e^i \wedge \overline{e}^j.$$

This section generalizes a result of Borthwick and Uribe [BU].

We denote by q the number of negative eigenvalues of the matrix $\{F_{ij}\}$. As in Subsection 3.8, the orientation of M defined by the symplectic form $iF^{\mathcal{L}}$ depends only on the parity of q. It coincides with the orientation defined by J if and only if q is even.

We will use the notation of Subsection 2.7. In particular, $\Lambda = \Lambda(T^{0,1}M)^*$ denotes the bundle of $(0,*)$-forms on M and $\mathcal{W}$ is a Hermitian vector bundle over M. Then $\mathcal{E} = \Lambda \otimes \mathcal{W}$ is a self-adjoint Clifford module endowed with a Hermitian Clifford connection $\nabla^{\mathcal{E}}$. The space $\Gamma(M, \mathcal{E} \otimes \mathcal{L}^k)$ of sections of the twisted Clifford module $\mathcal{E} \otimes \mathcal{L}^k$ coincides with the space $\mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$ of differential forms of type $(0,*)$ with values in $\mathcal{W} \otimes \mathcal{L}^k$. Let

$$D_k : \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k) \;\to\; \mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$$

denote the Dirac operator corresponding to the tensor product connection on $\mathcal{W} \otimes \mathcal{L}^k$.

For a form $\alpha \in \mathcal{A}^{0,*}(M, \mathcal{W}\otimes\mathcal{L}^k)$, we denote by $\|\alpha\|$ its L_2-norm and by α_i its component in $\mathcal{A}^{0,i}(M, \mathcal{W} \otimes \mathcal{L}^k)$.

Theorem 3.13. *Assume that the matrix $\{F_{ij}\}$ has at least q negative and at least p positive eigenvalues at any point $x \in M$. There exist a constant $C > 0$ such that, for any $k \gg 0$ and any $\alpha \in \operatorname{Ker} D_k$,*

$$\|\alpha_j\| \;\le\; \frac{C}{k}\|\alpha\|, \qquad \text{for} \quad j \neq q, q+1, \dots, n-p.$$

In particular, if the form $F^{\mathcal{L}}$ is non-degenerate and q is the number of negative eigenvalues of $\{F_{ij}\}$ (which is independent of $x \in M$), then $\alpha \in \operatorname{Ker} D_k$ implies

$$\|\alpha \;-\; \alpha_q\| \;\le\; \frac{C'}{k}\|\alpha_q\|.$$

Theorem 3.13 is proven in Subsection 5.5. The main ingredient of the proof is the following estimate on D_k, which also has an independent interest:

Proposition 3.14. *If the matrix $\{F_{ij}\}$ has at least q negative and at least p positive eigenvalues at any point $x \in M$, then there exists a constant $C_1 > 0$, such that*

$$\|D_k\,\alpha\| \;\ge\; C_1 k^{1/2}\,\|\alpha\|,$$

for any $k \gg 0$, $j \neq q, q+1, \dots, n-p$ and $\alpha \in \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$.

The proof is given in Subsection 5.2.

Remark 3.15. a. For the case when $\mathcal{L}$ is a positive line bundle, the Riemannian metric on M is almost Kähler and $\mathcal{W}$ is a trivial line bundle, Theorem 3.13 was established by Borthwick and Uribe [BU, Theorem 2.3].

b. Theorem 3.13 implies that, if $F^{\mathcal{L}}$ is non-degenerate, then $\operatorname{Ker} D_k$ is dominated by the component of degree q. If $\alpha \in \Gamma(M, \mathcal{E}^-)$ (resp. $\alpha \in \Gamma(M, \mathcal{E}^+)$) and q is even (resp. odd) then $\alpha_q = 0$. So, we obtain the vanishing result of Theorem 3.2 for the case when M is almost complex and $F^{\mathcal{L}}$ is a $(1,1)$ form.

c. Theorem 3.13 is an analogue of Theorem 3.9. Of course, the cohomology $H^j(M, \mathcal{O}(\mathcal{W} \otimes \mathcal{L}^k))$ is not defined if J is not integrable. Moreover, the square D_k^2 of the Dirac operator does not preserve the $\mathbb{Z}$-grading on $\mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$. Hence, one can not hope that the kernel of D_k belongs to $\oplus_{j=q}^{n-p} \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$. However, Theorem 3.13 shows, that for any $\alpha \in \operatorname{Ker} D_k$, "most of the norm" of α is concentrated in $\oplus_{j=q}^{n-p} \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$.

4. Proof of the vanishing theorem for the kernel of a Dirac operator

In this section we present a proof of Theorem 3.2 based on Propositions 4.3 and 4.4, which will be proved in the following sections.

The idea of the proof is to study the large k behavior of the square D_k^2 of the Dirac operator.

4.1. **The operator $\widetilde{J}$.** We need some additional definitions. Recall that $F^{\mathcal{L}}$ denotes the curvature of the connection $\nabla^{\mathcal{L}}$. In this subsection we do not assume that $F^{\mathcal{L}}$ is non-degenerate. For $x \in M$, define the skew-symmetric linear map $\widetilde{J}_x : T_xM \to T_xM$ by the formula

$$iF^{\mathcal{L}}(v, w) \;=\; g^{TM}(v, \widetilde{J}_x w), \qquad v, w \in T_xM.$$

The eigenvalues of $\widetilde{J}_x$ are purely imaginary. Note that, in general, $\widetilde{J}$ is not an almost complex structure on M.

Define

$$\tau(x) \;=\; \operatorname{Tr}^+ \widetilde{J}_x \;:=\; \mu_1 + \cdots + \mu_l, \qquad m(x) \;=\; \min_j \, \mu_j(x). \tag{4.1}$$

where $i\mu_j$, $j = 1, \ldots, l$ are the eigenvalues of $\widetilde{J}_x$ for which $\mu_j > 0$. Note that $m(x) = 0$ if and only if the curvature $F^{\mathcal{L}}$ vanishes at the point $x \in M$.

4.2. **Estimate on D_k^2.** Our estimate on the square D_k^2 of the Dirac operator is obtained in two steps: first we compare it to the *metric Laplacian*

$$\Delta_k \;:=\; (\nabla^{\mathcal{E} \otimes \mathcal{L}^k})^* \, \nabla^{\mathcal{E} \otimes \mathcal{L}^k},$$

and then we estimate the large k behavior of Δ_k. These two steps are the subject of the following two propositions.

Proposition 4.3. *Supposed that the differential form $F^{\mathcal{L}}$ is non-degenerate. If the orientation defined on M by the symplectic form $iF^{\mathcal{L}}$ coincides with (resp. is opposite to) the given orientation of M, then there exists a constant C such that, for any $s \in \Gamma(M, \mathcal{E}^- \otimes \mathcal{L}^k)$ (resp. for any $s \in \Gamma(M, \mathcal{E}^+ \otimes \mathcal{L}^k)$), one has an estimate*

$$\Big\langle (D_k^2 - \Delta_k)\, s,\, s \Big\rangle \;\geq\; -k \Big\langle (\tau(x) - 2m(x))\, s,\, s \Big\rangle - C\, \|s\|^2.$$

Here $\langle\cdot,\cdot\rangle$ denotes the L_2-scalar product on the space of sections and $\|\cdot\|$ is the norm corresponding to this scalar product.

The proposition is proven in Subsection 7.3 using the Lichnerowicz formula (7.7).

In the next proposition we do not assume that $F^{\mathcal{L}}$ is non-degenerate.

Proposition 4.4. *Suppose that $F^{\mathcal{L}}$ does not vanish at any point $x \in M$. For any $\varepsilon > 0$, there exists a constant C_ε such that, for any $k \in \mathbb{Z}$ and any section s of the bundle $\mathcal{E} \otimes \mathcal{L}^k$,*

$$\langle \Delta_k\, s, s \rangle \;\geq\; k\, \langle (\tau(x) - \varepsilon) s, s \rangle \;-\; C_\varepsilon\, \|s\|^2. \tag{4.2}$$

Proposition 4.4 is essentially proven in [BU, Theorem 2.1]. The only difference is that we do not assume that the curvature $F^{\mathcal{L}}$ has a constant rank. This forces us to use the original Melin inequality [Me] (see also [Ho, Theorem 22.3.3]) and not the Hörmander refinement of this inequality [Ho, Theorem 22.3.2]. That is the reason that $\varepsilon \neq 0$ appears in (4.2). Note, [BU], that if $F^{\mathcal{L}}$ has constant rank, then Proposition 4.4 is valid for $\varepsilon = 0$.

Proposition 4.4 is proven in Section 8.

4.5. **Proof of Theorem 3.2.** Assume that the orientation defined by $iF^{\mathcal{L}}$ coincides with the given orientation of M and $s \in \Gamma(M, \mathcal{E}^- \otimes \mathcal{L})$, or that the orientation defined by $iF^{\mathcal{L}}$ is opposite to the given orientation of M and $s \in \Gamma(M, \mathcal{E}^+ \otimes \mathcal{L})$. By Proposition 4.3,

$$\langle D_k^2\, s, s \rangle \;\geq\; \langle \Delta_k\, s, s \rangle \;-\; k \Big\langle (\tau(x) - 2m(x))\, s,\, s \Big\rangle \;-\; C\, \|s\|^2. \tag{4.3}$$

Choose

$$0 \;<\; \varepsilon \;<\; 2 \min_{x \in M} m(x)$$

and set

$$C' = 2 \min_{x \in M} m(x) - \varepsilon > 0.$$

Since the metric Laplacian Δ_k is a non-negative operator, it follows from (4.2) and (4.3) that

$$\langle D_k^2\, s,\, s \rangle \;\geq\; kC' \|s\|^2 \;-\; (C + C_\varepsilon)\, \|s\|^2.$$

Thus, for $k > (C + C_\varepsilon)/C'$, we have $\langle D_k^2\, s, s \rangle > 0$. Hence, $D_k s \neq 0$. □

5. Proof of the vanishing theorem for almost complex manifolds

In this section we prove Theorem 3.13 and Proposition 3.14. The proof is very similar to the proof of Theorem 3.2 (cf. Section 4). It is based on Proposition 4.4 and the following refinement of Proposition 4.3:

Proposition 5.1. *Assume that the matrix $\{F_{ij}\}$ (cf. Subsection 3.12) has at least q negative eigenvalues at any point $x \in M$. For any $x \in M$, we denote by $m_q(x) > 0$ the minimal positive number, such that at least q of the eigenvalues of $\{F_{ij}\}$ do not exceed $-m_q$. Then there exists a constant C such that*

$$\left\langle (D_k^2 - \Delta_k)\, \alpha,\ \alpha \right\rangle \ \geq\ -k \left\langle (\tau(x) - 2m_q(x))\, \alpha,\ \alpha \right\rangle \ -\ C\, \|\alpha\|^2$$

for any $j = 0, \dots, q-1$ and any $\alpha \in \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$.

The proposition is proven in Subsection 7.12.

5.2. **Proof of Proposition 3.14.** Choose $0 < \varepsilon < 2 \min_{x\in M} m_q(x)$ and set

$$C' \ =\ 2 \min_{x\in M} m_q(x) - \varepsilon.$$

Fix $j = 0, \dots, q-1$ and let $\alpha \in \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$. Since the metric Laplacian Δ_k is a non-negative operator, it follows from Propositions 4.4 and 5.1, that

$$\langle D_k^2\, \alpha, \alpha \rangle \ \geq\ kC'\, \|\alpha\|^2 \ -\ (C + C_\varepsilon)\, \|\alpha\|^2.$$

Hence, for any $k > 2(C + C_\varepsilon)/C'$, we have

$$\|D_k\, \alpha\|^2 \ =\ \langle D_k^2\, \alpha,\, \alpha \rangle \ \geq\ \frac{kC'}{2}\, \|\alpha\|^2.$$

This proves Proposition 3.14 for $j = 0, \dots, q-1$. The statement for $j = n-p+1, \dots, n$ may be proven by a verbatim repetition of the above arguments, using a natural analogue of Proposition 5.1. (Alternatively, the statement for $j = n-p+1, \dots, n$ may be obtained as a formal consequence of the statement for $j = 0, \dots, q-1$ by considering M with an opposite almost complex structure). □

5.3. If the manifold M is not Kähler, then the operator D_k^2 does not preserve the $\mathbb{Z}$-grading on $\mathcal{A}^{0,*}(M, \mathcal{W} \otimes \mathcal{L}^k)$. However, the next proposition shows that the *mixed degree operator* $\alpha_i \mapsto (D_k^2 \alpha_i)_j$ is *uniformly bounded in* k.

Proposition 5.4. *There exists a constant $C_2 > 0$ such that*

$$\|(D_k^2\, \alpha)_j\| \ \geq\ C_2\, \|\alpha\|,$$

for any $k \geq 0$, $i \neq j$ and $\alpha \in \mathcal{A}^{0,i}(M, \mathcal{W} \otimes \mathcal{L}^k)$.

The proof of the proposition, based on the Lichnerowicz formula, is given in Subsection 7.13.

5.5. **Proof of Theorem 3.13.** Let $\alpha \in \operatorname{Ker} D_k$ and fix $j \notin q, q+1, \dots, n-p$. Set $\beta = \alpha - \alpha_j$. Then

$$0 \;=\; \|D_k\,\alpha\|^2 \;=\; \|D_k\,\alpha_j\|^2 \;+\; 2\operatorname{Re}\langle\, D_k\,\alpha_j,\, D_k\,\beta\,\rangle \;+\; \|D_k\,\beta\|^2. \tag{5.1}$$

From Proposition 5.4, we obtain

$$|\langle\, D_k\,\alpha_j,\, D_k\,\beta\,\rangle| \;=\; |\langle\, D_k^2\,\alpha_j,\, \beta\,\rangle| \;\le\; C_2\,\|\alpha_j\|\,\|\beta\| \;\le\; C_2\,\|\alpha_j\|\,\|\alpha\|. \tag{5.2}$$

It follows from (5.1) and (5.2) that

$$\|D_k\,\alpha_j\|^2 \;\le\; C_2\,\|\alpha_i\|\,\|\alpha\|.$$

Hence, by Proposition 3.14, we get

$$C_1^2 k\,\|\alpha_j\|^2 \;\le\; C_2\,\|\alpha_i\|\,\|\alpha\|,$$

and $\|\alpha_j\| \le \frac{C_2}{kC_1^2}\|\alpha\|$. □

6. Proof of the Andreotti-Grauert theorem

In this section we use the results of Subsection 3.12 in order to get a new proof of the Andreotti-Grauert theorem (Theorem 3.9).

Note first, that, if the manifold M is Kähler, then the Andreotti-Grauert theorem follows directly from Theorem 3.13. Indeed, in this case the Dirac operator D_k is equal to the Dolbeault-Dirac operator $\sqrt{2}(\bar\partial + \bar\partial^*)$. Hence, the restriction of the kernel of D_k to $\mathcal{A}^{0,j}(M, \mathcal{O}(\mathcal{W} \otimes \mathcal{L}^k))$ is isomorphic to the cohomology $H^j(M, \mathcal{O}(\mathcal{W} \otimes \mathcal{L}^k))$.

In general, $D_k \ne \sqrt{2}(\bar\partial + \bar\partial^*)$. However, the following proposition shows that those two operators have the same "large k behavior".

Recall from Subsection 3.12 the notation

$$\mathcal{E} \;=\; \Lambda(T^{0,1}M) \otimes \mathcal{W}.$$

Proposition 6.1. *There exists a bundle map $A \in \operatorname{End}(\mathcal{E}) \subset \operatorname{End}(\mathcal{E} \otimes \mathcal{L}^k)$, independent on k, such that*

$$\sqrt{2}\,\left(\bar\partial + \bar\partial^*\right) \;=\; D_k \;+\; A. \tag{6.1}$$

Proof. Choose a holomorphic section $e(x)$ of $\mathcal{L}$ over an open set $U \subset \mathcal{L}$. It defines a section $e^k(x)$ of $\mathcal{L}^k$ over U and, hence, a holomorphic trivialization

$$U \times \mathbb{C} \;\xrightarrow{\sim}\; \mathcal{L}^k, \qquad (x, \phi) \mapsto \phi \cdot e^k(x) \in \mathcal{L}^k \tag{6.2}$$

of the bundle $\mathcal{L}^k$ over U. Similarly, the bundles $\mathcal{W}$ and $\mathcal{W} \otimes \mathcal{L}^k$ may be identified over U by the formula

$$w \;\mapsto\; w \otimes e^k. \tag{6.3}$$

Let $h^{\mathcal{L}}$ and $h^{\mathcal{W}}$ denote the Hermitian fiberwise metrics on the bundles $\mathcal{L}$ and $\mathcal{W}$ respectively. Let $h^{\mathcal{W}\otimes\mathcal{L}^k}$ denote the Hermitian metric on $\mathcal{W}\otimes\mathcal{L}^k$ induced by the metrics $h^{\mathcal{L}}, h^{\mathcal{W}}$. Set

$$f(x) \ := \ |e(x)|^2, \qquad x\in U,$$

where $|\cdot|$ denotes the norm defined by the metric $h^{\mathcal{L}}$. Under the isomorphism (6.3) the metric $h^{\mathcal{W}\otimes\mathcal{L}^k}$ corresponds to the metric

$$h_k(\cdot,\cdot) \ = \ f^k\, h^{\mathcal{W}}(\cdot,\cdot) \tag{6.4}$$

on $\mathcal{W}$.

By [BGV, p. 136], the connection $\nabla^{\mathcal{L}}$ on $\mathcal{L}$ corresponds under the trivialization (6.2) to the operator

$$\Gamma(U,\mathbb{C}) \ \to \ \Gamma(U,T^*U\otimes\mathbb{C}); \qquad s\mapsto ds+kf^{-1}\partial f\wedge s.$$

Similarly, the connection on $\mathcal{E}\otimes\mathcal{L}^k \ = \ \Lambda(T^{0,1}M)^*\otimes\mathcal{W}\otimes\mathcal{L}^k$ corresponds under the isomorphism (6.3) to the connection

$$\nabla_k:\ \alpha \ \mapsto \ \nabla^{\mathcal{E}}\alpha \ + \ kf^{-1}\partial f\wedge\alpha, \qquad \alpha\in\Gamma(U,\Lambda(T^{0,1}U)^*\otimes\mathcal{W}|_U)$$

on $\mathcal{E}|_U$. It follows now from (2.8) and (2.4) that the Dirac operator D_k corresponds under (6.3) to the operator

$$\widetilde{D}_k:\ \alpha \ \mapsto \ D_0\alpha \ - \ \sqrt{2}kf^{-1}\iota(\partial f)\alpha, \qquad \alpha\in\mathcal{A}^{0,*}(U,\mathcal{W}|_U). \tag{6.5}$$

Here $\iota(\partial f)$ denotes the contraction with the vector field $(\partial f)^*\in T^{0,1}M$ dual to the 1-form ∂f, and D_0 stands for the Dirac operator on the bundle $\mathcal{E}=\mathcal{E}\otimes\mathcal{L}^0$.

Let $\bar\partial_k^*:\mathcal{A}^{0,*}(U,\mathcal{W}|_U)\to\mathcal{A}^{0,*-1}(U,\mathcal{W}|_U)$ denote the adjoint of the operator $\bar\partial$ with respect to the scalar product on $\mathcal{A}^{0,*}(U,\mathcal{W}|_U)$ determined by the Hermitian metric h_k on $\mathcal{W}$ and the Riemannian metric on M. Then, it follows from (6.4), that

$$\bar\partial_k^* \ = \ \bar\partial \ + \ kf^{-1}\iota(\bar\partial f). \tag{6.6}$$

By (6.5) and (6.6), we obtain

$$\sqrt{2}\ \big(\bar\partial+\bar\partial_k^*\big) \ - \ \widetilde{D}_k \ = \ \sqrt{2}\ \big(\bar\partial+\bar\partial_0^*\big) \ - \ D_0.$$

Set $A=\sqrt{2}(\bar\partial+\bar\partial_0^*)-D_0$. By [Du, Lemma 5.5], A is a zero order operator, i.e. $A\in\operatorname{End}(\mathcal{E})$ (note that our definition of the Clifford action on $\Lambda(T^{0,1}M)^*$ and, hence, of the Dirac operator defers from [Du] by a factor of $\sqrt{2}$). □

6.2. **Proof of Theorem 3.9.** Let $A \in \mathrm{End}\,(\mathcal{E})$ be the operator defined in Proposition 6.1 and let

$$\|A\| \;=\; \sup_{\|\alpha\|=1} \|A\alpha\|, \qquad \alpha \in \mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k)$$

be the L_2-norm of the operator $A : \mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k) \to \mathcal{A}^{0,*}(M, \mathcal{E} \otimes \mathcal{L}^k)$. By Proposition 3.14, there exists a constant $C_1 > 0$ such that

$$\|D_k\,\alpha\| \;\geq\; C_1 k^{1/2}\,\|\alpha\|,$$

for any $k \gg 0$, $j \neq q, q+1, \dots, n-p$ and $\alpha \in \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$. Then, if $k > \|A\|^2/C_1^2$, we have

$$\|\sqrt{2}(\bar\partial + \bar\partial^*)\alpha\| \;=\; \|(D_k + A)\alpha\| \;\geq\; \|D_k\alpha\| \;-\; \|A\|\,\|\alpha\| \geq\; \big(C_1 k^{1/2} - \|A\|\big)\,\|\alpha\| \;>\; 0,$$

for any $j \neq q, q+1, \dots, n-p$ and $0 \neq \alpha \in \mathcal{A}^{0,j}(M, \mathcal{W} \otimes \mathcal{L}^k)$. Hence, the restriction of the kernel of the Dolbeault-Dirac operator to the space $\mathcal{A}^{0,j}(M, \mathcal{E} \otimes \mathcal{L}^k)$ vanishes for $j \neq q, q+1, \dots, n-p$. □

7. The Lichnerowicz formula. Proof of Propositions 4.3, 5.1 and 5.4

In this section we use the Lichnerowicz formula (cf. Subsection 7.3) to prove the Propositions 4.3, 5.1 and 5.4.

Before formulating the Lichnerowicz formula, we need some more information about the Clifford modules and Clifford connections (cf. [BGV, Section 3.3]).

7.1. **The symbol map and the quantization map.** Recall from Subsection 2.6 that the exterior algebra ΛT^*M has a natural structure of a self-adjoint Clifford module.

The Clifford bundle $C(M)$ is isomorphic to ΛT^*M as a bundle of vector spaces. The isomorphism is given by the *symbol map*

$$\sigma :\, C(M) \;\to \Lambda T^*M, \qquad \sigma :\, a \;\mapsto c(a)1.$$

The inverse of σ is called the *quantization map* and is denoted by $\mathbf{c}$. If $e^1, \dots, e^{2n}$ is an orthonormal basis of T_x^*M, then (cf. [BGV, Proposition 3.5])

$$\mathbf{c}(e^{i_1} \wedge \cdots \wedge e^{i_k}) \;=\; c(e^{i_1}) \cdots c(e^{i_k}).$$

Note that σ *is not* a map of algebras, i.e., $\sigma(ab) \neq \sigma(a)\sigma(b)$.

Assume now that $\mathcal{E}$ is a Clifford module. The composition of the quantization map with the Clifford action of $C(M)$ on $\mathcal{E}$ defines a map $\mathbf{c} : \Lambda T^*M \to \mathrm{End}\,(\mathcal{E})$. Though this map does not define the action of the exterior algebra on $\mathcal{E}$ (i.e. $\mathbf{c}(ab) \neq \mathbf{c}(a)\mathbf{c}(b)$) it plays an important role in differential geometry.

Let $\mathcal{A}(M) = \Gamma(M, \Lambda T^*M)$ denote the space of smooth sections of ΛT^*M, i.e., the space of smooth differential forms on M. The quantization map induces an isomorphism between $\mathcal{A}(M)$ and the space of sections of $C(M)$. More generally, for any bundle $\mathcal{E}$ over M, there is an isomorphism

$$\mathcal{A}(M,\mathcal{E}) \;\cong\; \Gamma(M, C(M)\otimes\mathcal{E}) \tag{7.1}$$

between the space of differential forms on M with values in $\mathcal{E}$ and the space of smooth sections of the tensor product $C(M)\otimes\mathcal{E}$. Combining this isomorphism with the Clifford action

$$c:\; \Gamma(M, C(M)\otimes\mathcal{E}) \to \Gamma(M,\mathcal{E}),$$

we obtain a map $\mathbf{c}:\; \mathcal{A}(M,\mathcal{E}) \to \mathcal{E}$. Similarly, we have a map

$$\mathbf{c}:\; \mathcal{A}(M, \mathrm{End}\,(\mathcal{E})) \;\to\; \mathrm{End}\,(\mathcal{E}). \tag{7.2}$$

In this paper we will be especially interested in the restriction of the above map to the space of 2-forms. In this case the following formula is useful

$$\mathbf{c}(F) \;=\; \sum_{i<j} F(e_i,e_j)\, c(e^i)\, c(e^j), \qquad F\in\mathcal{A}^2(M,\mathrm{End}\,(\mathcal{E})), \tag{7.3}$$

where $(e_1,\dots,e_{2n})$ is an orthonormal frame of the tangent space to M, and $(e^1,\dots,e^{2n})$ is the dual frame of the cotangent space.

7.2. **The curvature of a Clifford connection.** Let $\nabla^{\mathcal{E}}$ be a Clifford connection on a Clifford module $\mathcal{E}$ and let

$$F^{\mathcal{E}} \;=\; (\nabla^{\mathcal{E}})^2 \;\in\; \mathcal{A}^2(M,\mathrm{End}\,(\mathcal{E}))$$

denote the curvature of $\nabla^{\mathcal{E}}$.

Let $\mathrm{End}_{C(M)}\,(\mathcal{E})$ denote the bundle of endomorphisms of $\mathcal{E}$ commuting with the action of the Clifford bundle $C(M)$. Then the bundle $\mathrm{End}\,(\mathcal{E})$ of all endomorphisms of $\mathcal{E}$ is naturally isomorphic to the tensor product

$$\mathrm{End}\,(\mathcal{E}) \;\cong\; C(M)\otimes \mathrm{End}_{C(M)}\,(\mathcal{E}). \tag{7.4}$$

By Proposition 3.43 of [BGV], $F^{\mathcal{E}}$ decomposes with respect to (7.4) as

$$F^{\mathcal{E}} \;=\; R^{\mathcal{E}} \;+\; F^{\mathcal{E}/\mathcal{S}}, \qquad\qquad R^{\mathcal{E}}\in\mathcal{A}^2(M,C(M)),\;\; F^{\mathcal{E}/\mathcal{S}}\in\mathcal{A}^2(M,\mathrm{End}_{C(M)}\,(\mathcal{E})). \tag{7.5}$$

In this formula, $F^{\mathcal{E}/\mathcal{S}}$ is an invariant of $\nabla^{\mathcal{E}}$ called the *twisting curvature* of $\mathcal{E}$, and $R^{\mathcal{E}}$ is determined by the Riemannian curvature R of M. If $(e_1,\dots,e_{2n})$ is an orthonormal frame of the tangent space $T_xM,\ x\in M$ and $(e^1,\dots,e^{2n})$ is the dual frame of the cotangent space T^*M, then

$$R^{\mathcal{E}}(e_i,e_j) \;=\; \frac{1}{4}\sum_{k,l}\langle R(e_i,e_j)e_k,e_l\rangle\, c(e^k)\, c(e^l).$$

Assume that $\mathcal{S}$ is a spinor bundle, $\mathcal{E} = \mathcal{W} \otimes \mathcal{S}$ and the connection $\nabla^{\mathcal{E}}$ is given by (2.6). Then $\mathcal{A}(M, \mathrm{End}_{C(M)}(\mathcal{E})) \cong \mathcal{A}(M, \mathrm{End}(\mathcal{W}))$. The twisting curvature $F^{\mathcal{E}/\mathcal{S}}$ is equal to the curvature $F^{\mathcal{W}} = (\nabla^{\mathcal{W}})^2$ via this isomorphism (cf. [BGV, p. 121]). This explains why $F^{\mathcal{E}/\mathcal{S}}$ is called the twisting curvature.

Let $\mathcal{W}$ be a vector bundle over M with connection $\nabla^{\mathcal{W}}$ and let $F^{\mathcal{W}} = (\nabla^{\mathcal{W}})^2$ denote the curvature of this connection. The twisting curvature of the connection (2.1) on the tensor product $\mathcal{E} \otimes \mathcal{W}$ is the sum

$$F^{(\mathcal{E}\otimes\mathcal{W})/\mathcal{S}} \ = \ F^{\mathcal{W}} \ + \ F^{\mathcal{E}/\mathcal{S}}. \tag{7.6}$$

7.3. **The Lichnerowicz formula.** Let $\mathcal{E}$ be a Clifford module endowed with a Hermitian structure and let $D : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{E})$ be a self-adjoint Dirac operator associated to a Hermitian Clifford connection $\nabla^{\mathcal{E}}$. Consider the metric Laplacian (cf. Subsection 4.2)

$$\Delta^{\mathcal{E}} \ = \ (\nabla^{\mathcal{E}})^* \, \nabla^{\mathcal{E}} \ : \ \Gamma(M, \mathcal{E}) \ \to \Gamma(M, \mathcal{E}),$$

where $(\nabla^{\mathcal{E}})^*$ denotes the operator adjoint to $\nabla^{\mathcal{E}} : \Gamma(M, \mathcal{E}) \to \Gamma(M, T^*M \otimes \mathcal{E})$ with respect to the L_2-scalar product. Clearly $\Delta^{\mathcal{E}}$ is a non-negative self-adjoint operator.

The following *Lichnerowicz formula* (cf. [BGV, Theorem 3.52]) plays a crucial role in our proof of vanishing theorems:

$$D^2 \ = \ \Delta^{\mathcal{E}} \ + \ \mathbf{c}(F^{\mathcal{E}/\mathcal{S}}) \ + \ \frac{r_M}{4}, \tag{7.7}$$

where r_M stands for the scalar curvature of M and $F^{\mathcal{E}/\mathcal{S}}$ is the twisting curvature of $\nabla^{\mathcal{E}}$, cf. Subsection 7.2. The operator $\mathbf{c}(F^{\mathcal{E}/\mathcal{S}})$ is defied in (7.2) (see also (7.3)).

Let $\mathcal{L}$ be a Hermitian line bundle over M endowed with a Hermitian connection $\nabla^{\mathcal{L}}$ and let $\nabla_k = \nabla^{\mathcal{E}\otimes\mathcal{L}^k}$ denote the product connection (cf. (2.1)) on the tensor product $\mathcal{E} \otimes \mathcal{L}^k$. It is a Hermitian Clifford connection on $\mathcal{E} \otimes \mathcal{L}^k$. We denote by D_k and Δ_k the Dirac operator and the metric Laplacian associated to this connection. By (7.6), it follows from the Lichnerowicz formula (7.7), that

$$D_k^2 \ = \ \Delta_k \ + \ k \, \mathbf{c}(F^{\mathcal{L}}) \ + \ A, \tag{7.8}$$

where $F^{\mathcal{L}} = (\nabla^{\mathcal{L}})^2$ is the curvature of $\nabla^{\mathcal{L}}$ and

$$A \ := \ \mathbf{c}(F^{\mathcal{E}/\mathcal{S}}) \ + \ \frac{r_M}{4} \ \in \ \mathrm{End}(\mathcal{E}) \ \subset \ \mathrm{End}(\mathcal{E} \otimes \mathcal{L}^k) \tag{7.9}$$

is independent of $\mathcal{L}$ and k.

7.4. **Calculation of $\mathbf{c}(F^{\mathcal{L}})$.** To compare D_k^2 with the Laplacian Δ_k we now need to calculate the operator $\mathbf{c}(F^{\mathcal{L}}) \in \mathrm{End}(\mathcal{E}) \subset \mathrm{End}(\mathcal{E} \otimes \mathcal{L}^k)$. This may be reformulated as the following problem of linear algebra.

Let V be an oriented Euclidean vector space of real dimension $2n$ and let V^* denote the dual vector space. We denote by $C(V)$ the Clifford algebra of V^*. Let E be a module over $C(V)$. We will assume that E is endowed with a Hermitian scalar product such that the operator $c(v) : E \to E$ is skew-symmetric for any $v \in V^*$. In this case we say that E is a *self-adjoint* Clifford module over V.

The space E possesses a *natural grading* $E = E^+ \oplus E^-$, where E^+ and E^- are the eigenspaces of the chirality operator with eigenvalues $+1$ and -1 respectively, cf. Subsection 2.3.

In our applications V is the tangent space T_xM to M at a point $x \in M$ and E is the fiber of $\mathcal{E}$ over x.

Let F be an imaginary valued antisymmetric bilinear form on V. Then F may be considered as an element of $V^* \wedge V^*$. We need to estimate the operator $\mathbf{c}(F) \in \mathrm{End}\,(E)$. Here $\mathbf{c} : \Lambda V^* \to C(V)$ is the quantization map defined exactly as in Subsection 7.1 (cf. [BGV, §3.1]).

Let us define the skew-symmetric linear map $\widetilde{J} : V \to V$ by the formula

$$iF(v,w) \;=\; \langle v, \widetilde{J}w\rangle, \qquad v,w \in V.$$

The eigenvalues of $\widetilde{J}$ are purely imaginary. Let $\mu_1 \ge \cdots \ge \mu_l > 0$ be the positive numbers such that $\pm i\mu_1, \dots, \pm i\mu_l$ are all the non-zero eigenvalues of $\widetilde{J}$. Set

$$\tau \;=\; \mathrm{Tr}^+ \widetilde{J} \;:=\; \mu_1 + \cdots + \mu_l, \qquad m \;=\; \min_j \mu_j.$$

By the Lichnerowicz formula (7.7), Proposition 4.3 is equivalent to the following

Proposition 7.5. *Suppose that the bilinear form F is non-degenerate. Then it defines an orientation of V. If this orientation coincides with (resp. is opposite to) the given orientation of V, then the restriction of $\mathbf{c}(F)$ onto E^- (resp. E^+) is greater than $-(\tau - 2m)$, i.e., for any $\alpha \in E^-$ (resp. $\alpha \in E^+$)*

$$\langle c(F)\alpha, \alpha\rangle \;\ge\; -(\tau - 2m)\,\|\alpha\|^2.$$

We will prove the proposition in Subsection 7.11 after introducing some additional constructions. Since we need these constructions also for the proof of Proposition 5.1, we do not assume that F is non-degenerate unless this is stated explicitly.

7.6. **A choice of a complex structure on V.** By the Darboux theorem (cf. [Au, Theorem 1.3.2]), one can choose an orthonormal basis $f^1, \dots, f^{2n}$ of V^*, which defines the positive orientation of V (i.e., $f^1 \wedge \cdots \wedge f^{2n}$ is a positive volume form on V) and such that

$$iF^{\mathcal{L}}_x \;=\; \sum_{j=1}^{l} r_j\, f^j \wedge f^{j+n}, \tag{7.10}$$

for some integer $l \le n$ and some non-zero real numbers r_j. We can and we will assume that $|r_1| \ge |r_2| \ge \cdots \ge |r_l|$.

Let $f_1, \dots, f_{2n}$ denote the dual basis of V.

Remark 7.7. If the vector space V is endowed with a complex structure $J : V \to V$ compatible with the metric (i.e., $J^* = -J$) and such that F is a $(1,1)$ form with respect to J, then the basis $f_1, \dots, f_{2n}$ can be chosen so that $f_{j+n} = Jf_j,\ i = 1, \dots, n$.

Let us define a complex structure $J : V \to V$ on V by the condition $f_{i+n} = Jf_i,\ i = 1, \dots, n$. Then, the complexification of V splits into the sum of its holomorphic and anti-holomorphic parts

$$V \otimes \mathbb{C} \ = \ V^{1,0} \oplus V^{0,1},$$

on which J acts by multiplication by i and $-i$ respectively. The space $V^{1,0}$ is spanned by the vectors $e_j = f_j - if_{j+n}$, and the space $V^{0,1}$ is spanned by the vectors $\overline{e}_j = f_j + if_{j+n}$. Let $e^1, \dots, e^n$ and $\overline{e}^1, \dots, \overline{e}^n$ be the corresponding dual base of $(V^{1,0})^*$ and $(V^{0,1})^*$ respectively. Then (7.10) may be rewritten as

$$iF^{\mathcal{L}}_x \ = \ \frac{i}{2} \sum_{j=1}^{n} r_j\, e^j \wedge \overline{e}^j.$$

We will need the following simple

Lemma 7.8. *Let $\mu_1, \dots, \mu_l$ and $r_1, \dots, r_l$ be as above. Then $\mu_i = |r_i|$, for any $i = 1, \dots, l$. In particular,*

$$\operatorname{Tr}^+ \widetilde{J} \ = \ |r_1| + \cdots + |r_l|.$$

Proof. Clearly, the vectors $e_1, \dots, e_n; \overline{e}_1, \dots, \overline{e}_n$ form a basis of eigenvectors of $\widetilde{J}$ and

$$\begin{aligned} &\widetilde{J}\, e_j \ = \ ir_j\, e_j, \quad \widetilde{J}\, \overline{e}_j \ = \ -ir_j\, \overline{e}_j && \text{for} \quad j = 1, \dots, l, \\ &\widetilde{J}\, e_j \ = \ \widetilde{J}\, \overline{e}_j \ = \ 0 && \text{for} \quad j = l+1, \dots, n. \end{aligned}$$

Hence, all the nonzero eigenvalues of $\widetilde{J}$ are $\pm i|r_1|, \dots, \pm i|r_l|$. □

7.9. **Spinors.** Set

$$S^+ \ = \ \bigoplus_{j \text{ even}} \Lambda^j(V^{0,1}), \quad S^- \ = \ \bigoplus_{j \text{ odd}} \Lambda^j(V^{0,1}). \tag{7.11}$$

Define a graded action of the Clifford algebra $C(V)$ on the graded space $S = S^+ \oplus S^-$ as follows (cf. Subsection 2.7): if $v \in V$ decomposes as $v = v^{1,0} + v^{0,1}$ with $v^{1,0} \in V^{1,0}$ and $v^{0,1} \in V^{0,1}$, then its Clifford action on $\alpha \in E$ equals

$$c(v)\alpha \ = \ \sqrt{2}\, \left(v^{0,1} \wedge \alpha \ - \ \iota(v^{1,0})\, \alpha \right). \tag{7.12}$$

Then (cf. [BGV, §3.2]) S is the *spinor representation* of $C(V)$, i.e., the complexification $C(V)\otimes\mathbb{C}$ of $C(V)$ is isomorphic to $\mathrm{End}\,(S)$. In particular, the Clifford module E can be decomposed as

$$E \;=\; S\otimes W,$$

where $W=\mathrm{Hom}_{C(V)}\,(S,E)$. The action of $C(V)$ on E is equal to $a\mapsto c(a)\otimes 1$, where $c(a)$ $(a\in C(V))$ denotes the action of $C(V)$ on S. The natural grading on E is given by $E^\pm=S^\pm\otimes W$.

To prove Proposition 7.5 it suffices now to study the action of $\mathbf{c}(F)$ on S. The latter action is completely described be the following

Lemma 7.10. *The vectors $\bar{e}^{j_1}\wedge\cdots\wedge\bar{e}^{j_m}\in S$ form a basis of eigenvectors of $\mathbf{c}(F)$ and*

$$\mathbf{c}(F)\,\bar{e}^{j_1}\wedge\cdots\wedge\bar{e}^{j_m} \;=\; \Big(\sum_{j'\notin\{j_1,\dots,j_m\}} r_{j'} \;-\; \sum_{j''\in\{j_1,\dots,j_m\}} r_{j''}\Big)\,\bar{e}^{j_1}\wedge\cdots\wedge\bar{e}^{j_m}.$$

Proof. Obvious. □

7.11. **Proof of Proposition 7.5.** Recall that the orientation of V is fixed and that we have chosen the basis $f_1,\dots,f_{2n}$ of V which defines the same orientation. Suppose now that the bilinear form F is non-degenerate. Then $l=n$ in (7.10). It is clear, that the orientation defined by iF coincides with the given orientation of V if and only if the number q of positive numbers among $r_1,\dots,r_n$ is even. Hence, by Lemma 7.10, the restriction of $\mathbf{c}(F)\in\mathrm{End}\,(S)$ on $\Lambda^j(V^{0,1})\subset S$ is greater than $-(\tau-2m)$ if the parity of j and q are different. The Proposition 7.5 follows now from (7.11). □

7.12. **Proof of Proposition 5.1.** Assume that at least q of the numbers $r_1,\dots,r_l$ are negative and let $m_q>0$ be the minimal positive number such that at least q of these numbers are not greater than $-m_q$. It follows from Lemma 7.10, that

$$\langle\, c(F)\alpha,\alpha\,\rangle \;\ge\; -\,(\tau-2m_q)\,\|\alpha\|^2,$$

for any $j<q$ and any $\alpha\in\Lambda^j(V^{0,1})$. Proposition 5.1 follows now from the Lichnerowicz formula (7.7). □

7.13. **Proof of Proposition 5.4.** Clearly, the metric Laplacian Δ_k preserves the $\mathbb{Z}$-grading on $\mathcal{A}^{0,*}(M,\mathcal{E}\otimes\mathcal{L}^k)$. In other words, $(\Delta_k\alpha)_j=0$, for any $\alpha\in\mathcal{A}^{0,i}(M,\mathcal{E}\otimes\mathcal{L}^k)$ and any $j\ne i$. By Lemma 7.10, the same is true for the operator $\mathbf{c}(F^{\mathcal{L}})$. Hence, it follows from (7.8), that if $\alpha\in\mathcal{A}^{0,i}(M,\mathcal{E}\otimes\mathcal{L}^k)$, then

$$(D_k^2\,\alpha)_j \;=\; (A\,\alpha)_j, \quad \text{for any} \quad j\ne i.$$

Here $A\in\mathrm{End}\,(\mathcal{E})\subset\mathrm{End}\,(\mathcal{E}\otimes\mathcal{L}^k)$ is the operator defined in (7.9). Since, A is an independent of k bounded operator, the proposition is proven. □

8. Estimate of the metric Laplacian

In this section we prove Proposition 4.4.

8.1. **Reduction to a scalar operator.** In this subsection we construct a space $\mathcal{Z}$ and an operator $\widetilde{\Delta}$ on the space $L_2(\mathcal{Z})$ of $\mathcal{Z}$, such that the operator Δ_k is "equivalent" to a restriction of $\widetilde{\Delta}$ onto certain subspace of $L_2(\mathcal{Z})$. This allow to compare the operators Δ_k for different values of k.

Let $\mathcal{F}$ be the principal G-bundle with a compact structure group G, associated to the vector bundle $\mathcal{E} \to M$. Let $\mathcal{Z}$ be the principal $(S^1 \times G)$-bundle over M, associated to the bundle $\mathcal{E} \otimes \mathcal{L} \to M$. Then $\mathcal{Z}$ is a principle S^1-bundle over $\mathcal{F}$. We denote by $p : \mathcal{Z} \to \mathcal{F}$ the projection.

The connection $\nabla^{\mathcal{L}}$ on $\mathcal{L}$ induces a connection on the bundle $p : \mathcal{Z} \to \mathcal{F}$. Hence, any vector $X \in T\mathcal{Z}$ decomposes as a sum

$$X \ = \ X^{\rm hor} \ + \ X^{\rm vert}, \tag{8.1}$$

of its horizontal and vertical components.

Consider the *horizontal exterior derivative* $d^{\rm hor} : C^\infty(\mathcal{Z}) \to \mathcal{A}^1(\mathcal{Z}, \mathbb{C})$, defined by the formula

$$d^{\rm hor} f(X) \ = \ df(X^{\rm hor}), \qquad X \in T\mathcal{Z}.$$

The connections on $\mathcal{E}$ and $\mathcal{L}$, the Riemannian metric on M, and the Hermitian metrics on $\mathcal{E}, \mathcal{L}$ determine a natural Riemannian metrics $g^{\mathcal{F}}$ and $g^{\mathcal{Z}}$ on $\mathcal{F}$ and $\mathcal{Z}$ respectively, cf. [BU, Proof of Theorem 2.1]. Let $(d^{\rm hor})^*$ denote the adjoint of $d^{\rm hor}$ with respect to the scalar products induced by this metric. Let

$$\widetilde{\Delta} \ = \ (d^{\rm hor})^* d^{\rm hor} : C^\infty(\mathcal{Z}) \to C^\infty(\mathcal{Z})$$

be the *horizontal Laplacian* for the bundle $p : \mathcal{Z} \to \mathcal{F}$.

Let $C^\infty(\mathcal{Z})_k$ denote the space of smooth functions on $\mathcal{Z}$, which are homogeneous of degree k with respect to the natural fiberwise circle action on the circle bundle $p : \mathcal{Z} \to \mathcal{F}$. It is shown in [BU, Proof of Theorem 2.1], that to prove Proposition 4.4 it suffices to prove (4.2) for the restriction of $\widetilde{\Delta}$ to the space $C^\infty(\mathcal{Z})_k$.

8.2. **The symbol of $\widetilde{\Delta}$.** The decomposition (8.1) defines a splitting of the cotangent bundle $T^*\mathcal{Z}$ to $\mathcal{Z}$ into the horizontal and vertical subbundles. For any $\xi \in T^*\mathcal{Z}$, we denote by $\xi^{\rm hor}$ the horizontal component of ξ. Then, one easily checks (cf. [BU, Proof of Theorem 2.1]), that the principal symbol $\sigma_2(\widetilde{\Delta})$ of $\widetilde{\Delta}$ may be written as

$$\sigma_2(\widetilde{\Delta})(z, \xi) \ = \ g^{\mathcal{F}}(\xi^{\rm hor}, \xi^{\rm hor}). \tag{8.2}$$

The subprincipal symbol of $\widetilde{\Delta}$ is equal to zero.

On the *character set* $\mathcal{C} = \big\{(z,\xi) \in T^*\mathcal{Z}\backslash\{0\} : \xi^{\rm hor} = 0\big\}$ the principal symbol $\sigma_2(\widetilde{\Delta})$ vanishes to second order. Hence, at any point $(z,\xi) \in \mathcal{C}$, we can define the *Hamiltonian map* $F_{z,\xi}$ of $\sigma_2(\widetilde{\Delta})$, cf. [Ho, §21.5]. It is a skew-symmetric endomorphism of the tangent space $T_{z,\xi}(T^*\mathcal{Z})$. Set

$$\mathrm{Tr}^+ F_{z,\xi} \ = \ \nu_1 \ + \ \cdots \ + \ \nu_l,$$

where $i\nu_1,\dots,i\nu_l$ are the nonzero eigenvalues of $F_{z,\xi}$ for which $\nu_i > 0$.

Let $\rho : \mathcal{Z} \to M$ denote the projection. Then, cf. [BU, Proof of Theorem 2.1] [1],

$$\mathrm{Tr}^+ F_{z,\xi} \ = \ \tau(\rho(z))\,|\xi^{\rm vert}| \tag{8.3}$$

Here $\xi^{\rm vert}$ is the vertical component of $\xi \in T^*\mathcal{Z}$, and τ denotes the function defined in (4.1).

8.3. **Application of the Melin inequality.** Let $D^{\rm vert}$ denote the generator of the S^1 action on $\mathcal{Z}$. The symbol of $D^{\rm vert}$ is $\sigma(D^{\rm vert})(z,\xi) = \xi^{\rm vert}$. Fix $\varepsilon > 0$, and consider the operator

$$A \ = \ \widetilde{\Delta} \ - \big(\tau(\rho(z)) - \varepsilon\big)\, D^{\rm vert} : \, C^\infty(\mathcal{Z}) \ \to \ C^\infty(\mathcal{Z}).$$

The principal symbol of A is given by (8.2), and the subprincipal symbol

$$\sigma_1^s(A)(z,\xi) \ = \ -\big(\tau(\rho(z)) - \varepsilon\big)\, \xi^{\rm vert}.$$

It follows from (8.3), that

$$\mathrm{Tr}^+ F_{z,\xi} + \sigma_1^s(A)(z,\xi) \ \ge \ \varepsilon\,|\xi^{\rm vert}| \ > \ 0.$$

Hence, by the Melin inequality ([Me], [Ho, Theorem 22.3.3]), there exists a constant C_ε, depending on ε, such that

$$\langle\, Af, f\,\rangle \ \ge -\, C_\varepsilon\, \|f\|^2. \tag{8.4}$$

Here $\|\cdot\|$ denotes the L_2 norm of the function $f \in C^\infty(\mathcal{Z})$.

From (8.4), we obtain

$$\langle\, \widetilde{\Delta} f, f\,\rangle \ \ge \ \langle\, (\tau(\rho(z)) - \varepsilon) D^{\rm vert} f, f\,\rangle \ - \ C_\varepsilon\, \|f\|^2.$$

Noting that if $f \in C^\infty(\mathcal{Z})_k$, then $D^{\rm vert} f = kf$, the proof is complete. □

[1] The absolute value sign of $\xi^{\rm vert}$ is erroneously missing in [BU].

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Institute of Mathematics, The Hebrew University, Jerusalem 91904, Israel
E-mail address: maxim@math.huji.edu