SOME PROPERTIES OF SOLUTIONS OF THE SINGULAR HELMHOLTZ EQUATION IN THE HALF-SPACE

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Preprint No. 16
1997/1998

ABSTRACT. A singular Helmholtz equation in the half-space $\mathbb{R}^{n+1}_+ = \{(x,t) : x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ t > 0\},\ n \ge 1$, has the form

$$\Big(\Delta_{x,t} - \frac{2\nu - 1}{t}\frac{\partial}{\partial t} + k^2\Big)u(x,t) = 0$$

where $\Delta_{x,t} = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2 + \partial^2/\partial t^2$. For $k > 0$ and certain $\nu > 0$, we obtain sharp L^p-estimates of oscillatory integrals of the Poisson type, satisfying this equation with the Dirichlet boundary condition $\lim\limits_{t\to 0} u(x,t) = f(x) \in L^p(\mathbb{R}^n)$. An almost everywhere convergence of these integrals to f is established in the maximal range of p. The argument is based on known results for Bochner-Riesz means.

*Partially supported by the Edmund Landau Center for research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (Germany).

1 Introduction

The equation mentioned above is also known as the generalized axially symmetric reduced wave equation. Equations of this type were studied from different points of view in the pioneering papers by A. Weinstein, and later by D. Colton, A. Erdélyi, R.P. Gilbert, A.E. Heins, P. Henrici, J.S. Lowndes, R.F. Millar, P. Ramankutty and many other authors (see, e.g., [2, 12, 15, 16, 20, 27] and references therein). The case $\nu = 1/2$ corresponds to the usual Helmholtz equation, for which the Dirichlet problem

$$\Delta_{x,t} u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \quad u(x,0) = f(x) \tag{1.1}$$

has a unique solution satisfying the Sommerfeld outgoing radiation condition. This condition reads as follows: if $\tilde{x} = (x,t)$, $|\tilde{x}| = (|x|^2 + t^2)^{1/2} \to \infty$, then

$$u(x,t) = O(|\tilde{x}|^{-n/2}) \quad \text{and} \quad \frac{\partial u}{\partial |\tilde{x}|} - iku = o(|\tilde{x}|^{-n/2}) \tag{1.1$'$}$$

uniformly in all directions $\theta = \tilde{x}/|\tilde{x}| \in \mathbb{R}^{n+1}_+$. The solution of (1.1)-(1.1$'$) is represented as a convolution

$$u(x,t) \equiv (P_t f)(x) = (p_t * f)(x) \tag{1.2}$$

with the kernel

$$p_t(x) = c_n t \left(\frac{k}{\sqrt{t^2 + |x|^2}} \right)^{(n+1)/2} H^{(1)}_{(n+1)/2}(k\sqrt{t^2 + |x|^2}) \tag{1.3}$$

where $c_n = i\pi^{(1-n)/2} 2^{-(n+1)/2}$, $H^{(1)}_{(n+1)/2}(z)$ is the Hankel function of the first kind (cf. [6], Section 10.8, for n=1). The convolution (1.2) is called a metaharmonic continuation of f into $\mathbb{R}^{n+1}_+$. Properties of metaharmonic functions were studied by I.N. Vekua [25] (see also [4, 5]). In the limit case $k = 0$ the convolution (1.2) turns into the usual Poisson integral [24]. The Fourier transform of $p_t(x)$ has the form

$$\hat{p}_t(\xi) = \begin{cases} \exp(-t\sqrt{|\xi|^2 - k^2} & \text{if } |\xi| > k, \\ \\ \exp\ (it\sqrt{k^2 - |\xi|^2}) & \text{if } |\xi| < k. \end{cases} \tag{1.4}$$

More detailed information about convolutions (1.2) with the arbitrary complex wave number k can be found in [21], Section 22.1.

We shall consider a natural generalization of (1.2) given by

$$u(x,t) \equiv (P_t^\nu f)(x) = (p_t^\nu * f)(x), \tag{1.5}$$

$$p_t^\nu(x) = c_n^\nu t^{2\nu} \left(\frac{k}{\sqrt{t^2 + |x|^2}} \right)^{\nu+n/2} H^{(1)}_{\nu+n/2}(k\sqrt{t^2+|x|^2}), \tag{1.6}$$

where $c_n^\nu = i\pi^{1-n/2} 2^{-\nu-n/2}/\Gamma(\nu)$. The function (1.6) coincides with (1.3) for $\nu = 1/2$ and has the following asymptotics

$$p_t^\nu(x) = \frac{it^{2\nu}}{(2\pi)^{n/2}} \left(\frac{k}{|\tilde{x}|} \right)^{\nu+n/2} \sqrt{\frac{1}{k|\tilde{x}|}}\ e^{i(k|\tilde{x}|-(2\nu+n+1)\pi/4)}(1 + O(|\tilde{x}|^{-1})) =$$

$$= O(|\tilde{x}|^{-\nu-(n+1)/2}), \qquad |\tilde{x}| = (|x|^2 + t^2)^{1/2} \to \infty, \tag{1.7}$$

(cf. formula 7.13.1(1) from [9]). If $Re\ \nu > -1$, then the Fourier transform of $p_t^\nu(x)$ can be evaluated as follows:

$$(p_t^\nu)^\wedge(\xi) \stackrel{\text{def}}{=} \lim_{N\to\infty} \int\limits_{|x|<N} p_t^\nu(x) e^{ix\cdot\xi} dx =$$

$$= \frac{i\pi t^{2\nu} k^{\nu+n/2}}{2^\nu \Gamma(\nu) |\xi|^{n/2-1}} \int\limits_0^\infty J_{n/2-1}(s|\xi|)\ \frac{H^{(1)}_{\nu+n/2}(k\sqrt{t^2+s^2})}{(\sqrt{t^2+s^2}\ )^{\nu+n/2}} s^{n/2} ds =$$

$$= \frac{\pi^{1/2}(t\omega)^\nu}{2^{\nu-1/2}\Gamma(\nu)} \begin{cases} i(\pi/2)^{1/2} H_\nu^{(1)}(t\omega) & \text{if } |\xi| < k, \\ (2/\pi)^{1/2} K_\nu(t\omega) & \text{if } |\xi| > k, \end{cases} \qquad \omega = |k^2 - |\xi|^2|^{1/2}, \tag{1.8}$$

where K_ν is the modified Bessel function of the third kind (use, e.g., Theorem 3.3 from [24, Chapter 4] and the formula 7.14.2(48) from [9]). The behaviour of $(p_t^\nu)^\wedge(\xi)$ for $t \to 0$ and $|\xi| \to k$ becomes clear from the following general proposition.

Proposition 1.1 *Let*

$$c_\nu^{(1)} = \frac{2^\nu \Gamma(\nu)}{\pi i}; \qquad c_\nu^{(2)} = \frac{e^{-i\nu\pi} 2^{-\nu} \Gamma(-\nu)}{\pi i} \qquad (\nu \neq 0, -1, -2, \ldots).$$

For $r \to 0$, the following asymptotic reations hold:

$$r^\nu H_\nu^{(1)}(r) = \begin{cases} c_\nu^{(1)} + o(1), & Re\ \nu > 0, \\ c_\nu^{(1)} + c_\nu^{(2)} r^{2\nu} + o(1), & Re\ \nu = 0, \quad \nu \neq 0, \\ \dfrac{2}{\pi i} \log \dfrac{1}{r}\ (1 + o(1)), & \nu = 0, \\ r^{2\nu}(c_\nu^{(2)} + o(1)), & Re\ \nu < 0, \quad \nu \neq -1, -2, \ldots, \\ (-1)^{|\nu|} c_{|\nu|}^{(1)} r^{2\nu}(1 + o(1)), & \nu = -1, -2, \ldots; \end{cases} \tag{1.9}$$

$$r^\nu K_\nu(r) = \frac{\pi i}{2} \begin{cases} c_\nu^{(1)} + o(1), & Re\ \nu > 0, \\ c_\nu^{(1)} + c_\nu^{(2)} e^{i\nu\pi} r^{2\nu} + o(1), & Re\ \nu = 0, \quad \nu \neq 0, \\ \dfrac{2}{\pi i} \log \dfrac{1}{r}\ (1 + o(1)), & \nu = 0, \\ r^{2\nu} e^{i\nu\pi}(c_\nu^{(2)} + o(1)), & Re\ \nu < 0, \quad \nu \neq -1, -2, \ldots, \\ (-1)^{|\nu|} c_{|\nu|}^{(1)} e^{i\nu\pi} r^{2\nu}(1 + o(1)), & \nu = -1, -2, \ldots\ . \end{cases} \tag{1.10}$$

The validity of Proposition 1.1 can be easily checked using the standard properties of Bessel functions [9].

For fixed ξ, $|\xi| \neq k$, the function $\varphi : t \to (p_t^\nu)^\wedge(\xi)$ satisfies the differential equation

$$\left(\frac{d^2}{dt^2} - \frac{2\nu - 1}{t} \frac{d}{dt} + k^2 - |\xi|^2 \right) \varphi(t) = 0, \qquad t > 0,$$

of the Bessel type. Moreover, if $Re\,\nu > 0$, then, by (1.9) and (1.10), $\varphi(0) = 1$, and therefore the function (1.5) is a solution of the Dirichlet problem

$$\Big(\Delta_{x,t} - \frac{2\nu-1}{t}\frac{\partial}{\partial t} + k^2\Big)u(x,t) = 0 \quad \text{in} \quad \mathbb{R}^{n+1}_+, \quad u(x,0) = f(x) \qquad (1.11)$$

(at least for sufficiently good f). Note that for $Re\,\nu \le 0,\ \nu \ne 0$, the limit $\lim\limits_{t\to 0}\varphi(t)$ does not exist.

Up to my knowledge, the Dirichlet problems (1.1) and (1.11) with $k > 0$ has not been studied in the L^p-setting. The present article brings some light to this topic. Here one should mention the papers [8, 14, 19] related to the case $k = 0$. The case $k > 0$ seems to be more difficult and misterious than $k = 0$ or $Im\,k > 0$. In the last two cases the kernel (1.6) is integrable for $\nu > 0$.

Theorem 1.2 *For each $t > 0$, the operator P_t, defined by* (1.2), *is bounded on $L^p(\mathbb{R}^n)$, $n \ge 2$, if and only if $|1/p - 1/2| < 1/n$.*

This statement was obtained by the author jointly with W. Trebels in 1993 but was unpublished. It was announced in [21, p. 300] in connection with studying an inversion problem for the generalized oscillatory potentials in the half-space

$$(J^{\alpha,k}f)(x) = \zeta_{n,\alpha}\int\limits_{\mathbb{R}^n_+}\left(\frac{k}{|x-y|}\right)^{(n-\alpha)/2} H^{(1)}_{(n-\alpha)/2}(k|x-y|)f(y)dy, \qquad (1.12)$$

$x \in \mathbb{R}^n_+$, $\zeta_{n,\alpha} = 2^{(n+\alpha)/2}\pi^{1-n/2}i/\Gamma(\alpha/2)$. In the case $n = 2$, $\alpha = 1$, which is especially important in applications (e.g., in diffraction theory), we have

$$(J^{1,k}f)(x) = \frac{1}{4\pi}\int\limits_{\mathbb{R}^2_+}\frac{e^{ik|x-y|}}{|x-y|}f(y)dy. \qquad (1.13)$$

Potentials (1.12) admit the Wiener-Hopf factorization in terms of oscillatory fractional integrals involving the kernel (1.3). More precisely,

$$J^{\alpha,k}f = J_-^{\alpha/2,k} J_+^{\alpha/2,k} f \tag{1.14}$$

(at least for $0 < \alpha < 1$ and sufficiently goof f), where

$$(J_+^{\alpha/2,k}f)(x) = \frac{1}{\Gamma(\alpha/2)} \int\limits_{\mathbb{R}^n_+} (x_n - y_n)_+^{\alpha/2-1} \mathcal{P}_k(x-y) f(y)\, dy, \tag{1.15}$$

$$(J_-^{\alpha/2,k}f)(x) = \frac{1}{\Gamma(\alpha/2)} \int\limits_{\mathbb{R}^n_+} (y_n - x_n)_+^{\alpha/2-1} \mathcal{P}_k(x-y) f(y)\, dy. \tag{1.15$'$}$$

Here $\mathcal{P}_k(x-y) \equiv p_t(x'-y')$ denotes the Poisson-like kernel (1.6) with $t = |x_n - y_n|$, $x' = (x_1, \ldots, x_{n-1})$, $y' = (y_1, \ldots, y_{n-1})$, and with n replaced by $n-1$. The equality (1.14) is a consequence of the following relation between the corresponding symbols

$$(|\xi|^2 - k^2 - i0)^{-\alpha/2} = (\sqrt{|\xi'|^2 - k^2} - i\xi_n)^{-\alpha/2} (\sqrt{|\xi'|^2 - k^2} + i\xi_n)^{-\alpha/2} \tag{1.16}$$

(all expressions in this equality have a suitable interpretation). Further details concerning (1.12)-(1.16) can be found in [21], Section 22.

Let us state main results of the paper. We assume

$$n \geq 1, \quad \nu > 0, \quad \delta_\nu = \nu/n + 1/2n; \quad L^p = L^p(\mathbb{R}^n), \quad \Delta_p = |1/2 - 1/p|.$$

The letter c (sometimes with subscripts) designates a constant independent of t and f, which is not necessarily the same at each occurrence.

Theorem A

(i) *If $\nu > (n-1)/2$, then the operator P_t^ν defined by (1.5) is bounded on L^p for all $p \in [1, \infty]$, and*

$$\|P_t^\nu\|_{L^p \to L^p} \leq c(1+t)^q, \quad q = (2\nu + n - 1)\Delta_p + (1 - 2\Delta_p)\max(\nu - 1/2, 0). \tag{1.17}$$

(ii) *Let* $0 < \nu \le (n-1)/2,\ 1 < p < \infty$. *If* $\Delta_p \ge \delta_\nu$, *then* P_t^ν *is not bounded on* L^p *for any* $t > 0$. *If* $\Delta_p < \delta_\nu$, *then in the cases*

(*a*) $n = 2$,

(*b*) $n \ge 3,\ \nu > \dfrac{n-1}{2(n+1)}$,

the operator P_t^ν *is bounded on* L^p *and*

$$\|P_t^\nu\|_{L^p \to L^p} \le c(1+t)^s \quad \forall s > \frac{n\Delta_p}{\delta_\nu} + \left(1 - \frac{\Delta_p}{\delta_\nu}\right) \max(\nu - 1/2, 0). \quad (1.18)$$

Theorem A implies Theorem 1.2 for $\nu = 1/2$.

Theorem B *Let* $f \in L^p,\ 1 \le p < \infty$.

(i) *If* p *and* ν *satisfy the conditions of Theorem A, under which* P_t^ν *is bounded on* L^p, *then*

$$\lim_{t \to 0} \|P_t^\nu f - f\|_p = 0.$$

(ii) *If* $1/2 - 1/p < \delta_\nu$, *then*

(*a*) $\lim\limits_{t \to 0} \|P_t^\nu f - f\|_{L^p(K)} = 0$ *for each bounded domain* $K \subset \mathbb{R}^n$; (1.19)

(*b*) $(P_t^\nu f)(x)$ *converges to* $f(x)$ *as* $t \to 0$ *almost everywhere on* $\mathbb{R}^n$.

Theorems A and B will be proved in Sections 2 and 3 respectively.

Some comments are in order. The proof of Theorem A is based on known results for Bochner-Riesz means [7, 11, 22, 23] defined by

$$(S_k^\nu f)(x) = F^{-1}[(1 - |\xi|^2/k^2)_+^\nu Ff](x) = (f * b_k^\nu)(x), \quad (1.20)$$

$$b_k^\nu(x) = k^n b^\nu(kx), \qquad b^\nu(x) = 2^{\nu - n/2}\pi^{-n/2}\Gamma(\nu+1)|x|^{-\nu-n/2} J_{\nu+n/2}(|x|).$$

In spite of the fact that the operators P_t^ν and S_k^ν have different geneses (the Dirichlet problem for the singular Helmholtz equation and summation of multiple Fourier series), their properties are close. Note also, that for sufficiently nice real-valued f,

$$(S_k^\nu f)(x) = \lim_{t\to 0} \left(\frac{2}{kt}\right)^{2\nu} \frac{\Gamma(\nu+1)\,\Gamma(\nu)}{\pi} Im\,(P_t^\nu f)(x) \tag{1.21}$$

because $J_{\nu+n/2}(z) = 2^{-1}(H^{(1)}_{\nu+n/2}(z) + \overline{H^{(1)}_{\nu+n/2}(z)})$. The relation (1.21) enables us to interprete the wave number k as a dilation parameter in the corresponding Bochner-Riesz mean. Moreover, (1.21) partly explains why the proof of the a.e. convergence in Theorem B is simple and covers all reasonable p and ν, whereas the proof of the a.e. convergence of the Bochner-Riesz means is rather subtle (cf. [22, Section 2.4] and [23, Chapter IX, Section 6.8]).

Our next remark concerns the well-known gap in the celebrated Stein's conjecture that for all $0 < \nu \leq (n-1)/2$ and $1 < p < \infty$ the operator S_k^ν is bounded on L^p whenever $\Delta_p < \delta_\nu$ (see [23, p. 390]). This gap is inherited by P_t^ν. The restriction $\nu > (n-1)/2(n+1)$ in Theorem A(ii) can be seemingly reduced a bit due to the results by J. Bourgain [1] (see also [28]).

We conclude this introduction by sketching some open problems which might be of certain interest.

1. Let us write $P^\nu_{t,k} f$ and $p^\nu_{t,k}(x)$ for the expressions (1.5) and (1.6) respectively, and replace k by $k + i\varepsilon,\ \varepsilon > 0$. Since $p^\nu_{t,k+i\varepsilon}(x)$ has an exponential decay as $|x| \to \infty$, then $P^\nu_{t,k+i\varepsilon}$ is a bounded operator on L^p for all $p \in [1, \infty]$ and all $\nu > 0$. Moreover, due to (1.8) for t fixed and $f \in L^2$ we have

$$\lim_{\varepsilon\to 0} \|P^\nu_{t,k+i\varepsilon} f - P^\nu_{t,k} f\|_2 = 0 \tag{1.22}$$

(the generalized limiting absorption principle). Let now $f \in L^p$. What can one say about (a) the L^p-analogue of (1.22), (b) the almost everywhere convergence of $(P^\nu_{t,k+i\varepsilon} f)(x)$ as $\varepsilon \to 0$, and (c) the properties of the maximal operator $f \to \sup_{\varepsilon>0} |P^\nu_{t,k+i\varepsilon} f|$?

2. What is the exact behaviour of the norm $\|P_t^\nu\|_{L^p\to L^p}$ as $t \to \infty$?

3. How to prove the factorization formula (1.14) (e.g., for $f \in L^p(\mathbb{R}^n_+)$ with a suitable p) in the case $\alpha \geq 1$? Note that the standard use of the Fubini theorem fails because of oscillation, and the limiting absorption principle also does not work because we know almost nothing about mapping properties of the fractional integrals (1.15) and (1.15′). An answer to this question would enable us to invert the potential (1.12) (in particular, (1.13)) for $\alpha \geq 1$. The corresponding theory for $p=2$ and $0<\alpha<1$ was developed in [21].

2. Proof of Theorem A.

Without loss of generality one can assume $k = 1$.

Step 1 We start with $p = 2$. By using the estimate

$$r^\nu |H_\nu^{(1)}(r)| \leq c\,(1+r)^{\nu-1/2}, \tag{2.1}$$

which follows from (1.9) and the asymptotics of $H_\nu^{(1)}(r)$, and taking into account a boundedness of the function $r^\nu K_\nu(r)$, from (1.8) we obtain

$$(p_t^\nu)^\wedge(\xi) \leq c \begin{cases} (1+t)^{\nu-1/2} & \text{if} \quad |\xi|<1, \quad \nu \geq 1/2, \\ 1 \quad \text{if} \quad |\xi|<1, \quad \nu < 1/2, \\ 1 \quad \text{if} \quad |\xi|>1. \end{cases}$$

By the Parseval equality it follows that

$$\|P_t^\nu\|_{L^2\to L^2} \leq c\,(1+t)^{\max(\nu-1/2,0)}. \tag{2.2}$$

Step 2 Consider the case $\nu > (n-1)/2$. Denote $\tilde{x} = (x,t)$, $|\tilde{x}| = (|x|^2+t^2)^{1/2}$ (this notation will be used throughout the paper). By (2.1) (with ν replaced by $\nu + n/2$) we have

$$|(p_t^\nu)(x)| \leq c t^{2\nu}\,\frac{(1+|\tilde{x}|)^{\nu+(n-1)/2}}{|\tilde{x}|^{2\nu+n}} \leq c_1\,[a_t(x)+b_t(x)], \tag{2.3}$$

$$a_t(x) = \frac{t^{2\nu}}{|\tilde{x}|^{2\nu+n}} \in L^1 \quad \text{for} \quad \nu > 0,$$

$$b_t(x) = \frac{t^{2\nu}}{|\tilde{x}|^{\nu+(n+1)/2}} \in L^1 \quad \text{for} \quad \nu > (n-1)/2.$$

A simple calculation yields

$$\|f * a_t\|_p \le c\|f\|_p \quad \text{and} \quad \|f * b_t\|_p \le c t^{\nu+(n-1)/2}\|f\|_p.$$

Hence

$$\|P_t^\nu\|_{L^p \to L^p} \le c(1+t)^{\nu+(n-1)/2} \qquad \forall p \in [1,\infty]. \tag{2.4}$$

Now (1.17) follows from (2.2) and (2.4) by interpolation.

Step 3 Let us prove (ii). An unboundedness of the operator P_t^ν for $\nu \le (n-1)/2$, $\Delta_p \ge \delta_\nu$, can be proved directly by applying P_t^ν to the characteristic function of a small ball (cf. [10, p. 10] and [7, p. 93]). In order to prove the rest of the statements we proceed as follows. Denote by $\psi_t(x)$ the characteristic function of the interval $[0,t]$, and let $p_t^\nu(x) = u_t(x) + v_t(x)$ where

$$u_t(x) = \psi_t(|x|)\, p_t^\nu(x), \qquad v_t(x) = (1-\psi_t(|x|))\, p_t^\nu(x) \in L^1. \tag{2.5}$$

By (2.3), as above we get

$$\|f * u_t\|_p \le c\|f\|_p(1+t)^{\nu+(n-1)/2}. \tag{2.6}$$

In order to estimate $f * v_t$ we decompose $v_t(x)$ by using the asymptotic expansion

$$p_t^\nu(x) = c_\nu(t)\sqrt{2/\pi}\; e^{-i(2\nu+n+1)\pi/4}|\tilde{x}|^{-\nu-(n+1)/2}e^{i|\tilde{x}|} \times$$

$$\times \sum_{m=0}^{M-1}\Big(\nu+\frac{n}{2}, m\Big)(-2i|\tilde{x}|)^{-m} + c_\nu(t)\, O(|\tilde{x}|^{-M-\nu-(n+1)/2}),$$

$c_\nu(t) = i\pi^{1-n/2}t^{2\nu}2^{-\nu-n/2}/\Gamma(\nu)$ (cf. [9], formula 7.13.1(1)). This gives

$$v_t(x) = \sum_{m=0}^{M-1} c_m k_{t,m}(x) + R_{M,t}(x). \tag{2.7}$$

Here c_m are some coefficients, $c_0 \neq 0$,

$$k_{t,m}(x) = (1-\psi_t(|x|))\,\frac{t^{2\nu}e^{i|\tilde{x}|}}{|\tilde{x}|^{\nu+(n+1)/2+m}}, \quad 0 \leq m \leq M-1; \tag{2.8}$$

$$R_{M,t}(x) \leq c\,t^{2\nu}(1-\psi_t(|x|)) \begin{cases} |\tilde{x}|^{-2\nu-n} \text{ if } |\tilde{x}|<1, \ M \leq \nu+(n+1)/2, \\ \\ |\tilde{x}|^{-M-\nu-(n+1)/2} \text{ if } |\tilde{x}|>1 \end{cases} \tag{2.9}$$

(the estimate for $|\tilde{x}| < 1$ is derived from

$$R_{M,t} = v_t - \sum_{m=0}^{M-1} c_m k_{t,m}, \tag{2.10}$$

using the inequality $r^{\nu+n/2}|H^{(1)}_{\nu+n/2}(r)| \leq c$ for $0 < r < 1$, and comparing the terms in the right-hand side of (2.10)). Choose $M \in \mathbb{N}$ so that

$$-\nu + (n-1)/2 < M < \nu + (n+1)/2 \tag{2.11}$$

(for $\nu > 0$ such a choice is possible). Then the first estimate in (2.9) holds, $R_{M,t} \in L^1$, and we get $\|f * R_{M,t}\|_p \leq c\,A(t)\|f\|_p$. Here for $t \leq 1$,

$$A(t) = \int\limits_{t<|y|<1} \frac{t^{2\nu}\,dy}{(t^2+|y|^2)^{\nu+n/2}} + \int\limits_{|y|>1} \frac{t^{2\nu}\,dy}{(t^2+|y|^2)^{(\nu+(n+1)/2+M)/2}} \leq c(1+t^{2\nu}),$$

and for $t > 1$,

$$A(t) = \int\limits_{|y|>t} \frac{t^{2\nu}\,dy}{(t^2+|y|^2)^{(\nu+(n+1)/2+M)/2}} = c\,t^{\nu+(n-1)/2-M} \leq c\,t^{2\nu}$$

by (2.11). Thus we have

$$\|f * R_{M,t}\|_p \le c\,(1+t)^{2\nu}\|f\|_p \quad \forall p \in [1,\infty]. \tag{2.12}$$

The function (2.8) can be transformed as follows. Denote

$$\lambda = t^2/|x|^2 \ (<1), \qquad \beta = \nu + m + (n+1)/2.$$

Then $k_{t,m}(x) = (1-\psi_t(|x|))\, t^{2\nu}|x|^{-\beta}a(\lambda)$, $\quad a(\lambda) = (1+\lambda)^{-\beta/2}\exp(i|x|\sqrt{1+\lambda})$, and by the Taylor formula,

$$a(\lambda) = e^{i|x|}\sum_{l=0}^{L-1}\sum_{k=0}^{l} c_{k,l}\lambda^l|x|^k + \mathcal{R}_L, \qquad |\mathcal{R}_L| \le \frac{\lambda^L}{L!}\max_{\lambda\in[0,1]}|a^{(L)}(\lambda)|,$$

with some coefficients $c_{k,l}$, $c_{0.0} \ne 0$. After simple calculations we obtain

$$k_{t,m}(x) = \sum_{l=0}^{L-1}\sum_{j=0}^{l} c_{l-j,l}\, k^{l,j}_{t,m}(x) + \mathcal{R}^m_{L,t}(x) \tag{2.13}$$

where

$$k^{l,j}_{t,m}(x) = (1-\psi_t(|x|))\,\frac{t^{2\nu+2l}e^{i|x|}}{|x|^{\beta+l+j}}, \qquad |\mathcal{R}^m_{L,t}(x)| \le c\sum_{k=0}^{L}\mathcal{R}^{k,m}_{L,t}(x), \tag{2.14}$$

$$\mathcal{R}^{k,m}_{L,t}(x) = \frac{t^{2\nu+2L}(1-\psi_t(|x|))}{|x|^{\beta+L+k}} = \frac{t^{2\nu+2L}(1-\psi_t(|x|))}{|x|^{\nu+m+(n+1)/2+L+k}}.$$

Fix $L \in \mathbb{N}$ so that $(n-1)/2 - \nu < L \le (n+1)/2 - \nu$. Then $\mathcal{R}^{k,m}_{L,t} \in L^1$ for all $m \le M-1$ and $k \le L$. Furthermore, $\|f * \mathcal{R}^{k,m}_{L,t}\|_p \le B(t)\|f\|_p$, where

$$B(t) = \int_{|y|>t}\frac{t^{2\nu+2L}\,dy}{|y|^{\beta+L+k}} = ct^{\nu+L-k-m+(n-1)/2} \le c\begin{cases} t^{\nu+(n+1)/2-M} & \text{if} \quad t \le 1, \\ t^{\nu+L+(n-1)/2} & \text{if} \quad t > 1. \end{cases}$$

Owing to the choice of M and L, this yields

$$\|f * \mathcal{R}^{k,m}_{L,t}\|_p \le (1+t)^n\|f\|_p, \qquad 1 \le p \le \infty, \tag{2.15}$$

for all $k = 0, 1, \ldots, L$ and $m = 0, 1, \ldots, M - 1$.

At the last stage we write $k^{l,j}_{t,m}(x)$ (see (2.14)) in the form

$$k^{l,j}_{t,m}(x) = t^{2\nu+2l} k^{l,j}_{1,m}(x) + \omega_t(x), \tag{2.16}$$

where

$$k^{l,j}_{1,m}(x) = k^{l,j}_{t,m}(x)|_{t=1}, \qquad \omega_t(x) = [\psi_1(|x|)] - \psi_t(|x|)] \, \frac{t^{2\nu+2l} e^{i|x|}}{|x|^{\beta+l+j}}.$$

If $t < 1$, then $\|f * \omega_t\|_p \le C(t) \|f\|_p$,

$$C(t) = \int\limits_{t<|y|<1} \frac{t^{2\nu+2l} \, dy}{|y|^{\beta+l+j}} = c \, t^{2\nu+l+n-\beta-j} \int\limits_{1}^{1/t} r^{n-1-\beta-l-j} dr = O(1)$$

by (2.11). If $t > 1$, then $\|f * \omega_t\|_p \le D(t) \|f\|_p$,

$$D(t) = \int\limits_{1<|y|<t} \frac{t^{2\nu+2l} \, dy}{|y|^{\beta+l+j}} = c \, t^{2\nu+l+n-\beta-j} \int\limits_{1/t}^{1} r^{n-1-\beta-l-j} dr \le$$

$$\le c \left(t^{\nu+L-1+(n-1)/2} + t^{2\nu+2L-2} \right) \le c \, t^{n-1}$$

due to the choice of L. Thus we get

$$\|f * \omega_t\|_p \le c \, (1+t)^{n-1} \|f\|_p \quad \forall p \in [1, \infty]. \tag{2.17}$$

It remains to estimate convolutions with the kernel

$$k^{l,j}_{1,m}(x) = (1 - \psi_1(|x|)) \, \frac{e^{i|x|}}{|x|^{\nu+m+(n+1)/2+l+j}}.$$

Convolutions of this type were investigated by different authors in connection with the Bochner-Riesz operator (1.20). In the principal case $m = l = j = 0$ the inequality

$$\|f * k^{0,0}_{l,0}\|_p \le c \, \|f\|_p, \tag{2.18}$$

holds for $\Delta_p = |1/p - 1/2| < \nu/n + 1/2n$ in the cases (a) $n = 2$ and (b) $n \geq 3$, $\nu > (n-1)/2(n+1)$ (see, e.g., [7, 11, 22, 23 (Chapter IX, Section 2.2)]. Combining (2.18) with the similar estimates for $\|f * k_{1,m}^{l,j}\|_p$, in view of (2.16) we get

$$\|f * k_{t,m}^{l,j}\|_p \leq c\,[(1+t)^{n-1} + (1+t)^{2\nu+2L-2}]\,\|f\|_p \leq c\,(1+t)^{n-1}\|f\|_p$$

for all $m \leq M-1$ and $j \leq l \leq L-1$ (we recall that $L \leq (n+1)/2 - \nu$). The last inequality together with (2.6), (2.12) and (2.15), yields

$$\|P_t^\nu\|_{L^p \to L^p} \leq c\,(1+t)^n, \qquad \nu \leq (n-1)/2, \tag{2.19}$$

for $\Delta_p < \delta_\nu = \nu/n + 1/2n$ in the cases (a) $n = 2$ and (b) $n \geq 3$, $\nu > (n-1)/2(n+1)$. The estimate (2.19) implies (1.18) by interpolating with (2.2). ∎

3. Proof of Theorem B.

Let for simplicity $k = 1$. The basic idea is to represent P_t^ν as a sum $P_t^\nu = A_t^\nu + B_t^\nu$ where A_t^ν is a usual approximate identity with an integrable kernel, and B_t^ν tends to 0 in the required sense. We write $p_t^\nu(x)$ (see (1.6)) in the form

$$p_t^\nu(x) = \frac{i\pi^{1-n/2}t^{2\nu}}{2^{\nu+n/2}\Gamma(\nu)|\tilde{x}|^{2\nu+n}}\,h(|\tilde{x}|), \qquad h(r) = r^{\nu+n/2}H^{(1)}_{\nu+n/2}(r),$$

where $|\tilde{x}| = (t^2 + |x|^2)^{1/2}$. By taking into account that $h(0) \equiv \lim\limits_{r\to 0} h(r) = 2^{\nu+n/2}\Gamma(\nu+n/2)/\pi i$ and

$$h(r) = h(0) + \int\limits_0^r h'(\rho)d\rho, \qquad h'(\rho) = \rho^{\nu+n/2}H^{(1)}_{\nu+n/2-1}(\rho),$$

we have $p_t^\nu(x) = a_t^\nu(x) + b_t^\nu(x)$, where $a_t^\nu(x) = t^{-n}a^\nu(x/t)$,

$$a^\nu(x) = \frac{\pi^{-n/2}\Gamma(\nu+n/2)}{\Gamma(\nu)}(1+|x|^2)^{-\nu-n/2}, \qquad \int\limits_{\mathbb{R}^n} a^\nu(x)dx = 1, \tag{3.1}$$

$$b_t^\nu(x) = \frac{i\pi^{1-n/2}t^{2\nu}}{2^{\nu+n/2}\Gamma(\nu)|\tilde{x}|^{2\nu+n}} \int\limits_0^{|\tilde{x}|} h'(\rho)d\rho. \tag{3.2}$$

Denote $A_t^\nu f = f * a_t^\nu, \quad B_t^\nu f = f * b_t^\nu$. If $f \in L^p$, then by (3.1),

$$\|A_t^\nu f\|_p \leq \|f\|_p \quad \text{and} \quad \lim_{t\to 0} A_t^\nu f = f \tag{3.3}$$

in the L^p-norm and a.e. for all $p \in [1,\infty)$. In order to manage $B_t^\nu f$ we write

$$b_t^\nu(x) = \chi(x)b_t^\nu(x) + (1-\chi(x))b_t^\nu(x) = b_{t,1}^\nu(x) + b_{t,2}^\nu(x) \tag{3.4}$$

where $\chi(x) \equiv 1$ for $|x| < 1$ and $\chi(x) \equiv 0$ for $|x| > 1$. It suffices to assume that $t < 1$. Then by (3.2),

$$|b_{t,1}^\nu(x)| \leq \frac{c\,t^{2\nu}\chi(x)}{|\tilde{x}|^{2\nu+n}} \int\limits_0^{|\tilde{x}|} \rho^{\nu+n/2}|H^{(1)}_{\nu+n/2-1}(\rho)|d\rho.$$

Since the integrand is locally bounded for $\rho > 0$, then

$$|b_{t,1}^\nu(x)| \leq \frac{c\,t^{2\nu}\chi(x)}{|\tilde{x}|^{2\nu+n-1}} \leq \frac{c\,t^{\varepsilon}\chi(x)}{|x|^{n-1+\varepsilon}} \in L^1, \qquad 0 < \varepsilon < \min(2\nu, 1),$$

and therefore for $f \in L^p,\ 1 \leq p < \infty$,

$$(f * b_{t,1}^\nu)(x) \to 0 \quad \text{as} \quad t \to 0 \quad \text{in the } L^p\text{-norm and a.e.} \tag{3.5}$$

The second term in (3.4) can be represented in the form $b_{t,2}^\nu = b_{t,2}^{\nu,1} + b_{t,2}^{\nu,2}$ by splitting the integral $\int_0^{|\tilde{x}|}$ in (3.2) into $\int_0^1$ and $\int_1^{|\tilde{x}|}$. We have $|b_{t,2}^{\nu,1}(x)| \leq ct^{2\nu}(1-\chi(x))/|x|^{2\nu+n} \in L^1$, and therefore for $f \in L^p,\ 1 \leq p < \infty$,

$$(f * b_{t,2}^{\nu,1})(x) \to 0 \quad \text{as} \quad t \to 0 \quad \text{in the } L^p\text{-norm and a.e.} \tag{3.6}$$

For $b^{\nu,2}_{t,2}$, by putting $\int_1^{|\tilde{x}|} h'(\rho)d\rho = h(|\tilde{x}|) - h(1)$ we obtain

$$|b^{\nu,2}_{t,2}(x)| \le \frac{c\,t^{2\nu}(1-\chi(x))}{|\tilde{x}|^{\nu+(n+1)/2}} \le \frac{c\,t^{2\nu}(1-\chi(x))}{|x|^{\nu+(n+1)/2}}.$$

If $\nu > (n-1)/2$, the last expression is integrable on $\mathbb{R}^n$ and we get

$$(f * b^{\nu,2}_{t,2})(x) \to 0 \quad \text{as} \quad t \to 0 \quad \text{in the } L^p\text{-norm and a.e.} \tag{3.7}$$

If $\nu \le (n-1)/2$ (this case is the most important !), then for any α such that

$$(n-1)/2 - \nu < \alpha < (n-1)/2 + \nu, \tag{3.8}$$

we have

$$|b^{\nu,2}_{t,2}(x)| \le \frac{c\,t^{2\nu}(1-\chi(x))}{|\tilde{x}|^{n-\alpha}|\tilde{x}|^{\alpha+\nu-(n-1)/2}} \le \frac{c\,t^{\gamma}(1-\chi(x))}{|x|^{n-\alpha}} = c\,t^{\gamma}k_\alpha(x)$$

where $\gamma = \nu - \alpha + (n-1)/2 > 0,\ k_\alpha(x) = |x|^{\alpha-n}(1-\chi(x))$. If $f \in L^p,\ 1 \le p < n/\alpha$, then $(f * k_\alpha)(x)$ is well-defined for almost all $x \in \mathbb{R}^n$ and, by the Young inequality, $\|f * k_\alpha\|_q \le c\,\|f\|_p,\ q = np/(n-\alpha p)$. In order to satisfy the condition $p < n/\alpha$, by (3.8) we have to assume

$$(n-1)/2 - \nu < \alpha < \min\{n/p;\ (n-1)/2 + \nu\}. \tag{3.9}$$

The latter is possible provided $(n-1)/2 - \nu < n/p$, i.e. $\delta_\nu \equiv \nu/n + 1/2n > 1/2 - 1/p$. The last condition coincides with that in Theorem B(ii). Thus in the case $\nu \le (n-1)/2$ for $\delta_\nu > 1/2 - 1/p$ we obtain

$$|(f * b^{\nu,2}_{t,2})(x)| \le c\,t^{\gamma}(|f| * k_\alpha)(x) \to 0 \quad \text{a.e. as} \quad t \to 0,$$

and $\lim\limits_{t\to 0} \|f * b^{\nu,2}_{t,2}\|_q = 0$ for $q = np/(n-\alpha p)$ and any α satisfying (3.9). This assertion together with (3.3), (3.5), (3.6) and (3.7) implies the statements (ii)(a) and (ii)(b) and also the statement (i) for $\nu > (n-1)/2$.

The validity of (i) for $\nu \le (n-1)/2$ follows by interpolation. Indeed, since by the Theorem A the estimate $\|P_t^\nu f\|_p \le c\|f\|_p$ is uniform in $t \in (0, t_0]$ for all t_0, it suffices to prove the equality $\|P_t^\nu f - f\|_p = 0$ for f belonging to the Schwartz class $\mathcal{S}$ which is dense in L^p. Given an arbitrarily small $\varepsilon > 0$, let $p_1 = (1/2 + \delta_\nu - \varepsilon)^{-1}$ if $p < 2$, and $p_1 = (1/2 - \delta_\nu + \varepsilon)^{-1}$ if $p > 2$. Assuming $t \le t_0$ and $f \in \mathcal{S}$, for a suitable $\theta \in (0,1)$ we get

$$\|P_t^\nu f - f\|_p \le \|P_t^\nu f - f\|_2^\theta \|P_t^\nu f - f\|_{p_1}^{1-\theta} \le c(t_0)\|f\|_{p_1}^{1-\theta}\, \|P_t^\nu f - f\|_2^\theta.$$

By the Parseval equality (use (1.8)) this expression tends to 0 as $t \to 0$. ∎

Acknowledgements

I wish to thank Elias M. Stein and Walter Trebels for helpful inspiring discussions. I am also grateful to David Colton for useful communication. The work was partially sponsored by the Edmund Landau Center for research in Mathematical Analysis, supported by the Minerva Foundation (Germany).

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