STRONGLY ALMOST DISJOINT SETS AND WEAKLY UNIFORM BASES

Z. T. BALOGH, S. W. DAVIS, W. JUST , S. SHELAH, AND P. J. SZEPTYCKI

ABSTRACT. A combinatorial principle CECA is formulated and its equivalence with GCH + certain weakenings of $\Box_\lambda$ for singular λ is proved. CECA is used to show that certain "almost point-$< \tau$" families can be refined to point-$< \tau$ families by removing a small set from each member of the family. This theorem in turn is used to show the consistency of "every first countable T_1-space with a weakly uniform base has a point-countable base."

Preprint No. 12
1997/98

This research was originally inspired by the following question of Heath and Lindgren [4]: Does every first countable Hausdorff space X with a weakly uniform base have a point-countable base? The answer to this question is negative if MA + $2^{\aleph_0} > \aleph_2$ is assumed (see [2]). On the other hand, if CH holds and the space has at most $\aleph_\omega$ isolated points, then the answer is positive (see [1]). The starting point of this paper was the observation that if in addition to GCH also the combinatorial principle $\Box_\lambda$ holds for every singular cardinal λ, then no bound on the number of isolated points is needed. An analysis of the proof lead to the formulation of a combinatorial principle CECA. It turns out that CECA is equivalent to GCH + some previously known weakenings of $\Box_\lambda$, but CECA has a different flavor than $\Box_\lambda$-principles and may be easier to work with. The equivalence will be shown in Section 1.

1991 *Mathematics Subject Classification.* 03E05, 03E35, 03E75, 54D70.

Key words and phrases. GCH, $\Box$, strongly almost disjoint families, weakly uniform base, point countable base.

The first author's research was partially supported by NSF grant DMS-9623391. The third author's research was done during visits at Rutgers University and The Hebrew University, Jerusalem which were supported by NSF grant DMS-9704477 and the Edmund Landau Center for Research in Mathematical Analysis and Related Areas, sponsored by the Minerva Foundation (German). The fourth author was partially supported by the Israel Basic Research Fund. This is publication number 674 in Shelah's publication list.

In [3] it is shown that under certain conditions almost disjoint families $\{A_\alpha : \alpha < \kappa\}$ can be refined to disjoint families by removing small sets A'_α from each A_α. Let us say that a family $\{A_\alpha : \alpha < \kappa\}$ is *point-$<\tau$* if for every $I \in [\kappa]^\tau$ the intersection $\bigcap_{\alpha \in I} A_\alpha$ is empty. In particular, a family is disjoint iff it is point-<2. In Section 2 the main theorem of this paper (Theorem 2) is derived from CECA. Roughly speaking, Theorem 2 asserts that certain families $\{A_\alpha : \alpha < \kappa\}$ that are "almost point-$<\tau$" can be refined to point-$<\tau$ families by removing small sets A'_α from each A_α.

In Section 3, some related results for almost disjoint families are proved, and it is explored how much of Theorem 2 can be derived from GCH alone rather than from CECA.

In Section 4 we show that the positive answer to the question of [2], and more, follows already from Theorem 2.

1. The Closed Continuous $\in$-chain Axiom

Definition 1. Let τ be a regular cardinal. A set M is *τ-closed* if $[M]^{<\tau} \subset M$. We say that a set M is *weakly τ-closed* if for every $I \in [M]^\tau$ there exists $J \in [I]^\tau$ such that $[J]^{<\tau} \subset M$. We say that M is *weakly closed* if M is weakly τ-closed for all regular τ (equivalently: for all regular $\tau \leq |M|$). An *$\in$-chain* is a sequence $\langle M_\xi : \xi < \alpha\rangle$ such that $M_\xi \subset M_\eta$ and $\langle M_\xi : \xi \leq \eta\rangle \in M_{\eta+1}$ for all $\xi < \eta < \alpha$. The *Closed Continuous $\in$-chain Axiom for λ and τ* (abbreviated $\mathrm{CECA}_\tau(\lambda)$) is the following statement:

For every cardinal $\Theta > \lambda$, and for every pair of sets A, B with $|A| = \lambda$, $|B| < \lambda$, there exists a continuous $\in$-chain $\langle M_\xi\rangle_{\xi < cf(\lambda)}$ of weakly τ-closed elementary submodels of $H(\Theta)$ such that
– $|M_\xi| < \lambda$ *for every* $\xi < cf(\lambda)$;
– $B \in M_0$; *and*
– $A \subset \bigcup_{\xi < cf(\lambda)} M_\xi$.

The *Closed Continuous $\in$-chain Axiom for λ* (abbreviated $\mathrm{CECA}(\lambda)$) is obtained by requiring that the models M_ξ in the definition of $\mathrm{CECA}_\tau(\lambda)$ are weakly closed. The *Closed Continuous $\in$-chain Axiom* (abbreviated CECA) asserts that $\mathrm{CECA}(\lambda)$ holds for all uncountable cardinals λ.

Lemma 1. *Let λ be a regular uncountable cardinal. The following are equivalent:*
(a) $\lambda^{<\lambda} = \lambda$;
(b) *For every set A with $|A| \leq \lambda$ there exists a weakly closed M such that $A \subset M$ and $|M| = \lambda$;*
(c) *There exists a weakly λ-closed M such that $|M| = \lambda$.*

Proof: (a) $\Rightarrow$ (b)

Suppose $\lambda^{<\lambda} = \lambda$, and $|A| = \lambda$. Build recursively an increasing sequence $\langle M_\xi : \xi < \lambda\rangle$ such that $A \subset M_0$, $[M_\xi]^{<\lambda} \subset M_{\xi+1}$, and $|M_\xi| = \lambda$ for all $\xi < \lambda$. Let $M = \bigcup_{\xi<\lambda} M_\xi$. If $\tau \leq \lambda$ is regular, $I \in [M_\xi]^\tau$ and $K \in [I]^{<\tau}$, then $K \in [M_\xi]^{<\tau} \subseteq [M_\xi]^{<\lambda}$ for some $\xi < \lambda$, and hence $K \in M_{\xi+1} \subseteq M$, as required.

The implications (b) $\Rightarrow$ (c) and (c) $\Rightarrow$ (a) are obvious. □

Corollary 1. CECA $\Rightarrow$ GCH.

Lemma 2. *Assume GCH. If λ is an uncountable limit cardinal or if $\lambda = \kappa^+$ for some regular infinite cardinal κ, then CECA(λ) holds.*

Proof: Let λ be as above, let $\Theta > \lambda$, and let A, B be such that $|A| = \lambda$, $|B| < \lambda$. In both of these cases we can construct a continuous $\in$-chain $\langle M_\xi \rangle_{\xi < cf(\lambda)}$ of elementary submodels of $H(\Theta)$ such that

- $|M_\xi| < \lambda$ for every $\xi < cf(\lambda)$;
- $|M_{\xi+1}|$ is regular and $[M_{\xi+1}]^{<|M_{\xi+1}|} \subset M_{\xi+1}$ for every $\xi < cf(\lambda)$;
- $B \in M_0$; and
- $A \subset \bigcup_{\xi < cf(\lambda)} M_\xi$.

It remains to show that if $\delta < cf(\lambda)$ is limit, then M_δ is weakly closed. So assume $\delta < cf(\lambda)$ is limit, let τ be a regular cardinal, and let $I \in [M_\delta]^\tau$.

If $cf(\delta) < \tau$, then there is $\xi < \tau$ with $|M_\xi \cap I| = \tau$. Fix such ξ, and let $J = I \cap M_{\xi+1}$. Then $[J]^{<\tau} \subset [M_{\xi+1}]^{<\tau} \subseteq [M_{\xi+1}]^{<|M_{\xi+1}|} \subset M_{\xi+1} \subset M_\delta$.

If $cf(\delta) \geq \tau$, then $[I]^{<\tau} \subset \bigcup_{\xi<\delta} M_\xi = M_\delta$, and we can take $J = I$. □

Thus GCH implies that CECA(κ) holds for all κ, except perhaps if $\kappa = \lambda^+$ for a singular strong limit cardinal λ. Fortunately, it turns out that in this case CECA(κ) is equivalent to a weakening of $\square_\lambda$ that has been extensively studied by the fourth author. To prove the equivalence, let us introduce some notation.

We say that $Pr(\lambda, \tau)$ holds if there exists an increasing continuous chain $\bar{N} = \langle N_i : i < \lambda^+ \rangle$ such that $|N_i| = \lambda$ for each $i < \lambda^+$, each N_i is weakly τ-closed, and $\lambda^+ \subset \bigcup_{i<\lambda^+} N_i$.

We say that $Pr'(\lambda, \tau)$ holds if there exists an increasing continuous chain $\bar{N} = \langle N_i : i < \lambda^+ \rangle$ such that $|N_i| = \lambda$ for each $i < \lambda^+$, each N_i is weakly τ-closed with respect to sets of ordinals, and $\lambda^+ \subset \bigcup_{i<\lambda^+} N_i$.

Note that if $2^\lambda = \lambda^+$, then in $Pr(\lambda, \tau)$ and $Pr'(\lambda, \tau)$ we can demand that $\bigcup_{i<\lambda^+} N_i = H(\lambda^+)$.

In Definition 1.4 of [3], the following principle $Sp(\sigma, \lambda)$ was introduced:[1] *There exists a sequence $\langle \mathcal{P}_\xi : \xi < \lambda^+ \rangle$ such that for all $\xi < \lambda^+$ we have $\mathcal{P}_\xi \subset [\xi]^\sigma$ and $|\mathcal{P}_\xi| \leq \lambda$; moreover if $\xi < \lambda^+$ with $\sigma^+ = cf(\xi)$ and x is a cofinal subset of ξ of cardinality σ^+, then x can be written in the form $x = \bigcup\{x_\nu : \nu \in \sigma\}$ where for each $\nu \in \sigma$ we have $[x_\nu]^\sigma \subset \bigcup_{\eta<\xi} \mathcal{P}_\eta$.*

The ideal $I[\kappa]$ was defined in [5] and [6]. We use here two equivalent definitions of $I[\kappa]$ given in [7].

Definition 2. For a regular uncountable cardinal κ, let $I[\kappa]$ be the family of all sets $A \subseteq \kappa$ such that the set $\{\delta \in A : \delta = cf(\delta)\}$ is not stationary in κ and for some $\langle \mathcal{P}_\alpha : \alpha < \kappa \rangle$ we have:
(a) $\mathcal{P}_\alpha$ is a family of $< \kappa$ subsets of α;
(b) for every limit $\alpha \in A$ such that $cf(\alpha) < \alpha$ there is $x \subset \alpha$ such that
$otp(x) < \alpha = \sup x$ and $\forall \beta < \alpha\, (x \cap \beta \in \bigcup_{\gamma<\alpha} \mathcal{P}_\gamma)$.

The following characterization of $I[\kappa]$ appears as Claim 1.2 in [7]. The abbreviation $nacc(C)$ stands for "nonaccumulation points of C" (in the order topology).

Lemma 3. *Let κ be a regular uncountable cardinal. Then $D \in I[\kappa]$ iff there exists a sequence $\langle C_\beta : \beta < \kappa \rangle$ such that:*

[1] Actually, the principle introduced in [3] is more general and contains an extra parameter τ, but the case $\tau = \sigma^+$ is most relevant for the results in [3] and fits most neatly into the framework of the present paper.

(a) *C_β is a closed subset of β;*
(b) *if $\alpha \in nacc(C_\beta)$ then $C_\alpha = C_\beta \cap \alpha$;*
(c) *for some club $E \subseteq \kappa$, for every $\delta \in D \cap E$:*
$cf(\delta) < \delta$ and $\delta = \sup C_\delta$ and $otp(C_\delta) = cf(\delta)$;
(d) *$nacc(C_\delta)$ is a set of successor ordinals.*

It is clear from the above lemma that Jensen's principle $\Box_\lambda$ implies that $\lambda^+ \in I[\lambda^+]$.

For $\tau \leq \lambda$, let $S^{\lambda^+}_\tau = \{\alpha < \lambda^+ : cf(\alpha) = \tau\}$.

Theorem 1. *Let $\tau < \lambda$ be infinite cardinals with τ regular and λ singular strong limit. The following are equivalent:*
(a) $S^{\lambda^+}_\tau \in I[\lambda^+]$;
(b) $Pr(\lambda, \tau)$;
(c) $Pr'(\lambda, \tau)$;
(d) $\mathrm{CECA}_\tau(\lambda^+)$.
Moreover, if $\tau = \sigma^+$, then each of the above is also equivalent to:
(e) $Sp(\sigma, \lambda)$.

Proof: The implications from (b) to (c) and from (d) to (b) are obvious.

(c) $\Rightarrow$ (d)

Let $\theta > \lambda^+$, and let $\bar{N} = \langle N_i : i < \lambda^+ \rangle$ exemplify $Pr'(\lambda, \tau)$. Let $\bar{M} = \langle M_\xi : \xi < \lambda^+ \rangle$ be an increasing continuous sequence such that $\bar{N} \in M_0$, and for all $\xi < \lambda^+$ we have:
(1) $M_\xi \prec \langle H(\Theta), <^* \rangle$ (where $<^*$ is some wellorder relation on $H(\Theta)$);
(2) $|M_\xi| = \lambda$; and
(3) $\bar{M} | (\xi + 1) \in M_{\xi+1}$.

Let $M = \bigcup_{\xi < \lambda^+} M_\xi$. Note that (3) implies in particular that $\xi \in M_{\xi+1}$, and hence $\lambda^+ \subset M$. Fix a bijection $h : M \to \lambda^+$ such that $h|M_\xi \in M_{\xi+1}$ for all ξ. Such h can be found by conditions (1) and (3): Recursively, let $h|M_{\xi+1}$ be the $<^*$-smallest bijection from $M_{\xi+1}$ onto an ordinal that extends $h|M_\xi$. For every $\xi < \lambda^+$, let $\eta(\xi)$ be the smallest ordinal $\eta \geq \xi$ such that $h[M_\eta] = \eta = M_\eta \cap \lambda^+ = N_\eta \cap \lambda^+$. By (2), $\eta(\xi) < \lambda^+$ for all ξ. Define:

$$E = \{\delta \in \mathbf{LIM} \cap \lambda^+ : \forall \xi < \delta \, (\eta(\xi) < \delta)\}.$$

Then E is a closed unbounded subset of λ^+ that consists of fixed points of the function η. Let $\{\delta_\varepsilon : \varepsilon < \lambda^+\}$ be the increasing continuous enumeration of E.

We show that $\langle M_{\delta_\varepsilon} : \varepsilon < \lambda^+ \rangle$ witnesses $\mathrm{CECA}_\tau(\lambda^+)$. Let $\varepsilon < \lambda$ and let $I \in [M_{\delta_\varepsilon}]^\tau$. Note that we have $h[I] \in [\delta_\varepsilon]^\tau$ and $M_{\delta_\varepsilon} \cap \lambda^+ = N_{\delta_\varepsilon} \cap \lambda^+$. So there is $J \subseteq h[I]$ with $|J| = \tau$ and $[J]^{<\tau} \subseteq N_{\delta_\varepsilon} \cap \mathbf{ON}$. We distinguish two cases:

Case 1: There exists $J' \in [J]^\tau$ with $\sup J' < \delta_\varepsilon$.

Let $\alpha = \eta(\sup J')$. Then $\alpha < \delta_\varepsilon$, $M_\alpha \cap \lambda = \alpha = h[M_\alpha]$, and $M_\alpha \cap \lambda^+ = N_\alpha \cap \lambda^+$. Thus, by shrinking J' if necessary, we may assume that $[J']^{<\tau} \subset N_\alpha$. Let $I' = h^{-1}J'$. Then $I' \in [I]^\tau$. Moreover, if $K \in [I']^{<\tau}$, then $K = h^{-1}L$ for some $L \in [J']^{<\tau}$. Since $L, h|M_\alpha \in M_{\alpha+1}$ and $L \subset M_{\alpha+1}$, it follows that $K \in M_{\alpha+1} \subset M_{\delta_\varepsilon}$, as desired.

Case 2: There is no J' as in Case 1.

Then let $I' = h^{-1}J$. Note that we must have $otp(J) = \tau$ and $\sup J = \delta_\varepsilon$. Thus, if $K \in [I']^{<\tau}$, then $K = h^{-1}L$, where $L \in N_\alpha$ for some $\alpha < \delta_\varepsilon$ with $N_\alpha = M_\alpha \cap H(\lambda^+)$. Thus, arguing as in the previous case, one can show that $K \in M_{\alpha+1}$ and hence $[I']^{<\tau} \subset M_{\delta_\varepsilon}$.

(c) $\Rightarrow$ (a)

Let $\bar{N} = \langle N_\alpha : \alpha < \lambda^+ \rangle$ be a sequence that witnesses $Pr'(\lambda, \tau)$. By thinning out the chain if necessary, we may assume that $\alpha \subset N_\alpha$ for each $\alpha < \lambda^+$. For each α, let $\mathcal{P}_\alpha = N_\alpha \cap \mathcal{P}(\alpha)$. We claim that the sequence $\langle \mathcal{P}_\alpha : \alpha < \lambda^+ \rangle$ witnesses that $S^{\lambda^+}_\tau \in I[\lambda^+]$. Condition (a) of Definition 2 is obvious. To verify that (b) also holds, let $\alpha \in S^{\lambda^+}_\tau \setminus \{\tau\}$. Pick a subset I of α of order type τ such that $\sup I = \alpha$. Then $I \in [N_\alpha \cap \mathbf{On}]^\tau$, and there exists $J \in [I]^\tau$ such that $[J]^{<\tau} \subset N_\alpha$. Since α is a limit ordinal and the sequence $\bar{N}$ is continuous, the latter implies that $[J]^{<\tau} \subset \bigcup_{\beta<\alpha} \mathcal{P}_\beta$. On the other hand, regularity of τ implies that $otp(J) = \tau < \alpha$ and for every $\beta < \alpha$ the set $J \cap \beta$ has cardinality $< \tau$. Thus condition (b) holds and we have shown that $S^{\lambda^+}_\tau \in I[\lambda^+]$.

(a) $\Rightarrow$ (d)

We will actually prove something slightly more general. Assume $S^{\lambda^+}_\tau \subset D \in I[\lambda^+]$. Let $\langle C_\alpha : \alpha < \lambda \rangle$ and E be witnesses that $D \in I[\lambda^+]$ as in Lemma 3. Let $\mu = cf(\lambda)$, and let $(\kappa_i)_{i<\mu}$ be a sequence of cardinals with supremum λ and such that $\max\{\mu, \tau\} < \kappa_0$ and $2^{\kappa_i} \leq \kappa_{i+1}$ for all $i < \mu$. Let A, B be as in the assumptions of $\mathrm{CECA}_\tau(\lambda^+)$, and let $\Theta > \lambda^+$. Recursively construct a double sequence $\langle M^i_\alpha : \alpha < \lambda^+, i < \mu \rangle$ such that conditions 1–8 below are satisfied. For $\alpha < \lambda^+$, let $M_\alpha = \bigcup_{i<\mu} M^i_\alpha$.

1. $M_\alpha \prec H(\Theta)$;
2. $B \subset M_0$;
3. $A \subset \bigcup_{\alpha<\lambda^+} M_\alpha$;
4. the sequence $\langle M_\alpha : \alpha < \lambda^+ \rangle$ is continuous increasing;
5. $\langle M_\beta : \beta \leq \alpha \rangle \in M_{\alpha+1}$;
6. $|M^i_\alpha| = \kappa_i$ and $M^i_\alpha \subset M^j_\alpha$ for $i < j$;
7. $\mathcal{P}(M^i_{\alpha+1}) \subset M^{i+1}_{\alpha+1}$;
8. if $|C_\alpha| < \lambda$, then $\mathcal{P}(\bigcup_{\beta\in C_\alpha} M^i_\beta) \subset M_{\alpha+1}$.

The construction is straightforward. Note that condition 8 can be satisfied since $|\bigcup_{\beta\in C_\alpha} M^i_\beta| = |C_\alpha| \cdot \kappa_i$, and $|C_\alpha| < \lambda$ for the relevant α's.

Now let $\langle \alpha_\xi : \xi < \lambda^+ \rangle$ be the continuous increasing enumeration of E. We show that the sequence $\langle M_{\alpha_\xi} : \xi < \lambda^+ \rangle$ witnesses $\mathrm{CECA}_\tau(\lambda^+)$. For this it suffices to verify that each M_{α_ξ} is weakly τ-closed; the remaining requirements of $\mathrm{CECA}_\tau(\lambda^+)$ are already covered by conditions 1–6. If $\alpha_\xi = \beta + 1$ for some β, then M_{α_ξ} is τ-closed by condition 7. So consider the case when α_ξ is a limit ordinal, and let $I \in [M_{\alpha_\xi}]^\tau$. We distinguish four cases:

Case 1: $cf(\alpha_\xi) = \tau \neq \mu$.

Then there exist $J \in [I]^\tau$ and $i < \mu$ such that $J \subset \bigcup_{\beta \in C_{\alpha_\xi}} M^i_\beta$. We will show that $[J]^{<\tau} \subset M_{\alpha_\xi}$. Let $K \in [I]^{<\tau}$. Then $K \subset \bigcup_{\beta \in C_{\alpha_\xi} \cap \gamma} M^i_\beta$ for some $\gamma \in C_{\alpha_\xi}$. Without loss of generality, we may assume that $\gamma \in naccC_{\alpha_\xi}$. Then $C_\gamma = C_{\alpha_\xi} \cap \gamma$. Since $|C_{\alpha_\xi}| = \tau < \lambda$, also $|C_\gamma| < \lambda$. By condition 8, $K \in M_{\gamma+1}$, and hence $K \in M_{\alpha_\xi}$, as required.

Case 2: $cf(\alpha_\xi) = \tau = \mu$.
We will show that $[I]^{<\tau} \subset M_{\alpha_\xi}$. If $K \in [I]^{<\tau}$, then $K \subset M^i_{\beta+1}$ for some $\beta < \alpha_\xi$. By condition 6, $K \in M^{i+1}_{\beta+1} \subset M_{\alpha_\xi}$.

Case 3: $cf(\alpha_\xi) < \tau$.
Then there exists $\beta < \alpha_\xi$ such that $|I \cap M_{\beta+1}| = \tau$. Since $M_{\beta+1}$ is weakly τ-closed and contained in M_{α_ξ}, there exists $J \in [I \cap M_{\beta+1}]^\tau$ such that $[J]^{<\tau} \subset M_{\alpha_\xi}$.

Case 4: $cf(\alpha_\xi) > \tau$.
Let η be the smallest limit ordinal such that $|I \cap M_{\alpha_\eta}| = \tau$. Then $\eta < \xi$ and $cf(\eta) \leq \tau$. If $cf(\eta) = \tau$, then we are back to Case 1 or Case 2. If $cf(\eta) < \tau$, then we are back to Case 3.

(a) $\Rightarrow$ (e)

Assume that $\tau = \sigma^+$, and let $\langle M^i_\alpha : \alpha < \lambda^+, i < \mu \rangle$ be as in the previous part of the proof, i.e., such that conditions 1–8 hold, and let the sequence $\langle \alpha_\xi : \xi < \lambda^+ \rangle$ be defined as above. For each $\xi < \lambda^+$, let $\mathcal{P}_\xi = [\xi]^\sigma \cap M_{\xi+1}$.
Now suppose that $cf(\xi) = \tau$, and let $x \subset \xi$ be cofinal of cardinality τ. We distinguish two cases:

Case 1: $\sigma < \mu$.
Then we can let $x_\nu = x$ for all $\nu < \sigma$. Each $y \in [x]^\sigma$ is contained in $M^i_{\alpha_\eta+1}$ for some $i < \mu$ and $\eta < \xi$. Thus $[x]^\sigma \subset \bigcup_{\eta<\xi} \mathcal{P}_\eta$, as required.

Case 2: $\sigma \geq \mu$.
Then let $x_\nu = x \cap \bigcup_{\beta \in C_{\alpha_\xi}} M^\nu_\beta$ for $\nu < \mu$ and $x_\nu = \emptyset$ for $\nu \geq \mu$. If $y \in [x_\nu]^\sigma$, then $y \subset \gamma \cap \bigcup_{\gamma \cap C_{\alpha_\xi}}$ for some $\gamma \in naccC_{\alpha_\xi}$, and hence $y \in \mathcal{P}_\eta$ for some $\eta < \xi$ with $\gamma < \alpha_\eta$.

(e) $\Rightarrow$ (a)

Assume $\tau = \sigma^+$, and let $\langle \mathcal{P}_\xi : \xi < \lambda^+ \rangle$ witness that $Sp(\sigma, \lambda)$ holds. Let $\xi \in S^{\lambda^+}_\tau \setminus \{\tau\}$. Now if $x \subset \xi$ is any cofinal subset of ξ of order type τ, then $x = \bigcup_{\nu \in \sigma} x_\nu$, where $[x_\nu]^\sigma \subset \bigcup_{\eta<\xi} \mathcal{P}_\eta$ for each $\nu < \sigma$. At least one of these x_ν's must be cofinal in ξ and of order type τ, and this x_ν is exactly as required in Definition 2. □

Note that $\forall \tau \in \mathbf{Reg}$ $\mathrm{CECA}_\tau(\kappa)$ does not always imply $\mathrm{CECA}(\kappa)$. For example, if $2^{\aleph_n} = \aleph_{n+2}$ for all $n \in \omega$, then there are no weakly closed models of cardinality $\aleph_n$ for $n \in (0, \omega)$, and thus $\mathrm{CECA}(\aleph_\omega)$ fails. But if $\tau = \aleph_n$, then for each $m \geq n+1$

there are plenty of $\aleph_n$-closed models of cardinality $\aleph_m$, and thus $\mathrm{CECA}_{\aleph_n}(\aleph_\omega)$ holds. This anomaly cannot happen if κ is a successor cardinal: If $\kappa = \lambda^+$ for a regular λ, then $\mathrm{CECA}_\lambda(\kappa)$ implies the existence of weakly λ-closed models of cardinality λ, which in turn implies that $\lambda^{<\lambda}$. Now the proof of Lemma 2 can be adapted to derive CECA(κ). If κ is the successor of a singular limit cardinal λ and $\tau < \lambda$, then $\mathrm{CECA}_{\tau^+}(\kappa)$ implies that $2^\tau \leq \lambda$. Since for cofinally many τ we also will have $\tau > cf(\lambda)$, König's Theorem implies that 2^τ is strictly less than λ. In other words, if $\mathrm{CECA}_\tau(\lambda^+)$ holds and λ is singular, then λ must be a strong limit cardinal. For such λ we have the following corollary to the proof of Theorem 1.

Corollary 2. *Let λ be a singular strong limit cardinal. Then the following are equivalent:*
(a) $\lambda^+ \in I[\lambda^+]$*;*
(b) $S^{\lambda^+}_\tau \in I[\lambda^+]$ *for all regular $\tau < \lambda$;*
(c) $\mathrm{CECA}(\lambda^+)$*;*
(d) $\mathrm{CECA}_\tau(\lambda^+)$ *holds for all regular $\tau \in \lambda$.*

Proof: The equivalence between (a) and (b) follows from Shelah's observation that $I[\lambda^+]$ is a normal, and hence λ^+-complete ideal (see [5], [6], or [7]). The equivalence between (b) and (d) was established in Theorem 2. The implication (c) $\Rightarrow$ (d) is obvious. To see that (a) implies (c), note that the last part of the proof of Theorem 2 shows that if $\lambda^+ \in I[\lambda^+]$, then the sequence $\langle M_{\alpha_\xi} : \xi < \lambda^+\rangle$ constructed from $D = \lambda^+$ witnesses $\mathrm{CECA}_\tau(\lambda^+)$ simultaneously for all $\tau \in \lambda \cap \mathbf{Reg}$. □

2. The main theorem

Let τ be a cardinal. Recall that a sequence of sets $\langle A_\alpha\rangle_{\alpha<\kappa}$ is *point* $< \tau$ if for every $I \in [\kappa]^\tau$ the intersection $\bigcap_{\alpha\in I} A_\alpha$ is empty.

Theorem 2. *Assume CECA. Suppose that σ, τ are regular infinite cardinals, and let $\langle A_\alpha\rangle_{\alpha<\kappa}$ be a sequence of (not necessarily distinct) sets such that one of the following conditions is satisfied:*
(1.1) *For every $I \in [\kappa]^\tau$ there is $J \in [I]^{<\tau}$ such that $|\bigcap_{\alpha\in J} A_\alpha| < \sigma$.*
(1.2) *For every $I \in [\kappa]^\tau \exists J \in [I]^{<\tau}\, |\bigcap_{\alpha\in J} A_\alpha| \leq \sigma$ and*
$\forall S \in [\bigcup_{\alpha<\kappa} A_\alpha]^\sigma\, (|\{\alpha : S \subset A_\alpha\}| \leq \sigma)$.
Then there exist $\langle A'_\alpha\rangle_{\alpha<\kappa}$ such that $|A'_\alpha| \leq \sigma$ for each $\alpha < \kappa$ and the sequence $\langle A_\alpha \setminus A'_\alpha\rangle_{\alpha<\kappa}$ is point-$<\tau$.

Proof: Suppose the theorem is false, let κ be the smallest cardinal for which the theorem fails, fix a counterexample $\bar{A} = \langle A_\alpha\rangle_{\alpha<\kappa}$ and σ, τ that witness this fact. Throughout the proof, let Θ denote a "sufficiently large" cardinal, and let $A = \bigcup_{\alpha<\kappa} A_\alpha$. In the proof we will consider all possible ways in which κ, σ, τ can be related to each other, and we will derive a contradiction in each case. To begin with, note that we may without loss of generality assume that $\tau \leq \kappa$; otherwise the conclusion of the theorem is vacuously true. Now let us eliminate the case $\sigma \geq \kappa$.

Lemma 4. *Suppose that $M \prec H(\Theta)$ is weakly τ-closed with respect to sets of ordinals, i.e., for every $I \in [M \cap \kappa]^\tau$ there exists $J \in [I]^\tau$ with $[J]^{<\tau} \subset M$. Moreover, suppose $\bar{A} \in M$ and $\sigma \subset M$. Then the sequence $\langle A_\alpha \setminus M\rangle_{\alpha\in\kappa\cap M}$ is point-$<\tau$.*

Corollary 3. *$\sigma < \kappa$.*

Proof: Let $M \prec H(\Theta)$ be weakly closed and such that $\sigma \subset M$, $|M| = \sigma$, and M contains everything relevant. For each $\alpha < \kappa$, let $A'_\alpha = M$. If $\sigma \geq \kappa$, then $\kappa \cap M = \kappa$, and Lemma 4 implies the conclusion of Theorem 2. □

Proof of Lemma 4: Suppose towards a contradiction that there are a and $I \in [\kappa \cap M]^\tau$ such that $a \in \bigcap_{\alpha \in I} A_\alpha \setminus M$. Since M is weakly τ-closed with respect to sets of ordinals,[2] by passing to a subset of I if necessary, we may assume that $[I]^{<\tau} \subset M$. Since (1.1) or (1.2) hold, we can find $J \in [I]^{<\tau}$ such that

$$(2) \qquad B = |\bigcap_{\alpha \in J} A_\alpha| \leq \sigma.$$

Since $\bar{A}, J \in M$, it follows that $B \in M$. Since $\sigma \subset M$ and $|B| \subset \sigma$, we conclude that $B \subset M$, which contradicts the assumption that $a \in B \setminus M$. □

Now let us reveal how CECA will be used in the remainder of this proof.

Lemma 5. *Suppose that there is a continuous increasing chain $\langle M_\xi \rangle_{\xi < cf(\kappa)}$ of weakly closed elementary submodels of $H(\Theta)$ such that $|M_\xi| < \kappa$ for each $\xi < cf(\kappa)$, that M_0 contains everything relevant, $\sigma \subset M_0$, $[\kappa]^{<cf(\kappa)} \subset \bigcup_{\xi < cf(\kappa)} M_\xi$, and for every $\xi < \kappa$:*
(i) *The sequence $\langle A_\alpha \setminus M_\xi \rangle_{\alpha \in M_\xi \cap \kappa}$ is point-$< \tau$;*
(ii) *$\alpha \in \kappa \setminus M_\xi$ implies $|M_\xi \cap A_\alpha| \leq \sigma$;*
(iii) *If $\alpha \in M_\xi$ and $|A_\alpha| \leq |M_\xi|$, then $A_\alpha \subset M_\xi$.*
Then there are $A'_\alpha \in [A_\alpha]^{\leq \sigma}$ for $\alpha < \kappa$ such that the sequence $\langle A_\alpha \setminus A'_\alpha \rangle_{\alpha,\kappa}$ is point-$< \tau$.

Proof: Let the M_ξ's and A_α's be as in the assumption. By the choice of κ and (ii) we can pick, for every $\xi < \kappa$ and $\alpha \in \kappa \cap M_{\xi+1} \setminus M_\xi$, an $A'_\alpha \in [A_\alpha]^{\leq \sigma}$ in such a way that $A_\alpha \cap M_\xi \subset A'_\alpha$ and the sequence $\langle A_\alpha \setminus A'_\alpha \rangle_{\alpha \in \kappa \cap (M_{\xi+1} \setminus M_\xi)}$ is point-$< \tau$.

We claim that the sequence $\langle A_\alpha \setminus A'_\alpha \rangle_{\alpha < \kappa}$ is point-$< \tau$. To see this, let $a \in A$, and let $M^* = \bigcup_{\xi < cf(\kappa)} M_\xi$.

Case 1: $a \in M^*$.

Then there exists a unique $\xi < cf(\kappa)$ such that $a \in M_{\xi+1} \setminus M_\xi$. For this ξ, the following hold:

- By (i), $|\{\alpha \in \kappa \cap M_\xi : a \in A_\alpha\}| < \tau$.
- By construction, $|\{\alpha \in \kappa \cap (M_{\xi+1} \setminus M_\xi) : a \in A_\alpha \setminus A'_\alpha\}| < \tau$.
- If $\alpha \in \kappa \setminus M_{\xi+1}$, then $A_\alpha \cap M_{\xi+1} \subset A'_\alpha$, and hence no $\alpha \in \kappa \setminus M_{\xi+1}$ satisfies $a \in A_\alpha \setminus A'_\alpha$.

It follows that the sequence $\langle A_\alpha \setminus A'_\alpha \rangle_{\alpha < \kappa}$ is point-$< \tau$ at a.

Case 2: $a \notin M^*$.

We show that M^* is weakly τ-closed with respect to sets of ordinals. Let $I \in [M^* \cap \kappa]^\tau$. If $\tau = cf(\kappa)$, then $[\kappa]^{<\tau} \subset M^*$ and hence $[I]^{<\tau} \subset M^*$. If $\tau \neq cf(\kappa)$, then there exists $\xi < cf(\kappa)$ with $|M_\xi \cap I| = \tau$, and $J = M_\xi \cap I$ is as

[2] If M has only the property that for every subset $I \in [M \cap \kappa]^{\tau^+}$ there exists $J \in [I]^\tau$ with $[J]^{<\tau} \subset M$, then the argument presented here shows that the sequence $\langle A_\alpha \setminus M \rangle_{\alpha \in \kappa \cap M}$ is point-$< \tau^+$.

required. Now Lemma 4 implies that the sequence $\langle A_\alpha \setminus A'_\alpha\rangle_{\alpha<\kappa}$ is point-$<\tau$ at a.□

Since we may assume $\sigma < \kappa$, CECA clearly implies the existence of a sequence $\langle M_\xi\rangle_{\xi<cf(\kappa)}$ that satisfies all unnumbered assumptions of Lemma 5. By Lemma 4, this sequence will also satisfy condition (i). How can we make sure that condition (ii) also holds? This depends on τ.

Lemma 6. *Suppose that $M \prec H(\Theta)$ is weakly closed, $\bar{A} \in M$, and $\max\{\sigma,\tau\} \subset M$. Then $\alpha \in \kappa \setminus M$ implies $|A_\alpha \cap M| \leq \sigma$.*

Proof: Assume towards a contradiction that $\alpha \in \kappa \setminus M$ is such that $|A_\alpha \cap M| \geq \sigma^+$. Since M is weakly closed,[3] there exists $Z \in [A_\alpha \cap M]^\sigma \cap M$. We are going to prove that the set $I = \{\beta \in \kappa : Z \subset A_\beta\}$ has cardinality $> \max\{\sigma,\tau\}$, in contradiction with conditions (1.1) and (1.2) of Theorem 2. Suppose that $|I| \leq \max\{\sigma,\tau\}$. Note that since $\kappa, \bar{A}, Z \in M$, it follows that $I \in M$, and there is a one-to-one function $f \in M$ that maps I into $\max\{\sigma,\tau\} \subset M$. It follows that $I \subset M$. On the other hand, $\alpha \in I \setminus M$, which gives a contradiction. □

It follows from Lemma 6 that if $\tau < \kappa$, then we can find a sequence $\langle M_\xi\rangle_{\xi<cf(\kappa)}$ that satisfies the assumption of Lemma 5. So it remains to prove Theorem 2 for the case $\sigma < \tau = \kappa$. If τ happens to be equal to σ^+, then any chain we get from CECA will satisfy $|M_\xi| \leq \sigma$, and we get condition (ii) for free. If $\sigma^+ < \tau = \kappa$, then we need to take advantage of GCH. The following lemma shows how to handle this last remaining case.

Lemma 7. *Suppose $\kappa = \tau \geq \sigma^{++}$, and let $\langle M_\xi\rangle_{\xi<\kappa}$ be a continuous increasing sequence of elementary submodels of $H(\Theta)$ such that each M_ξ has cardinality less than κ and $[\kappa]^{<\kappa} \subset \bigcup_{\xi<\kappa} M_\xi$. Then there exists a continuous subsequence $\langle M_{\xi_\nu}\rangle_{\nu<\kappa}$ such that*

(3) *$\nu < \kappa$ and $\alpha \in \kappa \setminus M_{\xi_\nu}$ implies $|A_\alpha \cap M_{\xi_\nu}| \leq \sigma$.ff*

Proof: Let $A = \bigcup_{\alpha<\kappa} A_\alpha$. Note that since $\kappa \geq \tau$, conditions (1) in Theorem 2 implies that for every $S \in [A]^\sigma$ the set $h(S) = \{\alpha < \kappa : S \subset A_\alpha\}$ has cardinality $< \kappa$. Since $\kappa \geq \sigma^{++}$ and κ $(= \tau)$ is regular, GCH implies that there is a continuous subsequence $\langle M_{\xi_\nu}\rangle_{\nu<\kappa}$ of $\langle M_\xi\rangle_{\xi<\kappa}$ such that

(4) For every $\nu < \kappa$ and every $S \in [A]^\sigma \cap M_{\xi_{\nu+1}}$ we have $h(S) \subset M_{\xi_{\nu+1}}$.

We claim that $\langle M_{\xi_\nu}\rangle_{0<\nu<\kappa}$ is as required. Let N_ν denote M_{ξ_ν}. Suppose towards a contradiction that $0 < \mu < \kappa$ and $\alpha \in \kappa \setminus N_\mu$ are such that $|A_\alpha \cap N_\mu| \geq \sigma^+$. We distinguish two cases.

Case 1: $\mu = \nu + 1$ for some ν.

Since $N_{\nu+1}$ is weakly closed, there exists $S \in [A_\alpha]^\sigma \cap N_{\nu+1}$. By (4), $h(S) \subset N_{\nu+1}$. Since α in $h(S)$, we have $\alpha \in N_{\nu+1} = N_\mu$, contradicting the choice of α.

Case 2: μ is a limit ordinal.

We will show that

(5) There are $\nu < \mu$ and $S \in [A_\alpha]^\sigma$ such that $S \in N_{\nu+1}$.

---

[3]Note that in this argument, as well as in the proof of the next lemma, only the following consequence of weak closedness is used: If $I \in [M]^{\sigma^+}$, then there is $J \in [I]^\sigma$ such that $J \in M$.

Indeed, since N_ν is weakly closed and $|A_\alpha \cap N_\nu| \geq \sigma^+$, we can pick $S \in [A_\alpha]^\sigma \cap N_\nu$. Since $N_\nu = \bigcup_{\nu<\mu} N_\nu$, there is $\nu < \mu$ such that $S \in N_{\nu+1}$. Now we can derive a contradiction as in the previous case. □ □

3. Related Theorems and Examples

We first present a modification of Theorem 2 exhibiting when we can obtain point-$< \tau$ families where τ is finite. Note that a family of sets is point-< 2 if and only if it is pairwise disjoint.

Theorem 3. *Assume CECA and let $n \geq 1$. Suppose that σ is a regular cardinal and let $\langle A_\alpha \rangle_{\alpha<\kappa}$ be a sequence of (not necessarily distinct) sets such that $|A_\alpha| \leq \sigma^{+n}$ and one of the following conditions is satisfied:*
(1.3) $|A_\alpha \cap A_\beta| < \sigma$ *for all* $\alpha < \beta < \kappa$.
(1.4) $|A_\alpha \cap A_\beta| \leq \sigma$ *for all* $\alpha < \beta < \kappa$ *and*
$\forall S \in [\bigcup_{\alpha<\kappa} A_\alpha]^\sigma \, (|\{\alpha : S \subset A_\alpha\}| \leq \sigma)$.
Then there exist $\langle A'_\alpha \rangle_{\alpha<\kappa}$ such that $|A'_\alpha| \leq \sigma$ for each $\alpha < \kappa$ and the sequence $\langle A_\alpha \setminus A'_\alpha \rangle_{\alpha<\kappa}$ is point-$< n+1$.

Proof: First apply Theorem 2 for $\tau = \aleph_0$ to get sets $A''_\alpha \in [A_\alpha]^\sigma$ such that $\langle A_\alpha \setminus A''_\alpha \rangle_{\alpha<\kappa}$ is point-finite. The step from point-finite to point-$< n+1$ can be made in ZFC:

Lemma 8. *Let $n \geq 1$. Suppose that σ is a regular infinite cardinal and that $\{A_\alpha : \alpha < \kappa\}$ is a point-finite family of sets satisfying $|A_\alpha| \leq \sigma^{+n}$ for each α and $|A_\alpha \cap A_\beta| \leq \sigma$ for each pair of distinct $\alpha, \beta \in \kappa$. Then there are $A'_\alpha \in [A_\alpha]^\sigma$ for each α such that $\{A_\alpha \setminus A'_\alpha : \alpha < \kappa\}$ is point-$< n+1$.*

Proof Lemma 8: First let us prove the lemma for the case $n = 1$. Let κ be the minimal cardinal such that there is a family of size κ that forms a counterexample to the theorem. Clearly $\kappa > \sigma$. First assume that $\kappa > \sigma^+$. Fix an $\in$-chain of elementary submodels $\{M_\xi : \xi < cf(\kappa)\}$ such that

1. $\sigma^+ \subseteq M_0$ and $|M_\xi| < \kappa$, for each $\xi < \kappa$.
2. $\{A_\alpha : \alpha < \kappa\} \in M_0$ and
3. $\{A_\alpha : \alpha < \kappa\} \subseteq \bigcup\{M_\xi : \xi < cf(\kappa)\}$.

Using the minimality of κ, for each $\xi \in cf(\kappa)$ fix a sequence $\{A'_\alpha : \alpha \in \kappa \cap (M_{\xi+1} \setminus M_\xi)\}$ of sets of size σ such that $\{A_\alpha \setminus A'_\alpha : \alpha \in \kappa \cap M_{\xi+1}\}$ is pairwise disjoint. Note that by point-finiteness if $\alpha \notin M_\xi$ then $A_\alpha \cap M_\xi = \emptyset$. Also $\sigma^+ \subseteq M_\xi$ implies that $A_\alpha \subseteq M_\xi$ for each $\alpha \in M_\xi$. Therefore $\{A_\alpha \setminus A'_\alpha : \alpha \in \kappa\}$ is pairwise disjoint.

The case for $\kappa = \sigma^+$ is similar. Fix an $\in$-chain as above of elementary submodels $\{M_\xi : \xi < cf(\kappa)\}$ of size σ so that $\sigma \subseteq M_0$ and let $A'_\alpha = A_\alpha \cap M_{\xi+1}$ where $\alpha \in M_{\xi+1} \setminus M_\xi$. To see that $\{A_\alpha \setminus A'_\alpha : \alpha \in \kappa\}$ is pairwise disjoint it suffices to observe that, as above, if $\alpha \notin M_\xi$ then $A_\alpha \cap M_\xi \neq \emptyset$ and, in addition, if $\alpha, \beta \in M_\xi$ then $A_\alpha \cap A_\beta \subseteq M_\xi$.

Now suppose $m > 1$ and the lemma is true for all $n < m$. Let $\langle A_\alpha \rangle_{\alpha<\kappa}$ be a sequence of sets such that $|A_\alpha| \leq \sigma^{+m}$ for each α and $|A_\alpha \cap A_\beta| \leq \sigma$ for each pair of distinct $\alpha, \beta \in \kappa$. Let $\nu = \sigma^{+(m-1)}$. Then $\sigma^{+m} = \nu^+$, and hence the lemma for $n = 1$ implies that there are $A''_\alpha \in [A_\alpha]^\nu$ for each α such that the family $\{A_\alpha \setminus A'_\alpha : \alpha < \kappa\}$ is pairwise disjoint. Since the lemma is true for $n = m-1$, there are $A'_\alpha \in [A''_\alpha]^\sigma$ such that the family $\{A''_\alpha \setminus A'_\alpha : \alpha < \kappa\}$ is point-$< m$. Now we claim that the family $\{A_\alpha \setminus A'_\alpha : \alpha < \kappa\}$ is point-$< m+1$. To see this, suppose

$a \in ((A_{\alpha_0} \setminus A_{\alpha_0}) \cap \cdots \cap (A_{\alpha_k} \setminus A_{\alpha_k}))$. Then there are at most $m-1$ indices α_i such that $a \in A''_{\alpha_i} \setminus A'_{\alpha_i}$, since the family $\{A''_\alpha \setminus A'_\alpha : \alpha < \kappa\}$ is point-$< m$. Similarly, since the family $\{A_\alpha \setminus A''_\alpha : \alpha < \kappa\}$ is disjoint, there is at most one α_i with $a \in A_{\alpha_i} \setminus A''_{\alpha_i}$. Thus, a is a member of $A_\alpha \setminus A'_\alpha$ for at most m indices α, and we are done. □ □

The following example shows that Lemma 8 is in a sense the best possible.

Theorem 4. *For each $n \in \omega$ there is a point-$< n+1$ family $\mathcal{A}_n$ of sets of size $\aleph_n$ such that*

(a) *$A \cap B$ is finite for each pair of distinct $A, B \in \mathcal{A}_n$.*
(b) *$\mathcal{A}_n$ has no point-$< n$ refinement of the form $\{A \setminus A' : A \in \mathcal{A}_n\}$ where each A' is countable.*

Proof: For each $n \in \omega$, each $i \in n$ and each $s \in \omega_n^{n\setminus\{i\}}$ let $A_s = \{t \in \omega_n^n : t|(n \setminus \{i\}) = s\}$. Let $\mathcal{A}_n = \bigcup_{i\in n}\{A_s : s \in \omega_n^{n\setminus\{i\}}\}$. We will show that $\mathcal{A}_n$ is the required family for each $n \in \omega$.

($n=2$): $\mathcal{A}_2$ is the family of rows and columns in $\omega_2 \times \omega_2$. Clearly this family is point-< 3 and the intersection of any two elements is either disjoint or contains one element. To see that (b) holds, suppose that $A' \in [A]^\omega$ for each $A \in \mathcal{A}_2$. Let $A_{0,\alpha} = \{(\alpha,\beta) : \beta \in \omega_2\}$ and let $A_{1,\alpha} = \{(\beta,\alpha) : \beta \in \omega_2\}$. Fix an uncountable $\alpha < \omega_2$ so that $A'_{0,\beta} \subseteq \alpha^2$ for each $\beta \in \alpha$. Consider $A_{1,\alpha}$. As α is uncountable, we can fix $\beta < \alpha$ so that $(\beta,\alpha) \notin A'_{1,\alpha}$. But then also $(\beta,\alpha) \notin A'_{0,\beta}$ so the refined family is not point-< 2.

Assume by induction that $n > 2$ and $\mathcal{A}_k$ is as required for each $k < n$. Clearly $\mathcal{A}_n$ is point-$< n+1$ and almost disjoint. So suppose that $A' \in [A]^\omega$ for each $A \in \mathcal{A}_n$. Fix $\alpha < \omega_n$ such that $|\alpha| = \omega_{n-1}$ and so that $A'_s \subseteq \alpha^n$ for each $i \in n$ and each $s \in \alpha^{n\setminus\{i\}}$. Let

$$S = \{t \in \omega_n^n : t(n-1) = \alpha \text{ and } t|(n-1) \in \alpha^{n-1}\}.$$

For each $i < n-1$, and each $s \in \alpha^{n\setminus\{i\}}$ such that $s(n-1) = \alpha$, let $B_s = \{t \in S : t|(n-1 \setminus \{i\}) = s\} \subseteq A_s$. By the induction hypothesis, $\{B_s \setminus A'_s : s \in \alpha^{n\setminus\{i\}}, i \in n-1 \text{ and } s(n-1) = \alpha\}$ is not point-$< n-1$. Therefore, there is $t \in S$ such that for each $i < n-1$, $t \in A_{t|(n\setminus\{i\})} \setminus A'_{t|(n\setminus\{i\})}$. Finally, since $t|(n-1) \in \alpha^{n-1}$ we have $A'_{t|n-1} \subseteq \alpha^n$. Therefore $t \in A_{t|(n-1)} \setminus A'_{t|(n-1)}$, so $\{A \setminus A' : A \in \mathcal{A}_n\}$ is not point-$< n$. □

By taking $\mathcal{A} = \bigcup_{n\in\omega}\mathcal{A}_n$ we obtain a family satisfying the following:

Corollary 4. *There is a point-finite, almost disjoint family $\mathcal{A}$ such that for each sequence $\{A' : A \in \mathcal{A}\}$ of countable sets, the family $\{A \setminus A' : A \in \mathcal{A}\}$ is not point-$< n$ for any $n \in \omega$.* □

We now present the main applications of Theorem 2 and Theorem 3. Letting $\sigma = \tau = \aleph_0$ in (1.1), we get:

Corollary 5. *Assume CECA. Suppose that $\langle A_\alpha \rangle_{\alpha<\kappa}$ is a sequence of sets such that every $I \in [\kappa]^{\aleph_0}$ has a finite subset $J \subset I$ such that $|\bigcap_{\alpha\in J} A_\alpha| < \aleph_0$. Then there are $A'_\alpha \in [A_\alpha]^{\leq\aleph_0}$ for $\alpha < \kappa$ such that the sequence $\langle A_\alpha \setminus A'_\alpha \rangle_{\alpha<\kappa}$ is point-finite.*

Letting $\sigma = \aleph_0$ and $\tau = \aleph_1$ in (1.1), we get:

Corollary 6. *Assume CECA. Suppose that $\langle A_\alpha\rangle_{\alpha<\kappa}$ is a sequence of sets such that or every countably infinite set S, the set $\{\alpha < \kappa : S \subset A_\alpha\}$ is countable. Then there are $A'_\alpha \in [A_\alpha]^{\leq\aleph_0}$ for $\alpha<\kappa$ such that the sequence $\langle A_\alpha \setminus A'_\alpha\rangle_{\alpha<\kappa}$ is point-countable.*

Let us restate the first part of Theorem 3 in a more conventional way:

Corollary 7. *Assume CECA. Let σ be a regular infinite cardinal. Suppose that $\langle A_\alpha\rangle_{\alpha<\kappa}$ is a sequence of sets of cardinality σ^+ each such that $A_\alpha \cap A_\beta$ has cardinality less than σ for all $\alpha<\beta<\kappa$. Then there are $A'_\alpha \in [A_\alpha]^{\leq\sigma}$ for $\alpha<\kappa$ such that the sequence $\langle A_\alpha \setminus A'_\alpha\rangle_{\alpha<\kappa}$ consists of pairwise disjoint sets.*

The consistency of Corollary 7 was already derived from a statement similar to CECA as Theorem 2.6 in [3].

Condition (1.4) allows us in certain circumstances to relax the assumption that the intersection of each two sets A_α, A_β has cardinality strictly less than σ. In particular, if $\sigma = \aleph_0$, then we get:

Corollary 8. *Assume CECA. Suppose that $\langle A_\alpha\rangle_{\alpha<\kappa}$ is a sequence of sets of cardinality $\aleph_1$ such that for every countably infinite set S, the set $\{\alpha<\kappa : S \subset A_\alpha\}$ is countable and $|A_\alpha \cap A_\beta| \leq \aleph_0$ for $\alpha<\beta<\kappa$. Then there are $A'_\alpha \in [A_\alpha]^{\leq\aleph_0}$ for $\alpha<\kappa$ such that the sequence $\langle A_\alpha \setminus A'_\alpha\rangle_{\alpha<\kappa}$ consists of pairwise disjoint sets.*

If the existence of certain large cardinals is consistent, none of the above corollaries is a consequence of GCH alone. To see this, suppose that $\langle A_\alpha\rangle_{\alpha<\kappa}$ is a sequence of sets for which there are $A'_\alpha \in [A_\alpha]^{\aleph_0}$ such that the sequence $\langle A_\alpha \setminus A'_\alpha\rangle_{\alpha<\kappa}$ is point-countable. Then on can recursively construct for each $\xi<\kappa$ pairwise disjoint sets $B_\xi \in [\kappa]^{\aleph_0}$ and pairwise disjoint sets $C_\xi \subseteq \bigcup_{\alpha\in B_\xi} A_\alpha \setminus A'_\alpha$ such that $C_\xi \cap A_\alpha \setminus A'_\alpha$ is infinite for each $\alpha \in B_\xi$. Now it is easy to construct a function $c : [\bigcup_{\alpha<\kappa} A_\alpha]^2 \to \{0,1\}$ such that $|c([A_\alpha]^2)| = 2$ for all $\alpha<\kappa$. But, if the existence of a supercompact cardinal is consistent, one can construct a model of ZFC where GCH holds and where there exists a sequence $\langle A_\alpha\rangle_{\alpha<\aleph_{\omega+1}}$ of sets of size $\aleph_1$ each such that every two of these sets have finite intersection, and for every function $c : \bigcup_{\alpha<\aleph\omega+1} A_\alpha \to \{0,1\}$ there exists α with $|c[A_\alpha]| = 1$ (see Theorem 4.6 of [3]).

However, assuming only GCH we can get weaker versions of Theorem 2 that are also of interest. For example:

Theorem 5. *Suppose that σ is a regular infinite cardinal and assume that $\lambda^\sigma = \lambda$ for each cardinal λ of cofinality greater than σ. Suppose that $\mathcal{A} = \langle A_\alpha\rangle_{\alpha<\kappa}$ is a sequence of (not necessarily distinct) sets such that*
$(*)$ For every $I \in [\kappa]^{\aleph_0}$ there is a finite $J \subseteq I$ such that $|\bigcap_{\alpha\in J} A_\alpha| \leq \sigma$.

Then there exists $\langle A'_\alpha\rangle_{\alpha<\kappa}$ such that $|A'_\alpha| \leq \sigma^+$ for each $\alpha<\kappa$ and the sequence $\langle A_\alpha \setminus A'_\alpha\rangle_{\alpha<\kappa}$ is point-finite.

Proof: Assume by induction that κ is minimal such that there is a counterexample to the theorem. First we show that without loss of generality we may assume that $\sigma^+ < \kappa$. If not, fix an elementary submodel M of size $\sigma^+ \geq \kappa$ such that $\mathcal{A} \cup \{\mathcal{A}\} \cup \sigma^+ \cup \{\sigma^+\} \subseteq M$. For each $\alpha \in \kappa$ let $A'_\alpha = A_\alpha \cap M$. To see that $\langle A_\alpha \setminus A'_\alpha\rangle_{\alpha<\kappa}$ is point-finite, fix $x \notin M$ such that $I = \{\alpha : x \in A_\alpha\}$ is infinite. By $(*)$, fix a finite $J \subseteq I$ such that $|\bigcap_{\alpha\in J} A_\alpha| \leq \sigma$. Clearly $\bigcap_{\alpha\in J} A_\alpha \in M$ But $\sigma^+ \subseteq M$ implies that $\bigcap_{\alpha\in J} A_\alpha \subseteq M$ contradicting $x \notin M$.

We may also assume that $\kappa = \lambda^+$ for λ a cardinal of cofinality $\leq \sigma$. Otherwise the proof is easier (see the comment after the next paragraph).

Fix a continuous $\in$-chain $\{M_\xi : \xi < \kappa\}$ of elementary submodels of some $H(\Theta)$ such that

(a) $|M_\xi| = \lambda$ and $\lambda \subseteq M_\xi$ for each $\xi \in \kappa$.
(b) $\mathcal{A} \in M_0$ and $\mathcal{A} \subseteq \bigcup_{\xi\in\kappa} M_\xi$.
(c) For ξ a successor ordinal, $M_\xi = \bigcup_{i<cf(\lambda)} M_\xi^i$ where for each i,
  (i) $\sigma < |M_\xi^i| < \lambda$ and
  (ii) $[M_\xi^i]^\sigma \subseteq M_\xi^{i+1}$.

Clearly by our assumption such a sequence exists.

(In the case that κ is not a successor of a singular cardinal of cofinality $\leq \sigma$ we may fix a sequence as above with the stronger property that $[M_\xi]^\sigma \subseteq M_\xi$ for each successor ξ.)

Notice that as before, by $(*)$, we have that for any model M as above and for any $x \notin M$, $\{\alpha \in M : x \in A_\alpha\}$ is finite. Indeed, if not, we can fix a finite subset J for which $|\bigcap_{\alpha\in J} A_\alpha| \leq \sigma$. But then $S = \bigcap_{\alpha\in J} A_\alpha$ is an element of M but it is not a subset. This implies that $|S| \geq \lambda > \sigma$. A contradiction.

The main lemma we need is the following.

Lemma 9. $|A_\alpha \cap M_\xi| \leq \sigma^+$ *for each* $\xi < \kappa$ *and each* $\alpha \notin M_\xi$.

Proof: Suppose not. First consider the case that ξ is a successor. Then there is an $i < cf(\lambda)$ such that $|A_\alpha \cap M_\xi^i| > \sigma^+$. Therefore there is an $S \in [A_\alpha \cap M_\xi^i]^\sigma$ such that $S \in M_\xi^{i+1} \subset M_\xi$. But then $\{\alpha : S \subseteq A_\alpha\} \in M_\xi$ but it is not a subset. Therefore it is infinitec contradicting $(*)$. In the case that ξ is a limit consider two subcases: If $cf(\xi) = \sigma^{++}$, then there is a successor $\eta < \xi$ such that $|A_\alpha \cap M_\eta| \geq \sigma^+$. If $cf(\xi) \neq \sigma^{++}$, then there is a successor $\eta < \xi$ such that $|A_\alpha \cap M_\eta| \geq \sigma^{++}$. Reasoning as above we get a contradiction to $(*)$ in both cases. $\square$

Now, to complete the proof of the theorem, using the Lemma and our inductive assumption fix $A'_\alpha \in [A_\alpha]^{\leq\sigma^+}$ for each $\xi \in \kappa$ and $\alpha \in M_{\xi+1} \setminus M_\xi$ such that

(d) $\{A_\alpha \setminus A'_\alpha : \alpha \in M_{\xi+1} \setminus M_\xi\}$ is point-finite, and
(e) $A_\alpha \cap M_\xi \subseteq A'_\alpha$.

The proof that $\{A_\alpha \setminus A'_\alpha : \alpha \in \kappa\}$ is point-finite is now straightforward: Fix any $x \in \bigcup \mathcal{A}$. If $x \notin \bigcup_\xi M_\xi$ then consider $I = \{\alpha : x \in A_\alpha\}$. If I is infinite, fix a finite $J \subseteq I$ such that $|\bigcap_{\alpha\in J} A_\alpha| \leq \sigma$. Then there is a ξ such that $J \subseteq M_\xi$. But then $S = \bigcap_{\alpha\in J} A_\alpha \in M_\xi$ and $S \subseteq M_\xi$ since $|S| \leq \sigma$ contradicting $x \in S \setminus M_\xi$.

If ξ is such that $x \in M_{\xi+1} \setminus M_\xi$ then by our construction $\{\alpha : x \in A_\alpha\} \subseteq M_\xi$ must be finite as above. $\square$

A slightly more general version of Theorem 5 with a slightly weaker conclusion also follows from GCH alone:

Theorem 6. *Assume GCH, let σ, τ be regular infinite cardinals. Suppose that $\mathcal{A} = \langle A_\alpha \rangle_{\alpha<\kappa}$ is a sequence of (not necessarily distinct) sets such that*
$(*)$ For every $I \in [\kappa]^\tau$ there is $J \in [I]^{<\tau}$ such that $|\bigcap_{\alpha\in J} A_\alpha| \leq \sigma$.

Then there exist $\langle A'_\alpha \rangle_{\alpha<\kappa}$ such that $|A'_\alpha| \leq \sigma^+$ for each $\alpha < \kappa$ and the sequence $\langle A_\alpha \setminus A'_\alpha \rangle_{\alpha<\kappa}$ is point-$<\tau^+$.

Proof: We modify the proof of Theorem 5. First note that we may assume that $\sigma^+, \tau < \kappa$ and that κ is the successor of a limit cardinal λ of cofinality $\leq \sigma$. Fix M_ξ satisfying (a), (b) and (c) of the previous proof. In addition, assume

(f) $[M^i_\xi]^{\leq \tau} \subseteq M^{i+1}_\xi$, for each successor $\xi < \kappa$ and each $i < cf(\lambda)$.

Define A'_α as before also satisfying $\langle A_\alpha \setminus A'_\alpha : \alpha \in M_{\xi+1} \setminus M_\xi \rangle$ is point-$< \tau^+$. To see that the whole sequence $\langle A_\alpha \setminus A'_\alpha \rangle_{\alpha < \kappa}$ is point-$< \tau^+$, fix $a \in \bigcup_{\alpha < \kappa} A_\alpha$ and fix ξ maximal so that $a \notin M_\xi$ (where $M_\kappa = \bigcup_{\xi \in \kappa} M_\xi$). It suffices to prove

Lemma 10. *$I = \{\alpha \in M_\xi : a \in A_\alpha\}$ has cardinality $< \tau^+$.*

Proof: Suppose not.

Case 1: $cf(\lambda) \neq \tau$. Then there are a successor $\eta < \xi$ and an $i < cf(\lambda)$ such that $|I \cap M^i_\eta| \geq \tau$. Fix $J \in [I \cap M^i_\eta]^{< \tau}$ such that $|\bigcap_{\alpha \in J} A_\alpha| \leq \sigma$. By (f), $J \in M_\eta$. Therefore $\bigcap_{\alpha \in J} A_\alpha \in M_\eta$. Therefore $\bigcap_{\alpha \in J} A_\alpha \subseteq M_\eta$ since it is of size $\leq \sigma$. This contradicts $a \notin M_\xi$.

Case 2: $cf(\lambda) = \tau$. In the case that $cf(\xi) \neq \tau^+$ we can find a successor $\eta < \xi$ such that $|I \cap M^i_\eta| \geq \tau^+$. This gives a contradiction as in Case 1. Otherwise assume that $cf(\xi) = \tau^+$. First fix a successor η such that $|I \cap M_\eta| \geq \tau$. Then fix $J \in [I \cap M_\eta]^{< \tau}$ such that $|\bigcap_{\alpha \in J} A_\alpha| \leq \sigma$. Now find $i < cf(\lambda) = \tau$ such that $J \subseteq M^i_\eta$. So $J \in M_\eta$ giving a contradiction as before. □□

Taking $\sigma = \aleph_0$ in Theorem 5 gives the following.

Corollary 9. *Assume $\lambda^\omega = \lambda$ for each cardinal λ of uncountable cofinality. Suppose that $\langle A_\alpha \rangle_{\alpha < \kappa}$ is a sequence of sets such that $|A_\alpha \cap A_\beta| \leq \aleph_0$ for each pair of distinct $\alpha, \beta \in \kappa$. Then there are $A'_\alpha \in [A_\alpha]^{\leq \aleph_1}$ for $\alpha < \kappa$ such that the sequence $\langle A_\alpha \setminus A'_\alpha \rangle_{\alpha < \kappa}$ is point-finite.*

Finally, we remark that Theorem 2 cannot be improved by demanding that the sets A'_α be of cardinality $< \sigma$. Indeed, if σ is regular, then let $\{A_\alpha : \alpha < \sigma^+\}$ be a family of subsets of σ such that $|A_\alpha \cap A_\beta| < \sigma$ while $|A_\alpha| = \sigma$ for each $\alpha < \sigma^+$. No such family can be point-$< \sigma^+$. So no point-$< \sigma^+$ family can be obtained by deleting sets of size less than σ from each A_α.

4. Topological Applications

In this section, we present some topological applications of the results just proved. We are particularly interested in the question, raised in [4], of whether a first countable space with a weakly uniform base must have a point-countable base.

Definition 3. If $n \in \omega$ and $\mathcal{B}$ is a base for a topological space X, then we say $\mathcal{B}$ is an *n-weakly uniform base for X* provided that if $A \subseteq X$ with $|A| = n$, then $\{B \in \mathcal{B} : A \subseteq B\}$ is finite. We say a base $\mathcal{B}$ for X is a *$< \omega$-weakly uniform base* provided that if $A \subseteq X$ with $|A| \geq \aleph_0$, then there is a finite subset $F \subseteq A$ with $\{B \in \mathcal{B} : F \subseteq B\}$ finite.

The notion of weakly uniform base, introduced in [4], is just what we have called 2-weakly uniform base. We shall see later in this section that the properties defined above are all distinct. To avoid trivialities we will be interested only in the case

$n \geq 2$. Clearly, for $n < m$, n-weakly uniform base implies m-weakly uniform base which implies $< \omega$-weakly uniform base.

It was shown in [4] that if X is a space with a 2-weakly uniform base $\mathcal{B}$ and $x \in X$ is in the closure of a countable subset of $X \setminus \{x\}$, then $\mathcal{B}$ is point-countable at x. In particular, if X is a first countable space with a weakly uniform base $\mathcal{B}$, then $\mathcal{B}$ is point-countable at all nonisolated points of X. Using that result, it was shown in [4] that a first countable space in which the boundary of the set of isolated points is separable has a point-countable base.

In [2], it is shown that a first countable space with a weakly uniform base and no more than $\aleph_1$ isolated points has a point-countable base. In [1] this is improved (consistently) to the result that assuming CH, every first countable space with a weakly uniform base and no more than $\aleph_\omega$ isolated points has a point-countable base.

An example is constructed in [2], assuming MA and $\aleph_2 < 2^{\aleph_0}$, of a normal Moore space with a weakly uniform base which has no point-countable base. Such a space also could not be metalindelöf.

Our next result establishes the independence of the existence of first countable spaces with weakly uniform bases but without point-countable bases, and it removes the cardinality restriction on the set of isolated points.

Theorem 7. *Assume CECA. If X is a T_1-space in which each non-isolated point is a cluster point of a countable set and X has a $< \omega$-weakly uniform base, then X has a point-countable base.*

Proof: Suppose $\mathcal{B}$ is a $< \omega$-weakly uniform base for X. Suppose x is a non-isolated point of X. We will show that $\mathcal{B}$ is point-countable at x. Choose a countable set $C \subseteq X \setminus \{x\}$ with $x \in \bar{C}$. For each $B \in \mathcal{B}$ with $x \in B$, the set $C \cap B$ is infinite, and hence there exists $F_B \in [C \cap B]^{<\aleph_0}$ such that $\{A \in B \ : F_B \subset A\}$ is finite. Since $[C]^{<\aleph_0}$ is countable and for each $F \in [C]^{<\aleph_0}$ there are only finitely many $B \in \mathcal{B}$ with $F_B = F$, we conclude that $\{B \in \mathcal{B} \ : x \in B\}$ is countable.

Let $\langle x_\alpha \ : \alpha < \kappa\rangle$ list the isolated points of X, and let $A_\alpha = \{B \in \mathcal{B} \ : x_\alpha \in B\}$. Notice that since $\mathcal{B}$ is a $< \omega$-weakly uniform base, we have that if I is an infinite subset of κ, then there exists a finite $J \subseteq I$ such that $|\{B \in \mathcal{B} \ : \{x_\alpha \ : a \in J\} \subseteq B\}| < \aleph_0$, i.e. $|\bigcap_{\alpha \in J} A_\alpha| < \aleph_0$. By Corollary 5, for each α there is $A^1_\alpha \in [A_\alpha]^{\leq\aleph_0}$ such that $\langle A_\alpha \setminus A^1_\alpha \ : \alpha < \kappa\rangle$ is point-finite, i.e. for each $B \in \mathcal{B}$ we have $B \in A_\alpha \setminus A^1_\alpha$ for only finitely many α's. Hence for each $B \in \mathcal{B}$, there is a finite set $I(B)$ of isolated points such that for every $\alpha < \kappa$, $\{B \in \mathcal{B} \ : x_\alpha \in B \setminus I(B)\} \subseteq A^1_\alpha$, and $|A^1_\alpha| \leq \aleph_0$. Hence $\mathcal{B} = \{B \setminus I(B) \ : B \in \mathcal{B}\}$ is a point-countable open family of subsets of X which forms a base at every non-isolated point of X. Thus $\mathcal{B}^* = \mathcal{B}_1 \cup \{\{x_\alpha\} \ : \alpha < \kappa\}$ is a point-countable base for X. □

Corollary 10. *It is consistent with ZFC that every first countable T_1-space with a weakly uniform base has a point-countable base.*

Combining that corollary with the example of [2] completes the independence result.

Assuming only $\lambda^\omega = \lambda$ for each cardinal λ of uncountable cofinality, we can obtain a weaker version of Theorem 7. The proof is identical, using Corollary 9 in place of Corollary 5.

Theorem 8. *Assume $\lambda^\omega = \lambda$ for each cardinal λ of uncountable cofinality. If X is a T_1-space in which each non-isolated point is a cluster point of a countable set and X has a $< \omega$-weakly uniform base, then X has a point-$< \aleph_2$ base.*

Corollary 11. *Assume $\lambda^\omega = \lambda$ for each cardinal λ of uncountable cofinality. Every first countable T_1-space with a weakly uniform base has a point-$< \aleph_2$ base.*

Definition 4. If $n \in \omega$, then we say that a topological space X is *n-metacompact* provided that for every open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V} \prec \mathcal{U}$ such that if $A \subseteq X$ with $|A| = n$, then $\{V \in \mathcal{V} : A \subseteq V\}$ is finite. We say X is $< \omega$-metacompact provided that for every open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{V} \prec \mathcal{U}$ such that if $A \subseteq X$ with $|A| \geq \aleph_0$, then there is a finite subset $F \subseteq A$ with $\{V \in \mathcal{V} : F \subseteq V\}$ finite.

It is clear that 1-metacompact is just metacompact and that for $n < m$, n-metacompact implies m-metacompact which implies $< \omega$-metacompact. We will call a refinement $\mathcal{V}$ as in the definition above an *n-weakly uniform refinement*, or *$< \omega$-weakly uniform refinement* respectively.

Theorem 9. *Assume CECA. If X is a $< \omega$- metacompact T_1-space, then X is metalindelöf.*

Proof: Suppose X is $< \omega$-metacompact and $\mathcal{U}$ is an open cover of X. Let $\mathcal{V}$ be an open, $< \omega$-weakly uniform refinement of $\mathcal{U}$. List the points of X as $\langle x_\alpha : \alpha < \kappa\rangle$. For each $\alpha < \kappa$, let $A_\alpha = \{V \in \mathcal{V} : x_\alpha \in V\}$. By Corollary 5, for each $\alpha < \kappa$ there is $A^1_\alpha \in [A_\alpha]^{\leq\aleph_0}$ such that $\langle A_\alpha \setminus A^1_\alpha : \alpha \in \kappa\rangle$ is point-finite on the set $\mathcal{V}$. For each $V \in \mathcal{V}$, we let $I(V) = \{x_\alpha \in V : V \in A_\alpha \setminus A^1_\alpha\}$. Note that $|I(V)| < \aleph_0$. Further, for each $\alpha < \kappa$, $\{V \in \mathcal{V} : x_\alpha \in V \setminus I(V)\} \subseteq A^1_\alpha$ and $|A^1_\alpha| \leq \aleph_0$. For each $\alpha < \kappa$, choose $V_\alpha \in \mathcal{V}$ such that $x_\alpha \in V_\alpha$. For $V \in \mathcal{V}$, let $V^* = (V \setminus I(V)) \cup \{x_\alpha \in I(V) : V = V_\alpha\}$. Note that $V^* \subseteq V$ and $V \setminus V^*$ is finite. Let $\mathcal{V}^* = \{V^* : V \in \mathcal{V}\}$. Then $\mathcal{V}^*$ is an open refinement of $\mathcal{U}$ and if $x_\alpha \in V^*$, then $V \in A^1_\alpha \cup \{V_\alpha\}$. Thus $\mathcal{V}^*$ is point-countable.

Corollary 12. *It is consistent with ZFC that every T_1-space with a weakly uniform base is metalindelöf.*

Again, combining this corollary with the example in [2] completes the independence result.

Example 1. *For each natural number $n \geq 1$, there is a Moore space X_n of scattered height 2 which has an $(n+1)$-weakly uniform base, but X_n does not have an n-weakly uniform base.*

Proof: Let L be a subset of $\mathbf{R} \times \{\mathbf{0}\}$ with $|L| = \aleph_1$, and let $D = \{p_n : n \in \omega\}$ be a countable subset of $\mathbf{R} \times (\mathbf{0}, \infty)$ which is dense in the Euclidean topology. It is shown in [2] that there is a collection $\mathcal{H}$ of countably infinite subsets of L and a partition $\{\mathcal{H}_n : n \in \omega\}$ of $\mathcal{H}$ such that

1. if $H_1, H_2 \in \mathcal{H}$ and $H_1 \neq H_2$, then $|H_1 \cap H_2| < \aleph_0$, and
2. if $Y \subseteq L$ and $|Y| = \aleph_1$, then for each $n \in \omega$, there exists $H \in \mathcal{H}_n$ such that $|Y \cap H| = \aleph_0$.

For each $n \in \omega$, let $K_n = \{(p_n, H) : H \in \mathcal{H}_n\}$ and let $K = \bigcup_{n\in\omega} K_n$. Let $X = L \cup K$. For each $n \in \omega$ and each $x \in L$, let $B_n(x) = \{(p_i, H) : (p_i, H) \in K_i$, $x \in H$, and p_i is an element of the Euclidean open ball in $\mathbf{R} \times (\mathbf{0}, \infty)$ of radius 2^{-n}

which is tangent to the axis $\mathbf{R} \times \{\mathbf{0}\}$ at the point $x\}$. If $x \in L$ and $n \in \omega$, then we define $G_n(x) = \{x\} \cup B_n(x)$ and let $\{G_n(x) \ : n \in \omega\}$ be a neighborhood base at x. If $y \in K$, then $\{y\}$ is open. It is shown in [2] that with this topology X is a Moore space with a weakly uniform base, and if $\mathcal{U}$ is any open cover of X which refines the open cover $\{G_0(x) \ : x \in L\} \cup \{\{y\} \ : y \in K\}$, then there exists $y \in K$ such that $\{U \in \mathcal{U} \ : y \in U\}$ is infinite.

Suppose k is a natural number and $k \geq 1$. We shall use the space X to construct a Moore space X_k of scattered height 2 with a $(k+1)$-weakly uniform base but no k-weakly uniform base. Let $X_k = L \cup (K \times k)$. Points of $K \times k$ will be isolated. If $x \in L$ and $n \in \omega$, let $G^k_n(x) = \{x\} \cup (B_n(x) \times k)$. Notice that $\langle G^k_n(x) \ : n \in \omega\rangle$ is a decreasing sequence, and thus this is a valid assignment of neighborhoods. Letting $\mathcal{G}_n = \{G^k_n(x) \ : x \in L\} \cup \{\{y\} : y \in K \times k\}$, we see that $\langle \mathcal{G}_n \ : n \in \omega\rangle$ is a development for X_k, and since each $G^k_n(x)$ is clopen, X_k is a 0-dimensional space of scattered height 2.

We now show that $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{G}_n$ is a $(k+1)$-weakly uniform base for X_k. Suppose $A \subseteq X_k$ with $|A| = k+1$. If $|A \cap L| \geq 2$, then $|\{B \in \mathcal{B} : A \subseteq B\}| = 0$. If $|A \cap L| = 1$. Then let $\{x\} = A \cap L$. Choose $n \in \omega$ such that $G^k_n(x) \cap A = \{x\}$. Then $|\{B \in \mathcal{B} : A \subseteq B\}| \leq n$, since for each $m \in \omega$ there is only one element of $\mathcal{G}_m$ which contains x. Finally, if $|A \cap L| = 0$, then let $A = \{(y_1, n_1), \ldots, (y_{k+1}, n_{k+1})\}$ where $y_i \in K$ and $n_i \in k$ for $1 \leq i \leq k+1$. Choose i, $j \leq k+1$ such that $n_i = n_j$. Then $\{B \in \mathcal{B} \ : A \subseteq B\} \subseteq \{B \in \mathcal{B} \ : \{(y_i, n_i), (y_j, n_j)\} \subseteq B\}$. Now for $B = G^k_n(x)$, we have that $(y_j, n_j) \in G^k_n(x)$ implies $y_i \in G_n(x)$. So $|\{B \in \mathcal{B} \ : \{(y_i, n_i), (y_j, n_j)\} \subseteq B\}| = |\{G_n(x) \ : \{y_i, y_j\} \subseteq G_n(x)\}| < \aleph_0$, since $\{G_n(x) \ : n \in \omega, x \in L\}$ is shown in [2] to be a weakly uniform collection.

We now show that X_k has no k-weakly uniform base. Suppose $\mathcal{V}$ is any open cover of X_k which refines $\mathcal{G}_1$. For each $x \in L$, choose $V_x \in \mathcal{V}$ and $n(x) \in \omega$ with $G^k_{n(x)}(x) \subseteq V_x$. Now in the space X, $\mathcal{U} = \{G_{n(x)}(x) \ : x \in L\} \cup \{\{y\} \ : y \in K\}$ refines $\{G_1(x) \ : x \in L\} \cup \{\{y\} \ : y \in K\}$. Hence there is a point $y \in K$ so that $\{U \in \mathcal{U} \ : y \in U\}$ is infinite, and so $\{x \in L \ : y \in G_{n(x)}(x)\}$ is infinite. Let $A = \{y\} \times k$. Then $|A| = k$ and for each $i \in k$, $(y, i) \in G^k_{n(x)}(x) \subseteq V_x$. So $\{x \in L \ : A \subseteq V_x\}$ is infinite, and thus $\mathcal{V}$ is not a k-weakly uniform base for X_k. $\square$

Example 2. *There is a space Y which has $< \omega$-weakly uniform base, but does not have n-weakly uniform base for any $n \in \omega$.*

Proof: For each natural number $n \geq 1$, let X_n be as constructed in Example 1. Let Y be the disjoint union of the spaces X_n. The natural base is easily seen to be $< \omega$-weakly uniform since if $A \subseteq Y$ and $|A| = \aleph_0$, then either $A \subseteq X_n$ for some n in which case any subset of A of size $n+1$ would be contained in only finitely many elements of the base, or A contains two points from distinct X_n's in which case no element of the base contains that two-point subset. To see that Y cannot have an n-weakly uniform base, it is enough to observe that X_n is an open subspace of Y. $\square$

Thus we have shown that these properties are all distinct. We leave the reader with one question regarding covering properties.

Question 1. *Is it true or consistent that every space with a weakly uniform base must be submetacompact?*

References

[1] A. V. Arhangel'skii, W. Just, Reznichenko, and P. J. Szeptycki; *Sharp bases and weakly uniform bases versus point-countable bases*, to appear in Topology and its Applications.

[2] S. W. Davis, G. M. Reed, and M. L. Wage; *Further results on weakly uniform bases*, Houston Journal of Mathematics, 2 (1976) 57–63.

[3] A. Hajnal, I. Juhász, and S.Shelah [HJSh249]; *Splitting strongly almost disjoint families,* TAMS 295 (1986) 369–387.

[4] R. W. Heath and W. F. Lindgren; *Weakly uniform bases,* Houston J. Math. 2 (1976) 85–90.

[5] S. Shelah [Sh108]; *On Successors of Singular Cardinals,* in: Logic Colloquium 78; M. Boffa, D. van Dalen, K. McAloon, eds., 357–380, North-Holland 1979.

[6] S. Shelah [Sh88a]; *Appendix: On stationary sets (to "Classification of nonelementary classes II. Abstract elementary classes"),* in: Classification Theory, Proceedings of the USA-Israel Conference on Classification Theory, Chicago, December 1985, J. T. Baldwin, ed., LNM 1292, 483–495, Springer 1987.

[7] S. Shelah [Sh420]; *Advances in Cardinal Arithmetic,* Finite and Infinite Combinatorics in Sets and Logic, N. W. Sauer et al. (eds.), Kluwer 1993, 355–383.

AUTHOR'S ADDRESSES:

Balogh and Davis
Department of Mathematics
Miami University
Oxford, OH 45056
U.S.A.

Just and Szeptycki
Department of Mathematics
Ohio University
Athens, OH 45701
U.S.A.

Shelah
Institute of Mathematics
The Hebrew University
Givat Ram
91904 Jerusalem
ISRAEL