The Positive Prekernel of a Cooperative Game*†

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Abstract

The positive prekernel, a solution of cooperative transferable utility games, is introduced. It is shown that this solution inherits many properties of the prekernel and of the core, which both are subsolutions. We prove that the positive prekernel on the set of games with players belonging to a universe of at least three possible members can be axiomatized by nonemptiness, anonymity, reasonableness, the weak reduced game property, the converse reduced game property, and a weak version of unanimity for two-person games. Additionally, we show that anonymity and reasonableness can be replaced by covariance and the strong nullplayer property, if at least four potential players are present.

Key words: TU-game, core, kernel.

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0 Introduction

The positive prekernel is a set-valued solution of cooperative transferable utility games. Its definition is strongly related to the definition of the prekernel. A preimputation belongs to the prekernel of a game, if for distinct players i and j the maximum surplus of i over j coincides with that of j over i. The only difference that occurs in the definition of the **positive** prekernel is that the maximum surplus is replaced by its **positive** part. Therefore the positive prekernel is a supersolution of both, the prekernel and the core, thus it is a nonempty supersolution of the core. The positive prekernel has all characterizing properties of the prekernel except the equal treatment property. Especially it satisfies the converse reduced game property and the nullplayer property. That may be regarded as an advantage over the prebargaining sets mentioned below which do not have these properties. Moreover, it is a subsolution of the prebargaining set and even of the prereactive bargaining set in the sense of Granot and Maschler (1997) (see Section 2). In the special case of the market game discussed in Maschler (1976) the positive kernel coincides with the bargaining set. Orshan (1994) showed that every nonsymmetric prekernel is a subsolution of the positive prekernel.

Our main results (see Sections 4 and 6) show that the positive prekernel has axiomatizations that are similar to an axiomatization of the core of totally balanced games (see Peleg (1989)). Hence the positive prekernel is the only known solution that is nonempty for every game, contains the core, and is axiomatized.

The paper is organized as follows: In Section 1 the notation and some definitions are presented.

In Section 2 it is shown that the positive prekernel coincides with the reactive bargaining set in the sense of Granot and Maschler (1997) for both the 7-person projective game and a 5-person market game. Moreover, an example of a balanced game is presented, in which the positive prekernel is strictly placed between the prereactive bargaining set and the union of the core and the prekernel. In this example the prereactive bargaining set is a proper subset of the prebargaining set.

In Section 3 it is shown that a preimputation belongs to the positive prekernel, if and only if there is a preimputation of the prekernel which yields the same positive part of the excess of every coalition. Alike the prekernel, the positive prekernel is a finite union of convex compact convex polytopes. The positive kernel, an individually rational modification, coincides with the positive **pre**kernel for weakly superadditive games. Moreover, it is proved that the positive prekernel satisfies the reduced game property and its converse.

Section 4 presents two characterizations of the positive prekernel on the set of games with player set contained in some universe of at least three members. This solution concept is uniquely determined by nonemptiness, anonymity, reasonableness (a preimputation is reasonable, if it assigns to every player at least her minimal and at most her maximal marginal contribution), the weak reduced game property, the converse reduced game property, and weak unanimity for two-person games (a solution concept satisfies this last property, if it contains the set of all imputations for every two-person game). If covariance under strategic equivalence is added, then we can replace reasonableness by some weaker property which resembles individual rationality in an obvious way.

In Section 5 the logical independence of the axioms of the first characterization is proved.

Section 6 presents an additional axiomatization which essentially arises from the first one by replacing reasonableness and anonymity by the strong nullplayer property and covariance.

1 Notation and Definitions

A **cooperative game with transferable utility** – a **game** – is a pair (N, v), where N is a finite nonvoid set and

$$v : 2^N \to I\!R,\ v(\emptyset) = 0$$

is a mapping. Here $2^N = \{S \subseteq N\}$ is the set of **coalitions** of (N, v).

If (N, v) is a game, then N is the **grand** coalition or the set of **players** and v is called **coalitional function** of (N, v).

The set of **feasible payoff vectors** of G is denoted by

$$X^*(N, v) = \{x \in I\!R^N \mid\ x(N) \leq\ v(N)\},$$

whereas

$$X(N, v) = \{x \in I\!R^N \mid\ x(N) = v(N)\}$$

is the set of **preimputations** of (N, v) (also called set of **Pareto optimal** feasible payoffs of (N, v)). Here

$$x(S) = \Sigma_{i \in S} x_i \quad (x(\emptyset) = 0)$$

for each $x \in I\!R^N$ and $S \subseteq N$. Additionally, let x_S denote the restriction of x to S, i.e.

$$x_S = (x_i)_{i \in S} \in I\!R^S.$$

For disjoint coalitions $S, T \subseteq N$ and $x \in I\!R^N$ let $(x_S, x_T) = x_{S \cup T}$.

A **solution** σ **on** a set Γ of games is a mapping that associates with every game $(N, v) \in \Gamma$ a set $\sigma(N, v) \subseteq X^*(N, v)$.

If $\bar{\Gamma}$ is a subset of Γ, then the canonical restriction of a solution σ on Γ is a solution on $\bar{\Gamma}$. We say that σ is a solution on $\bar{\Gamma}$, too. If Γ is not specified, then σ is a solution on every set of games.

Some convenient and well-known properties of a solution σ on a set Γ of games are as follows.

(1) σ is **anonymous** (satisfies **AN**), if for each $(N, v) \in \Gamma$ and each bijective mapping $\tau : N \to N'$ with $(N', \tau v) \in \Gamma$

$$\sigma(N', \tau v) = \tau(\sigma(N, v))$$

holds (where $(\tau v)(T) = v(\tau^{-1}(T))$, $\tau_j(x) = x_{\tau^{-1} j}$ $(x \in I\!R^N,\ j \in N',\ T \subseteq N')$).

In this case (N, v) and $(N', \tau v)$ are **isomorphic** games.

(2) σ satisfies the **nullplayer property (NPP)** if for every $(N, v) \in \Gamma$ every $x \in \sigma(N, v)$ satisfies $x_i = 0$ for every nullplayer $i \in N$. Here i is **nullplayer** if $v(S \cup \{i\}) = v(S)$ for $S \subseteq N$.

(3) σ is **covariant under strategic equivalence** (satisfies **COV**), if for $(N, v), (N, w) \in \Gamma$ with $w = \alpha v + \beta$ for some $\alpha > 0, \beta \in I\!R^N$

$$\sigma(N, w) = \alpha\sigma(N, v) + \beta$$

holds. The games v and w are called **strategically equivalent**.

(4) σ satisfies **nonemptiness (NE)**, if $\sigma(N, v) \neq \emptyset$ for $(N, v) \in \Gamma$.

(5) σ is **Pareto optimal** (satisfies **PO**), if $\sigma(N, v) \subseteq X(N, v)$ for $(N, v) \in \Gamma$.

(6) σ satisfies **reasonableness (on both sides) (REAS)**, if

$$x_i \geq \min\{v(S \cup \{i\}) - v(S) \mid S \subseteq N \setminus \{i\}\} = d_i^{\min}(N, v) \tag{1.1}$$

and

$$x_i \leq \max\{v(S \cup \{i\}) - v(S) \mid S \subseteq N \setminus \{i\}\} = d_i^{\max}(N, v) \tag{1.2}$$

for $i \in N, (N, v) \in \Gamma$, and $x \in \sigma(N, v)$.

With the help of assertion (1.2) Milnor (1952) defined his notion of reasonableness.

It should be remarked (see Shapley (1953)) that the Shapley value φ (to be more precise the solution σ given by $\sigma(N, v) = \{\varphi(N, v)\}$) satisfies all above properties.

Some more notation will be needed. Let (N, v) be a game and $x \in I\!R^N$. The **excess of** a coalition $S \subseteq N$ **at** x is the real number

$$e(S, x, v) = v(S) - x(S).$$

For **different** players $i, j \in N$ let

$$s_{ij}(x, v) = \max\{e(S, x, v) \mid i \in S \subseteq N \setminus \{j\}\}$$

denote the **maximum surplus of** i **over** j **at** x.

The **core** of (N, v) is the set

$$\mathcal{C}(N, v) = \{x \in X^*(N, v) \mid e(S, x, v) \leq 0 \ \forall S \subseteq N\}$$

of feasible payoff vectors which generate nonpositive excesses. The **prekernel** of (N, v) is the set

$$\mathcal{PK}(N, v) = \{x \in X(N, v) \mid s_{ij}(x, v) = s_{ji}(x, v) \ \forall i, j \in N \text{ with } i \neq j\}$$

of preimputations that **balance** the maximum surplusses of the pairs of players.

Definition 1.1 *The* **positive prekernel** *of a game* (N, v) *is the set*

$$\mathcal{POPK}(N, v) = \{x \in X(N, v) \mid s_{ij}(x, v) \leq (s_{ji}(x, v))_+ \ \forall i, j \in N \text{ with } i \neq j\},$$

where $r_+ = \max\{r, 0\}$ *denotes the positive part of a real number* r.

Remark 1.2 *(1) It is well-known (see, e.g., Davis and Maschler (1965) and Peleg (1986)) that both the prekernel as well as the core (restricted, of course, to balanced games) satisfy all above properties.*

(2) The positive prekernel of a game contains both the core and the prenucleolus of the game by definition.

(3) The positive prekernel satisfies anonymity, the nullplayer property, covariance, non-emptiness, Pareto optimality, and reasonableness. A proof of AN and COV is straightforward and skipped. The other properties can be shown by suitably modifying the corresponding proofs used for the prenucleolus.

2 Examples

The positive prekernel of a game (N, v) is contained in the prereactive bargaining set, as defined in Granot and Maschler (1997), of the game. Indeed, if $x \in \mathcal{POPK}(N, v)$ and i, j are distinct players in N, then player i has an objection against player j at x, if and only if $s_{ij}(x, v) > 0$. In this case such objection can be countered by any coalition attaining $s_{ji}(x, v)$, because $s_{ij}(x, v) = s_{ji}(x, v)$. We start with two examples discussed in Sections 3 and 4 of the paper mentioned before.

Example 2.1 *For the 7-person projective game, which is a monotone simple game, whose minimal winning coalitions are*

$$\{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 7\}, \{1, 5, 6\}, \{2, 6, 7\} \text{ and } \{1, 3, 7\},$$

the reactive bargaining set coincides with the (pre)kernel of the game. This game is weakly superadditive, thus its positive (pre)kernel coincides with its reactive bargaining set by Remark 3.8.

Example 2.2 *For the 5-person market game* (N, v)*, defined by* $N = \{1, 2, 3, 4, 5\}$ *and*

$$v(S) = \min\{|S \cap \{1, 2\}|, \ a|S \cap \{3, 4, 5\}|\},$$

where $a \geq 0$*, Granot and Maschler (1997) showed that the reactive bargaining set is the union of the core and of the kernel of the game. Again by weak superadditivity the positive (pre)kernel coincides with the reactive bargaining set in this case.*

The following example shows the existence of games for which the core is nonempty and for which the union of the core and the prekernel is a proper subset of the positive prekernel, which itself is a proper subset of the prereactive bargaining set. Moreover, the prereactive bargaining set is a proper subset of the bargaining set.

Example 2.3 *Let $N = P \cup Q \cup R$, where $P = \{1,2\}$, $Q = \{3\}$, and $R = \{4,5,6,7,8\}$. Let (N,v) be defined by*

$$v(S) = \begin{cases} 0, & \text{if } S = \emptyset \text{ or } S = N \\ 2, & \text{if } |S \cap R| = 3 \text{ and } S \cap (P \cup Q) \in \{P, Q\} \\ -1, & \text{if } S = \{i\} \cup \{j\} \text{ for some } i \in P,\ j \in R \\ -2, & \text{if } S = Q \cup T \text{ for some } T \subseteq R \text{ with } |T| = 2 \\ -5, & \text{if } S = \{i\} \text{ for some } i \in P \\ -40, & \text{otherwise} \end{cases}.$$

(1) Claim: $\mathcal{C}(N,v) \neq \emptyset$

As the reader may check, $(-5,-5,-10,4,4,4,4,4)$ is in the core of the game. (Indeed, the core is a singleton.)

(2) Claim: The union of the core and the prekernel is a proper subset of the positive kernel.

In order to show this claim observe that $x^1 = (-1,1,0,0,0,0,0,0)$ is not a member of the core (because $e(\{3,4,5,6\}, x^1, v) = 2$) and not a member of the prekernel (because $s_{12}(x,v) > s_{21}(x,v)$). However, a coalition S satisfies $e(S,x^1,v) > 0$, iff $e(S,x^1,v) = v(S) = 2$. The proof that both "types" of coalitions of positive excess balance the maximum surplus of players i,j satisfying $\{i,j\} \neq P$ is straightforward. Pareto optimality of x^1 together with the fact that a coalition of positive excess either contains P or does not intersect P shows Claim 2.

(3) Claim: The prereactive bargaining set $\mathcal{PM}^r(N,v)$ is a proper subset of the prebargaining set $\mathcal{PM}_1^{(i)}$.

Let $x^2 = (0,0,0,0,0,0,-2/3,2/3)$. First we show that x^2 does not belong to the prereactive bargaining set $\mathcal{PM}^r(N,v)$ by verifying that player 7 has a justified objection against player 8 in the sense of the reactive bargaining set. Precisely the coalitions $S = P \cup T \cup \{8\}$ and $S = Q \cup T \cup \{8\}$, where $T \subseteq \{4,5,6\}$ with $|T| = 2$, are the coalitions with nonnegative excess containing 8 and not containing 7. Of course player 7 can take every player $i \in T$ to define a justified objection against S by using the coalition $(N \setminus S) \cup \{i\}$.

In order to show that $x^2 \in \mathcal{PM}_1^{(i)}(N,v)$ note that $(s_{ij}(x^2,v))_+ = (s_{ji}(x^2,v))_+$ for distinct players with $i,j \notin \{7,8\}$. Of course $s_{7j}(x^2,v) \geq s_{j7}(x^2,v)$ for $j \neq 7$ and $s_{i8}(x^2,v) \geq s_{8i}(x^2,v)$ for $i \neq 8$. It remains to show that there is no player $i \neq 8$ who has a justified objection against 8 and that 7 does not have a justified objection *against any player $j \neq 7$ in the sense of the prebargaining set. Every objection (S, y) of a player $i \neq 8$ against 8 using a coaltion S* **not** *containing player 7 can be countered by the coalition $(N \setminus S) \cup \{k\}$, which has the same excess as S, where $k \in S \cap \{4,5,6\} \setminus \{i\}$. If $7 \in S$, then $e(S, x^2, v) = 8/3$. Nevertheless there are at least two distinct players $k, l \in R \setminus \{i\}$, thus one of them improves by at most $4/3$, let us say $y_k \leq x_k^2 + 4/3$. The excess of $(N \setminus S) \cup \{k\}$ is $4/3$, thus this coalition can be used to counterobject. A similar argument shows that 7 does not possess a justified objection.*

(4) *Claim: The positive prekernel is a proper subset of the prereactive bargaining set.*

Let $x^3 = (-10, -10, -20, 8, 8, 8, 8, 8)$. Then

$$e(S, x^3, v) \left\{ \begin{array}{ll} = & \left\{ \begin{array}{ll} 1, & \textit{if } S = \{i\} \cup \{j\} \textit{ for some } i \in P,\ j \in R \\ 2, & \textit{if } S = Q \cup T \textit{ for some } T \subseteq R \textit{ with } |T| = 2 \\ 5, & \textit{if } S = \{i\} \textit{ for some } i \in P \end{array} \right. \\ \leq & 0, \textit{ otherwise} \end{array} \right. . \qquad (2.1)$$

Players inside P (or R) do not possess justified objections against players of P (or R), because the are interchangeable and they are treated equally. Moreover, every objection of player 1 or 2 can be countered by using some coalition of the "second type" (coalitions that occur in the second row of (2.1)). Every objection against 1 or 2 can be countered using $\{1\}$ or $\{2\}$ respectively. Every objection of 3 against some player of R can be countered using a coalition of the same second type. Finally, every objection of some player in R against 3 can be countered by some coalition of the second type.

3 Properties of the Positive Prekernel

In this section we prove that the positive prekernel satisfies the reduced game property and its converse. Moreover, we show that every preimputation of the positive prekernel of a game can be "supported" (in the sense of Theorem 3.1) by some member of the prekernel of the game.

Theorem 3.1 *If (N, v) is a game, then*

$$\mathcal{POPK}(N, v) = \{y \in \mathbb{R}^N \mid \exists x \in \mathcal{PK}(N, v) \textit{ such that } (e(S, x, v))_+ = (e(S, y, v))_+ \ \forall S \subseteq N\}.$$

Proof:

(1) $\supseteq$: This inclusion is a direct consequence of the corresponding definitions.

(2) $\subseteq$: Let $y \in \mathcal{POPK}(N,v)$ and define

$$X = \{x \in X(N,v) \mid (e(S,x,v))_+ = (e(S,y,v))_+ \ \forall S \subseteq N\}.$$

It remains to show that X intersects $\mathcal{PK}(N,v)$. Let $\mathcal{N}(N,v;X)$ denote the nucleolus of (N,v) with respect to (w.r.t.) X, i.e.,

$$\mathcal{N}(N,v;X) = \{x \in X \mid \theta(e(S,x,v)_{S\subseteq N}) \leq_{lex} \theta(e(S,y,v)_{S\subseteq N}) \ \forall y \in X\},$$

where $\theta(z) \in \mathbb{R}^{2^{|N|}}$ is the vector whose components are those of $z \in \mathbb{R}^{2^N}$ arranged in nonincreasing order. The set X is a nonvoid compact polyhedron, thus $\mathcal{N}(N,v;X)$ consists of a unique member ν by Schmeidler (1969). In order to show that $\nu \in \mathcal{PK}(N,v)$ let $i,j \in N$, $i \neq j$. If $s_{ij}(\nu,v) > s_{ji}(\nu,v)$, then $s_{ij}(\nu,v) \leq 0$ by the definition of X. Therefore there exists $\epsilon > 0$ such that $\nu^\epsilon \in X$ and $\theta(e(S,\nu^\epsilon,v)_{S\subseteq N}) <_{lex} \theta(e(S,\nu,v)_{S\subseteq N})$, where $\nu^\epsilon \in \mathbb{R}^N$ is defined by

$$\nu_k^\epsilon = \begin{cases} \nu_i + \epsilon, & \text{if } k = i \\ \nu_j - \epsilon, & \text{if } k = j \\ \nu_k, & \text{otherwise} \end{cases},$$

which is impossible. q.e.d.

Remark 3.2 *Let (N,v) be a game. Then $\mathcal{POPK}(N,v)$ is a finite union of convex polytopes. Indeed, there is only a finite number of sets*

$$X^x = \left\{ y \in X(N,v) \,\middle|\, \forall S,T \subseteq N : \begin{array}{l} \big(e(S,x,v) \geq e(T,x,v) \geq 0 \Rightarrow e(S,y,v) \geq e(T,y,v) \geq 0\big) \\ \textit{and } \big(e(S,x,v) \leq 0 \Rightarrow e(S,y,v) \leq 0\big) \end{array} \right\},$$

where $x \in X(N,v)$. If $x \in \mathcal{POPK}(N,v)$ then X^x is a polytope containing x and contained in $\mathcal{POPK}(N,v)$.

We recall the definitions of the reduced game (see Davis and Maschler (1965)), of the reduced game property and its converse (see Sobolev (1975) and Peleg (1986)).

Definition 3.3 *Let (N,v) be a game, let $\emptyset \neq S \subseteq N$, and $x \in X^*(N,v)$. The* **reduced game** *w.r.t. S and x is the game $(S, v^{S,x})$ defined by*

$$v^{S,x}(T) = \begin{cases} 0, & \textit{if } T = \emptyset \\ v(N) - x(N \setminus S), & \textit{if } T = S \\ \max\{v(T \cup Q) - x(Q) \mid Q \subseteq N \setminus S\}, & \textit{otherwise} \end{cases}.$$

Definition 3.4 *Let σ be a solution on a set Γ of games. Then σ satisfies the*

(1) **reduced game property** *(***RGP***), if the following condition holds: If* $(N,v) \in \Gamma,\ \emptyset \neq S \subseteq N$, *and* $x \in \sigma(N,v)$, *then* $(S, v^{S,x}) \in \Gamma$ *and* $x_S \in \sigma(S, v^{S,x})$.

(2) **weak reduced game property** *(***WRGP***), if the following condition holds: If* $(N,v) \in \Gamma,\ \emptyset \neq S \subseteq N,\ |S| \leq 2$, *and* $x \in \sigma(N,v)$, *then* $(S, v^{S,x}) \in \Gamma$ *and* $x_S \in \sigma(S, v^{S,x})$.

(3) **converse reduced game property** *(***CRGP***), if the following condition holds: If* $(N,v) \in \Gamma,\ x \in X(N,v)$, *and for every* $S \subseteq N$ *with two members* $(S, v^{S,x}) \in \Gamma$ *and* $x_S \in \sigma(S, v^{S,x})$, *then* $x \in \sigma(N,v)$.

Note that Definition 3.4(2) is due to Peleg (1989) and that RGP implies WRGP. Furthermore, note that the prekernel and the core satisfy CRGP and RGP, if the set Γ of games is rich enough. The following lemmata show that the same properties hold in the case of the positive prekernel. If U is a set (the universe of players), then let Γ_U denote the set of games with player set contained in U.

Lemma 3.5 *The positive prekernel on* Γ_U *satisfies RGP.*

Proof: If $(N,v) \in \Gamma_U,\ x \in \mathcal{POPK}(N,v)$, and $\emptyset \neq S \subseteq N$, then $(S, v^{S,x})$ is a game, thus it is a game of Γ_U. Let $i,j \in S,\ i \neq j$. Then Definition 3.3 implies

$$s_{ij}(x_S, v^{S,x}) = s_{ij}(x,v), \tag{3.1}$$

thus the positive prekernel satisfies RGP. q.e.d.

Lemma 3.6 *The positive prekernel on* Γ_U *satisfies CRGP.*

Proof: Let (N,v) be a game and $x \in X(N,v)$ be a preimputation. If $x \notin \mathcal{POPK}(N,v)$, then distinct players $i,j \in N$ exist such that $0 < s_{ij}(N,v) > (s_{ji}(N,v)_+)$, thus equation (3.1) implies that $x_{\{i,j\}} \notin \mathcal{POPK}(\{i,j\}, v^{\{i,j\}})$. q.e.d.

Remark 3.7 *For every two-person game* (N,v) *the positive prekernel is either the prekernel, i.e., consists of the* **standard solution** x^v, *(defined by*

$$x_i = (v(\{i\}) - v(N \setminus \{i\}) + v(N))/2 \ \forall i \in N)$$

only, or it coincides with the core of the game.

The **kernel** of a game (N,v) is the set

$$\mathcal{K}(N,v) = \left\{ x \in X(N,v) \;\middle|\; \begin{array}{l} x_i \geq v(\{i\})\ \forall i \in N \text{ and} \\ \big(s_{ij}(x,v) \leq s_{ji}(x,v) \text{ or } x_j = v(\{j\})\ \forall i,j \in N,\ i \neq j\big) \end{array} \right\}.$$

In view of this definition we define the **positive kernel** of (N,v) by

$$\mathcal{POK}(N,v)=\left\{x\in X(N,v)\;\middle|\;\begin{array}{l}x_i\geq v(\{i\})\ \forall i\in N \text{ and}\\ (s_{ij}(x,v)\leq (s_{ji}(x,v))_+ \text{ or } x_j=v(\{j\})\ \forall i,j\in N,\ i\neq j)\end{array}\right\}.$$

Of course, the positive kernel of a game is contained in the (reactive) bargaining set of the game.

Remark 3.8 *If* (N,v) *is a game that is* **weakly superadditive**, *i.e.,*

$$d_i^{\min}(N,v)=v(\{i\})\ \forall i\in N,$$

then

$$\mathcal{POPK}(N,v)=\mathcal{POK}(N,v).$$

Indeed, by REAS, $\mathcal{POPK}(N,v)\subseteq\mathcal{POK}(N,v)$. *To show the other inclusion, let* $x\in\mathcal{POK}(N,v)$. *If* $s_{ij}(x,v)>(s_{ji}(x,v))_+$, *then* $x_j=v(\{j\})$. *Let* $S\subseteq N$ *be a coalition of maximal excess. Then* i *must be a member of* S, *because otherwise*

$$s_{ji}(x,v)\geq e(S\cup\{j\},x,v)\geq e(S,x,v)\geq s_{ij}(x,v).$$

Let $|S|$ *be of maximal size. Then* $S\neq N$ *by Pareto optimality of* x. *Moreover,* $x_k>d_k^{\min}(N,v)=v(\{k\})$ *for every* $k\in N\setminus S$ *by maximality of* S. *Take* $k\in N\setminus S$ *and a coalition* $T\subseteq N$ *with* $i\notin T\ni k$ *attaining* $s_{ki}(x,v)$. *The observation*

$$s_{ji}(x,v)\geq e(T\cup\{j\},x,v)\geq e(T,x,v)\geq s_{ik}(x,v)$$

directly leads to a contradiction.

4 A Characterization of the Positive Prekernel

In this section we shall assume that the universe U of players contains at least 3 members. We recall Peleg's (1989) notion of **unanimity for two-person games** (**UTPG**). A solution σ on a set Γ of games satisfies UTPG, if

$$\sigma(N,v)=\{x\in X(N,v)\mid x_i\geq v(\{i\})\ \forall i\in N\}$$

holds true for every two-person game $(N,v)\in\Gamma$. This property, together with WRGP, CRGP, and individual rationality ($x\in X^*(N,v)$ is **individually rational**, if $x_i\geq v(\{i\})$ for every $i\in N$) can be used to axiomatize the core of the set of markets games with player set in U (see, Peleg (1989)). If Γ contains a two-person game with an empty core, then there is no solution satisfying NE and UTPG. A weaker property will be used.

Definition 4.1 *A solution* σ *on a set* Γ *of games satisfies* **weak unanimity for two-person games** *(***WUTPG***), if*

$$\sigma(N,v)\supseteq\{x\in X(N,v)\mid x_i\geq v(\{i\})\ \forall i\in N\}$$

holds true for every two-person game $(N,v)\in\Gamma$.

Now we present the main result of this section.

Theorem 4.2 *The positive prekernel is the unique solution on Γ_U that satisfies NE, AN, REAS, WRGP, CRGP, and WUTPG.*

The following lemmata are useful in the proof of Theorem 4.2.

Lemma 4.3 *Let σ^1, σ^2 be solutions on Γ_U. If σ^1 satisfies WRGP, σ^2 satisfies CRGP, and if $\sigma^1(N,v) \subseteq \sigma^2(N,v)$ for every game $(N,v) \in \Gamma$ with at most two persons, then σ^1 is a subsolution of σ^2, i.e.,*

$$\sigma^1(N,v) \subseteq \sigma^2(N,v)\ \forall (N,v) \in \Gamma_U.$$

Proof: It suffices to show $\sigma^1(N,v) \subseteq \sigma^2(N,v)\ \forall (N,v) \in \Gamma$ with $|N| \geq 3$. If $x \in \sigma^1(N,v)$, then $x_S \in \sigma^1(S, v^{S,x})$ for every coalition $\emptyset \neq S \subseteq N$ with $|S| \leq 2$ by WRGP of σ^1. Therefore $x_S \in \sigma^2(S, v^{S,x})$ for these coalitions by the assumption, thus $x \in \sigma^2(N,v)$ by CRGP of σ^2. q.e.d.

Lemma 4.4 *If σ is a solution on Γ_U that satisfies NE, AN, REAS, WRGP, and CRGP, then σ is a subsolution of the positive prekernel.*

Proof: By REAS (only condition (1.1) is needed here) σ is Pareto optimal on one-person games. WRGP directly implies that σ satisfies PO. In view of Lemma 4.3 applied to $\sigma^1 = \sigma$ and $\sigma^2 = \mathcal{POPK}$ it suffices to show that $\sigma(N,v) \subseteq \mathcal{POPK}(N,v)$ for every two-person game (N,v) with $N \subseteq U$. If $\mathcal{C}(N,v) \neq \emptyset$, then it coincides with the core (see Remark 3.7), thus $\sigma(N,v) \subseteq \mathcal{POPK}(N,v)$ by REAS and PO in this case. Let $(N,v) \in \Gamma_U$ with $|N| = 2$ and $\mathcal{C}(N,v) = \emptyset$. Let $x = x^v \in \mathbb{R}^N$ denote the standard solution. We have to show that $\sigma(N,v) = \{x\}$.

Claim 1: $x \in \sigma(N,v)$

Assume, on the contrary, $x \notin \sigma(N,v)$. Take $* \in U \setminus N$ (which is possible by $|U| \geq 3$), let $N = \{i,j\}$, and define $(N \cup \{*\}, w)$ by

$$w(S) = \begin{cases} 0, & \text{if } S = \emptyset \\ v(N), & \text{if } S = N \cup \{*\} \\ x(S \cap N) + \alpha, & \text{otherwise} \end{cases},$$

where $\alpha = e(\{1\}, x, v) = e(\{2\}, x, v)$. A straightforward argument shows that $y = (x,0) \in \mathbb{R}^{N \cup \{*\}}$ is an element of the positive prekernel of w. (Indeed, it is well-known (see, e.g., Sudhölter (1993)) that the prekernel of a three-person game is a singleton, thus Theorem 3.1 also shows that y is the unique element of the positive prekernel.)

The reduced game $(N, w^{N,y})$ coincides with (N, v), thus $y \notin \sigma(N \cup \{*\}, w)$. Take $z \in \sigma(N \cup \{*\}, w)$ (which is possible by NE) and assume for simplicity reasons $N \cup \{*\} = \{1, 2, 3\}$ and

$$y_1 - z_1 \geq y_2 - z_2 \geq y_3 - z_3. \tag{4.1}$$

By condition (1.1) of REAS $y_i - \alpha \leq z_i \ \forall i = 1, 2, 3$. By PO $z_1 < y_1,\ z_3 > y_3$ and one of the inequalities of (4.1) is strict. Two cases may occur.

(1) $y_1 - z_1 > y_2 - z_2$

If $y_2 \geq z_2$, then define $(\{1, 2, 3\}, u)$ by

$$u(S) = \begin{cases} w(\{2,3\}) - z_2, & \text{if } S = \{3\} \\ v(N) - z_1 - 1, & \text{if } S = \{2,3\} \\ w(S), & \text{otherwise} \end{cases}$$

and observe that $u^{S,z} = w^{S,z}$ for every proper nonvoid subcoalition $\emptyset \neq S \subseteq N \cup \{*\},\ S \neq N \cup \{*\}$, because

$$e(\{2,3\}, z, w) < e(\{2\}, z, w),\ e(\{3\}, z, w) \leq e(\{2,3\}, z, w) < e(\{1,3\}, z, w)$$

and $v(N) - z_1 - 1 < w(\{2,3\})$. By CRGP $z \in \sigma(\{1,2,3\}, u)$. However,

$$\begin{aligned} & d_1^{\min}(\{1,2,3\}, u) \\ = \ & \min_{S \subseteq \{2,3\}} u(S \cup \{1\}) - u(S) \\ = \ & \min\{u(\{1\}), u(\{1,2\}) - u(\{2\}), u(\{1,3\}) - u(\{3\}), u(\{1,2,3\}) - u(\{2,3\})\} \\ = \ & \min\{w(\{1\}), w(\{1,2\}) - w(\{2\}), w(\{1,3\}) - w(\{2,3\}) + z_2, z_1 + 1\} \\ = \ & \min\{\alpha + y_1, y_1, y_1 - y_2 + z_2, z_1 + 1\} > z_1, \end{aligned}$$

which yields a contradiction to condition (1.1) of REAS.

If $y_2 \leq z_2$, then define $(\{1, 2, 3\}, u)$ by

$$u(S) = \begin{cases} q, & \text{if } S = \{2,3\} \\ w(S), & \text{otherwise} \end{cases},$$

where $q < \min\{w(\{3\}) + z_2,\ z_2 + z_3\}$, and observe that $u^{S,z} = w^{S,z}$ for every proper nonvoid subcoalition $\emptyset \neq S \subseteq N \cup \{*\},\ S \neq N \cup \{*\}$, because

$$e(\{2,3\}, z, w) < e(\{2\}, z, w),\ e(\{2,3\}, z, w) \leq e(\{3\}, z, w).$$

By CRGP $z \in \sigma(\{1,2,3\}, u)$. However,

$$d_1^{\min}(\{1,2,3\}, u) = \min\{\alpha + y_1, y_1, v(N) - q\} > z_1,$$

which yields a contradiction to condition (1.1) of REAS.

(2) $y_1 - z_1 = y_2 - z_2$

This implies $z_2 < y_2$. Therefore $s_{31}(z,w)$ is attained by $\{2,3\}$ and $s_{32}(z,w)$ is attained by $\{1,3\}$. Define $(\{1,2,3\},u)$ by

$$u(S) = \begin{cases} w(S), & \text{if } S \neq \{3\} \\ z_3 - 1, & \text{if } S = \{3\} \end{cases}$$

and observe that $u^{S,z} = w^{S,z}$ for every proper nonvoid subcoalition $\emptyset \neq S \subseteq N \cup \{*\},\ S \neq N \cup \{*\}$. By CRGP $z \in \sigma(\{1,2,3\},u)$. However,

$$\begin{aligned} d_3^{\max}(\{1,2,3\},u) &= \max\nolimits_{S \subseteq \{1,2\}} u(S \cup \{3\}) - u(S) \\ &= \max\{z_3 - 1, y_3, y_3 - \alpha\} < z_3, \end{aligned}$$

which yields a contradiction to condition (1.2) of REAS.

Claim 2: $\sigma(N,v) = \{x\}$

Assume, on the contrary, there exists $y \in \sigma(N,v) \setminus \{x\}$. Assume without loss of generality $N = \{1,2\},\ 3 \in U$, and $y_1 < x_1,\ y_2 > x_2$. Define $(\{1,2,3\},w)$ by $w(\{1\}) = w(\{2\}) = v(\{2\}),\ w(\{1,2\}) = v(N) + v(\{2\}) - v(\{1\})$, and $w(S \cup \{3\}) = w(S) + d$, for $S \subseteq N$, where $d = v(\{1\}) - v(\{2\}) + y_2 > y_1$. Moreover, define $z \in \mathbb{R}^{N \cup \{3\}}$ by $z_1 = z_2 = y_2$ and $z_3 = y_1$. Then

$$w^{N,z}(S) = \begin{cases} 0, & \text{if } S = \emptyset \\ 2y_2, & \text{if } S = N \\ \max\{v(\{2\}), v(\{1\}) + y_2 - y_1\}, & \text{otherwise} \end{cases},$$

thus

$$w^{N,z}(N) = v(N) + y_2 - y_1 < v(\{2\}) + v(\{1\}) + y_2 - y_1 \leq w^{N,z}(\{1\}) + w^{N,z}(\{2\}).$$

This last observation shows that $\mathcal{C}(N, w^{N,z}) = \emptyset$, thus $z_N \in \sigma(N, w^{N,z})$ by Claim 1. By construction $(\{1,3\}, w^{\{1,3\},z})$ and $(\{2,3\}, w^{\{2,3\},z})$ are isomorphic and it can easily be checked that they are isomorphic to (N,v). Therefore $z_S \in \sigma(S, w^{S,z})$ for every two-person subcoalition of $\{1,2,3\}$ by AN. CRGP directly implies $z \in \sigma(\{1,2,3\},w)$, but player 3 is **inessential** (i.e., strategically equivalent to a nullplayer) of worth d, where

$$d_3^{\min}(\{1,2,3\},w) = d = d_3^{\max}(\{1,2,3\},w),$$

thus $z_3 < d$ establishes a contradiction to condition (1.1) of REAS. q.e.d.

Corollary 4.5 *The positive prekernel is the maximum solution on Γ_U that satisfies NE, AN, REAS, WRGP, and CRGP.*

Proof: The positive prekernel satisfies the required properties by Remark 1.2, Lemma 3.5, and Lemma 3.6. Lemma 4.4 completes the proof. q.e.d.

Proof of Theorem 4.2: By Remark 1.2 the positive prekernel satisfies WUTPG. Corollary 4.5 and Lemma 4.3 complete the proof. q.e.d.

The core on the set of market games contained in Γ_U is the unique solution that satisfies individual rationality (IR), WRGP, CRGP, and UTPG. We used WRGP, CRGP, and WUTPG, a property that is weaker than UTPG, in our characterization. In some sense REAS replaces IR. However, it is possible to weaken REAS in such a way that a weak version of IR is obtained. A solution σ on a set Γ of games satisfies **reasonableness from below** (**REASB**), if condition (1.1) of REAS is satisfied. Now the positive prekernel can be characterized by weakening the axioms for the core and adding some "standard axioms".

Theorem 4.6 *The positive prekernel is the unique solution on Γ_U that satisfies NE, AN, COV, REASB, WRGP, CRGP, and WUTPG.*

Proof: Only the uniqueness part has to be shown. In the proof of Lemma 4.4 condition (1.2) of REAS is only used once, namely in Claim 1, part (2). This case directly leads to a contradiction, because $z_{\{1,2\}}$ is the standard solution of the game $(\{1,2\}, w^{\{1,2\},z})$ with an empty core. This game is isomorphic to a game that is strategically equivalent to (N, v), thus WRGP, AN and COV establish a contradiction. q.e.d

5 On the Independence of the Axioms

The following examples show that the properties used in Lemma 4.4 and Theorem 4.2 are logically independent. We start showing that these results are not valid, if $|U| = 2$.

Example 5.1 *If $|U| = 2$, then define*

$$\sigma^0(N, v) = \{x \in X(N, v) \mid x \textit{ is reasonable}\} \ \forall (N, v) \in \Gamma_U.$$

Then σ^0 satisfies NE, AN, REAS, COV, RGP, CRGP, and WUTPG, but it is not a subsolution of the positive prekernel.

From now on we assume that the universe U of players contains at least three members.

Example 5.2 *The solution σ^1 on Γ_U is defined by distinguishing cases.*

(1) If $N = \{i\}$, then $\sigma^1(N, v) = X(N, v) = \{v(\{i\})\}$.

(2) If $|N| = 2,\ N = \{i, j\}$*, then*

$$\sigma^1(N, v) = \begin{cases} \mathcal{C}(N, v), & \text{if } \mathcal{C}(N, v) \neq \emptyset \\ \{x \in X(N, v) \mid x_i = v(\{i\}) \text{ or } x_j = v(\{j\})\}, & \text{otherwise} \end{cases}$$

assigns to every two-person game with an empty core the extreme points of its set of preimputations.

(3) $\sigma^1(N, v) = \{x \in X(N, v) \mid x_S \in \sigma^1(S, v^{S,x})\ \forall S \subseteq N,\ |S| = 2\}$*, if* $|N| > 2$.

This solution satisfies AN, COV, and CRGP by definition. The core is a subsolution of σ^1*, thus it satisfies WUTPG. Moreover,* σ^1 *satisfies RGP by the* **transitivity of reducing:**

$$v^{T,x} = (v^{S,x})^{T,x_S}\ \forall (N, v) \in \Gamma_U,\ \emptyset \neq T \subseteq S \subseteq N,\ x \in X^*(N, v)$$

σ^1 *does not satisfy NE (even if* $|U| = 3$*), because its application to the game* $(N \cup \{*\}, w)$ *of Claim 1 of the proof of Lemma 4.4 yields the empty set.*

Claim: σ^1 *satisfies REAS.*

Assume, on the contrary, there is a game (N, v) *and* $x \in \sigma^1(N, v)$ *that does not satisfy reasonableness. Then there is a player* $i \in N$ *such that (1.1) or (1.2) of REAS is violated at* i.

(1) $x_i < d_i^{\min}(N, v)$

Then $s_{ij}(x, v) > 0\ \forall j \in N \setminus \{i\}$*, thus* $s_{ji}(x, v) = 0$ *by definition of* σ^1*. Take* $S \subseteq N \setminus \{i\}$ *which is maximal (under* $\subseteq$*) such that* $e(S, x, v) = 0$*. Then* $S \neq N \setminus \{i\}$*. Take* $T \subseteq N \setminus \{i\}$ *attaining* $0 = s_{ji}(x, v)$ *for some* $j \in N \setminus (S \cup \{i\})$*. By maximality of* S *there exists* $k \in S \setminus T$*. Therefore*

$$s_{kj}(x, v) \geq e(S \cup \{i\}, x, v) > 0 < e(T \cup \{i\}, x, v) \leq s_{jk}(x, v),$$

a contradiction.

(2) $x_i > d_i^{\max}(N, v)$

Then $s_{ji}(x, v) > 0\ \forall j \in N \setminus \{i\}$*, thus* $s_{ij}(x, v) = 0$ *by definition of* σ^1*. Take* $S \subseteq N \setminus \{i\}$ *which is minimal such that* $e(S \cup \{i\}, x, v) = 0$*. Then* $S \neq \emptyset$*. Take* $j \in S$ *and* $T \subseteq N \setminus \{i\}$ *such that* $s_{ij}(x, v)$ *is attained by* $T \cup \{i\}$*. By minimality of* S *we have* $T \setminus S \neq \emptyset$*. Let* $k \in T \setminus S$*. Then*

$$s_{kj}(x, v) \geq e(T, x, v) > 0 < e(S, x, v) \leq s_{jk}(x, v),$$

a contradiction.

Example 5.3 *In order to show that AN is independent we proceed similarly to Example 5.2 by defining σ^2. The only difference in the definition occurs for two-person games (N, v) with an empty core. Choose two different players, let us say 1 and 2, of U and define*

$$\sigma^2(N,v) = \begin{cases} \{x \in X(N,v) \mid x_1 = v(\{1\}) \text{ or } x_2 = v(\{2\}) \text{ or } x = x^v\}, & \text{if } N = \{1,2\} \\ x^v, & \text{otherwise} \end{cases},$$

where x^v is the standard solution of (N, v). As in the last example it is easy to verify that σ^2 satisfies NE, COV, RGP, CRGP, WUTPG. It does not satisfy AN.

Claim: *σ^2 satisfies REAS.*

By construction $x \in \sigma^2(N, v)$ implies

$$(s_{kl}(x,v))_+ = (s_{lk}(x,v))_+ \ \forall k,l \in N \text{ with } k \neq l,\ \{k,l\} \neq \{1,2\}. \tag{5.1}$$

Therefore $x \in \mathcal{POPK}(N, v)$ for every N satisfying $\{1, 2\} \not\subseteq N$. Assume, on the contrary, $x \in \sigma^2(N, v)$ is not reasonable, thus, $x \notin \mathcal{POPK}(N, v)$ and $|N| > 2$. Property (5.1) implies that (1.1) and (1.2) of REAS are satisfied for $i \in N \setminus \{1, 2\}$. Moreover, $s_{12}(x, v) > 0$ and $s_{21}(x, v) = 0$ can be assumed (otherwise exchange 1 and 2). Hence

$$x_2 > d_2^{\max}(N,v) \text{ or } x_1 < d_1^{\min}(N,v).$$

Two cases may occur:

(1) $x_2 > d_2^{\max}(N, v)$
Then
$$s_{i2}(x,v) = s_{2i}(x,v) \ \forall i \in N \setminus \{1,2\} \tag{5.2}$$
If $S \subseteq N$ has maximal excess, then $S \subseteq N \setminus \{2\}$, thus $S = \{1\}$ by (5.2). This observation contradicts $|N| > 2$.

(2) $x_1 < d_1^{\min}(N, v)$
The fact that $s_{1i}(x, v) = s_{i1}(x, v) > 0$ fir $i \neq 1, 2$ implies that $N \setminus \{2\}$ is the unique coalition of maximal excess. Take $j \in N \setminus \{1, 2\}$ and observe that $s_{j2}(x, v) = s_{2j}(x, v) > 0$ yields a contradiction.

Example 5.4 *The solution σ^3, defined by $\sigma^3(N, v) = X(N, v)$, shows that REAS is logically independent.*

Example 5.5 *The solution σ^4 is defined inductively on $|N|$. If $|N| \leq 2$, then $\sigma^4(N, v) = \mathcal{POPK}(N, v)$. If $|N| > 2$, then two cases are distinguished.*

(1) If (N, v) does not contain inessential players, then $\sigma^4(N, v)$ is the set of all reasonable preimputations of (N, v).

(2) If $i \in N$ is an inessential player of (N, v), then

$$\sigma^4(N, v) = \{x \in X(N, v) \mid x_{N \setminus \{i\}} \in \sigma^4(N \setminus \{i\}, v_{N \setminus \{i\}})\}.$$

An inductive argument shows that σ^4 is well-defined. In view of the fact that $\mathcal{POPK}$ is a subsolution of σ^4 which coincides on games with at most two persons, σ^4 satisfies NE, WUTPG, and CRGP. COV, AN are straightforward and WRGP is violated.

Example 5.6 *Define $\sigma^5(N, v) = \{x \in X(N, v) \mid x_S$ is reasonable for $(S, v^{S,x})$ $\forall \emptyset \neq S \subseteq N\}$. Then σ^5 satisfies NE, AN, REAS, COV, and WUTPG. Moreover, RGP is a direct consequence of the transitivity of reducing. Of course σ^5 violates CRGP.*

Example 5.7 *In order to show that WUTPG is independent, let σ be some subsolution of $\mathcal{POPK}$ on Γ_U satisfying NE, COV, RGP, and CRGP. We define the* **anonymous extension** *$\bar{\sigma}$ of σ by the following conditions:*

(1) $\bar{\sigma}(N, v) = \sigma(N, v)$, *if* $|N| = 1$.

(2) $\bar{\sigma}(N, v) = \{\tau(x) \mid \tau : N' \to N$ *is bijective,* $\tau v' = v,\ (N', v') \in \Gamma_U,\ x \in \sigma(N', v')\}$, *if* $|N| = 2$.

(3) $\bar{\sigma}(N, v) = \{x \in X(N, v) \mid x_S \in \bar{\sigma}(S, v^{S,x})\ \forall S \subseteq N,\ |S| = 2\}$, *if* $|N| > 2$.

By AN of $\mathcal{POPK}$, the transitivity of reducing, and its construction, the solution $\bar{\sigma}$ satisfies NE, AN, REAS, RGP, and CRGP. In order to present an explicit example, define σ^6 on games with at most two players by

$$\sigma^6(N, v) = \begin{cases} x^v, & \textit{if } \mathcal{C}(N, v) = \emptyset \\ \{x \in X(N, v) \mid x_i = v(\{i\}) \textit{ for some } i \in N\}, & \textit{if } \mathcal{C}(N, v) \neq \emptyset \end{cases}$$

(compare with Sudhölter (1993)). Note that σ^6 is a special case of an anonymous extension of a nonsymmetric prekernel in the sense of Orshan (1994). Furthermore, note that the prekernel itself may serve as a further example that does not satisfy WUTPG.

Note that we do not know whether Theorem 4.6 remains valid, if COV is dropped as a condition.

6 An Additional Axiomatization

A strong version of NPP is employed to axiomatize the positive prekernel. We assume that U contains at least four members.

Definition 6.1 *A solution σ on a set Γ of games satisfies the* **strong nullplayer property** *(***SNPP***), if it satisfies NPP and if the following condition is fulfilled: If $(N, v) \in \Gamma$ is a game with at least two persons and if $i \in N$ is a nullplayer of v, then $\sigma(S, v_S) = \{x_S \mid x \in \sigma(N, v)\}$, where $S = N \setminus \{i\}$ and v_S is the coalitional function of the corresponding subgame, whenever this subgame belongs to Γ.*

Note that the positive prekernel satisfies SNPP. The following result resembles Lemma 4.4.

Lemma 6.2 *If σ is a solution on Γ_U that satisfies COV, SNPP, WRGP, and CRGP, then σ is a subsolution of the positive prekernel.*

Proof: COV and SNPP imply that σ is Pareto optimal on one-person games, thus σ satisfies PO by WRGP. Let $(N, v) \in \Gamma_U$ be any two-person game. In view of Lemma 4.3 it suffices to show that $\sigma(N, v) \subseteq \mathcal{POPK}(N, v)$.

We assume that $\{1, 2, 3, 4\}$ is contained in U and that σ is not a subsolution of the core. Let

$$\Gamma^2 = \{(N, v) \in \Gamma_U \mid |N| \ = 2\}$$

and

$$\Gamma^{2,\emptyset} = \{(N, v) \in \Gamma^2 \mid \mathcal{C}(N, v) = \emptyset\}.$$

Claim 1: $\sigma(N, v) \neq \emptyset \ \forall (N, v) \in \Gamma^{2,\emptyset}$

Take $(\bar{N}, \bar{v}) \in \Gamma^2$, let us say $\bar{N} = \{1, 2\}$, with $\sigma(\bar{N}, \bar{v}) \nsubseteq \mathcal{C}(\bar{N}, \bar{v})$ and take $\bar{x} \in \sigma(\bar{N}, \bar{v}) \setminus \mathcal{C}(\bar{N}, \bar{v})$. Define $(\{1, 2, 3\}, \bar{w})$ to be the game which arises from $(\bar{N}, \bar{v})$ by adding the nullplayer 3, i.e., $\bar{w}(S) = \bar{v}(S \cap \bar{N})$. Then $\bar{y} = (\bar{x}, 0) \in \sigma(\{1, 2, 3\}, \bar{w})$ by SNPP. Let $(\tilde{N}, \tilde{u}) = (\{1, 3\}, \bar{w}^{\{1,3\},\bar{y}})$ denote the reduced game. By WRGP $\bar{y}_{\tilde{N}} \in \sigma(\tilde{N}, \tilde{u})$. Note that

$$\tilde{u}(\{1\}) = \max\{\bar{v}(1), \bar{x}_1\}, \tag{6.1}$$

$$\tilde{u}(\{3\}) = \max\{0, \bar{v}(\{2\}) - \bar{x}_2\}, \tag{6.2}$$

$$\tilde{u}(\tilde{N}) = \bar{x}_1. \tag{6.3}$$

In order to complete the proof of Claim 1 we proceed by showing that $(\tilde{N}, \tilde{u})$ has an empty core.

(1) Case: $\mathcal{C}(\bar{N}, \bar{v}) \neq \emptyset$

Then $\bar{x}_1 < \bar{v}(\{1\})$ or $\bar{x}_1 > \bar{v}(\bar{N}) - \bar{v}(\{2\})$.

(a) $\bar{x}_1 < v(\{1\})$

Then $\bar{x}_2 > \bar{v}(\{2\})$, hence

$$\tilde{u}(\{1\}) + \tilde{u}(\{3\}) = \bar{v}(\{1\}) + 0 > \bar{x}_1 = \tilde{u}(\tilde{N})$$

by (6.1,6.2,6.3), thus $(\tilde{N}, \tilde{u}) \in \Gamma^{2,\emptyset}$.

(b) $\bar{x}_1 > \bar{v}(\bar{N}) - \bar{v}(\{2\})$
Then $\bar{x}_2 < \bar{v}(\{2\})$, hence

$$\tilde{u}(\{1\}) + \tilde{u}(\{3\}) = \bar{x}_1 + \bar{v}(\{2\}) - \bar{x}_2 > \bar{x}_1 = \tilde{u}(\tilde{N})$$

by (6.1,6.2,6.3), thus $(\tilde{N}, \tilde{u}) \in \Gamma^{2,\emptyset}$.

(2) Case: $\mathcal{C}(\bar{N}, \bar{v}) = \emptyset$
Then $\bar{x}_1 < \bar{v}(\{1\})$ or $\bar{x}_2 < \bar{v}(\{2\})$.

(a) $\bar{x}_1 < \bar{v}(\{1\})$
Then

$$\tilde{u}(\{1\}) + \tilde{u}(\{3\}) \geq \bar{v}(\{1\}) + 0 > \bar{x}_1 = \tilde{u}(\tilde{N})$$

by (6.1,6.2,6.3), thus $(\tilde{N}, \tilde{u}) \in \Gamma^{2,\emptyset}$.

(b) $\bar{x}_2 < \bar{v}(\{2\})$
Then

$$\tilde{u}(\{1\}) + \tilde{u}(\{3\}) \geq \bar{x}_1 + \bar{v}(\{2\}) - \bar{x}_2 > \bar{x}_1 = \tilde{u}(\tilde{N})$$

by (6.1,6.2,6.3), thus $(\tilde{N}, \tilde{u}) \in \Gamma^{2,\emptyset}$.

By COV $\sigma(N, v) \neq \emptyset$ holds for all $(N, v) \in \Gamma^{2,\emptyset}$ with $|N \cap \bar{N}| = 1$. Repeating the same procedure (only the second case has to be considered) for a game $(N, v) \in \Gamma^{2,\emptyset}$ with $|N \cap \bar{N}| = 1$ yields Claim 1.

Claim 2: The standard solution x^v belongs to $\sigma(N, v)$ $\forall (N, v) \in \Gamma^{2,\emptyset}$

Assume $N = \{1, 2\}$ and define $(N \cup \{3, 4\}, w)$ by

$$w(S) = v(\{j \in N \mid j + 2 \in S\}).$$

By Claim 1 there exists $x \in \sigma(\{3, 4\}, w_{\{3,4\}})$, thus $y = (0, 0, x) \in \sigma(N \cup \{3, 4\}, w)$ by SNPP. However, with $u = w^{N,y}$ we come up with

$$u(\{1\}) = u(\{2\}) = \max\{v\{1\} - x_3, v\{2\} - x_4\},$$

thus $(0, 0) = y_N$ is the standard solution of $(N, u) \in \Gamma^{2,\emptyset}$, thus COV and WRGP show this claim.

Claim 3: If there is $(\bar{N}, \bar{v}) \in \Gamma^{2,\emptyset}$ and some $\bar{x} \in \sigma(\bar{N}, \bar{v})$ which is not reasonable, then there exists $(N, v) \in \Gamma^{2,\emptyset}$ such that at least one extreme point of the set of reasonable preimputations belongs to $\sigma(N, v)$.

Let $(\tilde{N}, \tilde{u})$ be defined as in Claim 1. A careful inspection of (b) of the second case implies this claim.

Claim 4: $\sigma(N, v) = \{x^v\}$ for all games $(N, v) \in \Gamma^{2,\emptyset}$.

Assume the contrary. Then, by Claims 1 and 3, there exists $(N, v) \in \Gamma^{2,\emptyset}$ and some reasonable member $y \in \sigma(N, v)$ which does not coincide with the standard solution x^v.

Without loss of generality we may assume that $N = \{1,2\}$ and $y_1 < x_1^v,\ y_2 > x_2^v$. Add an inessential player 3 with worth y_2, i.e., define $(\{1,2,3\}, w)$ by

$$w(S) = \begin{cases} v(S), & \text{if } S \subseteq N \\ v(S \cap N) + y_2, & \text{otherwise} \end{cases},$$

and define $z \in \mathbb{R}^{\{1,2,3\}}$ by $z_1 = y_1$ and $z_2 = z_3 = y_2$. Then $z \in \sigma(\{1,2,3\}, w)$ by SNPP and COV. By WRGP $z_{\{1,3\}} \in \sigma(\{1,3\}, w^{\{1,3\},z})$. Moreover, this reduced game is isomorphic to (N, v). Define $(\{1,2,3\}, u)$ by

$$u(\{2\}) = u(\{3\}) = v(\{2\}),\ u(\{2,3\}) = v(N) + v(\{2\}) - v(\{1\})$$

and

$$u(S \cup \{1\}) = u(S) + y_2 + v(\{1\}) - v(\{2\})\ \forall S \subseteq N.$$

By reasonableness and Pareto optimality of y we obtain

$$u^{N,z}(\{1\}) = \max\{u(\{1\}), u(\{1,3\}) - y_2\} = \max\{y_2 + v(\{1\}) - v(\{2\}), v(\{1\})\} = v(\{1\})$$

and

$$u^{N,z}(\{2\}) = \max\{u(\{2\}), u(\{2,3\}) - y_2\} = \max\{v(\{2\}), v(\{2\}) - v(\{1\}) + y_1\} = v(\{2\}),$$

thus $(N, u^{N,z}) = (N, v)$. Moreover, $(\{1,3\}, u^{\{1,3\},z}) = (\{1,3\}, w^{\{1,3\},z})$ and $z_{\{2,3\}}$ is the standard solution of $(\{2,3\}, u^{\{2,3\},z})$, thus $z \in \sigma(N \cup \{3\}, w)$ by Claim 2 and CRGP. Player 1 is inessential with worth $y_2 + v(\{1\}) - v(\{2\}) \neq y_1$. Hence SNPP and COV yield a contradiction.

Claim 5: It remains to show that $\sigma(N, v) \subseteq \mathcal{C}(N, v)$ for every two-person game $(N, v) \in \Gamma_U$ with a nonempty core.

Assume, on the contrary, there exists a two-person game $(\bar{N}, \bar{v})$ with $\mathcal{C}(\bar{N}, \bar{v}) \neq \emptyset$ and some $\bar{x} \in \sigma(\bar{N}, \bar{v}) \setminus \mathcal{C}(\bar{N}, \bar{v})$. Let $(\tilde{N}, \tilde{u})$ be defined as in the proof of Claim 1. A careful inspection of (b) of the first case shows that $(\bar{x}_1, 0)$ is an extreme point of the set of reasonable preimputations of $(\tilde{N}, \tilde{u})$. This observation contradicts Claim 4. q.e.d.

As in Section 4, using Lemma 6.2 instead of Lemma 4.4, we get the following result.

Theorem 6.3 *The positive prekernel is the unique solution on Γ_U that satisfies NE, COV, SNPP, WRGP, CRGP, and WUTPG.*

The following examples show that the axioms in Theorem 6.3 are logically independent. The solution σ^2 (see Example 5.3) satisfies all properties except SNPP. Moreover, σ^2 satisfies NPP, thus SNPP cannot be weakened to NPP. σ^4 and σ^5 (Examples 5.5 and 5.6) satisfy all properties except WRGP and CRGP respectively. Example 5.7 shows the independence WUTPG. The core satisfies all properties except NE. Independence of COV is shown by the following solution.

Example 6.4 *Let σ be defined by*

$$\sigma(N,v) = \{x \in X^*(N,v) \mid s_{ij}(x,v) \leq (s_{ji}(x,v))_+ \ \forall i,j \in N, \ i \neq j\}.$$

Define σ^7 by the requirements $x \in \sigma^7(N,v)$, if

(1) $x \in \sigma(N,v)$ and

(2) If there exists $\emptyset \neq S \subseteq N$ such that i is a nullplayer of $(S, v^{S,x})$, then $x_i = 0$.

The positive prekernel is a subsolution of σ^7, thus σ^7 satisfies NE and WUTPG. RGP is satisfied by definition. To show CRGP, let $x \in X(N,v)$ satisfy $x_S \in \sigma^7(S, v^{S,x})$ $\forall S \subseteq N$ with $|S| = 2$. By Pareto optimality of x, thus of x_S w.r.t. the reduced game, we have $x_S \in \mathcal{POPK}(S, v^{S,x})$. By CRGP (of $\mathcal{POPK}$) $x \in \mathcal{POPK}(N,v)$, thus $x \in \sigma^7(N,v)$. In order to show that σ^7 satisfies SNPP, it suffices to show $x \in \sigma^7(N,v)$ implies $(x,0) \in \sigma^7(N \cup \{*\}, w)$, where $(N \cup \{*\}, w)$ arises from (N,v) by adding the nullplayer $*$. Assume that there is $i \in N$ and $S \subseteq N \cup \{i\}$ with $i \in S$ such that i is a nullplayer in the reduced game $(S, w^{S,x})$. Then i is also a nullplayer in $(S \cap N, v^{S \cap N, x})$, thus SNPP follows immediately.

Of course these examples can be used to show that the properties of Lemma 6.2 are logically independent. Example 5.2 shows that this Lemma is false in the case $|U| = 3$. However, we do not know whether Theorem 6.3 remains true for a universe U of three players.

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