Special values of L-Functions of twisted automorphic representations
(Informal Notes on work in progress)

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Abstract: *We prove a formula for the values of the automorphic* L*-Function* $L(\pi \otimes \chi, k)$, π *a cuspidal representation of* $\mathrm{GL}_n(\mathbb{A})$, *as the character* χ *varies. In the case of the group* GL_2 *this formula coincides with a classical formula of A. Weil. If* π *has non-trivial cohomology, this formula has an expression in the cohomology of symmetric spaces and can therefore hopefully be used to prove the algebraicity of special* L*-values (cf [* $\mathcal{A}$ *] for the case* GL_3 *) or to construct* p*-adic* L*-functions.*

1 Introduction

Notations. We denote by $\mathbb{A}$ the Adeles of $\mathbb{Q}$ and by $\mathcal{A}(n) := \mathcal{A}(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}))$ resp. $\mathcal{A}_0(n) := \mathcal{A}_0(\mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A}))$ the space of automorphic forms resp. cusp forms on $\mathrm{GL}_n(\mathbb{A})$. By $\alpha : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ we denote the character, which sends an idele x to its norm $\alpha(x) := |x|$ and $\tau : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}^*$ denotes the additive standard character. Finally by $K_0(n-1,f) \leq \mathrm{GL}_{n-1}(\mathbb{Z}_p)$ we denote the subgroup of $k = (k_{i,j}) \in \mathrm{GL}_{n-1}(\mathbb{Z}_p)$ with $k_{n-1,i} \equiv 0\,(f),\, i \leq n-2$ and by $\mathcal{B} \leq \mathrm{GL}_n(\mathbb{Z}_p)$ we denote the Iwahori subgroup consisting of all elements $k \in \mathrm{GL}_n(\mathbb{Z}_p)$ which are congruent to an upper triangular matrix mod p. Finally let $e \in \mathrm{GL}_n$ be the unit matrix.

Let π be a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. We let $p > 2$ be a prime number and assume that the p-component π_p is unramified. Let $L(\pi, s) = \prod_\ell L(\pi_\ell, s)$ denote the automorphic L-Function attached to π.

We are interested in the values $L(\pi \otimes \chi, s)$ for fixed s (e.g. $s \in \mathbb{N}$ a critical integer), as the character $\chi : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ runs over all characters of finite order with conductor $f = f_\chi = p^e,\, e \in \mathbb{N}$ an arbitrary p-power and fixed infinity component, i.e. $\chi_\infty = \mathrm{sgn}$ or $\chi_\infty = \mathrm{id}$.

Our interest for this comes from proving the algebraicity of special values of L-functions and from p-adic interpolation of special values of L-functions.

To examine the values of the L-Function, we prove an integral formula for $L(\pi \otimes \chi, s)$ (modulo some factors, which are constant in χ) as the character χ runs as above. The idea is as follows:

We let $\sigma(\chi) := \mathrm{Ind}(\chi\alpha^{k_1}, \alpha^{k_2}, \ldots, \alpha^{k_{n-1}})$ be the unitarily induced representation. The theory of Eisenstein series provides an embedding

$$Eis : \sigma(\chi) \hookrightarrow \mathcal{A}(n-1)$$

(of course one has to convince oneself that the analytic continuation of the Eisenstein series has no pole at $\sigma(\chi)$). Then, the Rankin-Selberg convolution $L(\pi \times \sigma(\chi), s)$ attached to the pair of automorphic representations $\pi \times \sigma(\chi)$ decomposes into the product

$$L(\pi \times \sigma(\chi), s) = L(\pi \otimes \chi, s + k_1) \prod_{i=2,\dots,n-1} L(\pi, s + k_i)$$

(cf. [J-P-S 1], (9.4)). Since the factors $L(\pi, s + k_i)$, $i = 2, \dots, n-1$ are constant in χ we may use the zeta integral of the Rankin-Selberg convolution

$$I(\phi, E, s) = \int_{\mathrm{GL}_{n-1}(\mathbb{Q})\backslash \mathrm{GL}_{n-1}(\mathbf{A})} \phi(\begin{pmatrix} g & \\ & 1 \end{pmatrix}) E(g) |\det g|^{s-1/2}\, dg \qquad \phi \in \mathcal{A}_0(\pi),\ E \in Eis(\sigma(\chi))$$

to derive an integral representation for the value $L(\pi \otimes \chi, s + k_1)$. Here, by $\mathcal{A}_0(\pi)$ we understand the space of automorphic forms belonging to π . The problem is therefore to find automorphic forms $\phi_\chi \in \mathcal{A}_0(\pi)$ and $E_\chi \in Eis(\sigma(\chi))$ such that $I(\phi_\chi, E_\chi, s)$ equals $L(\pi \times \sigma(\chi), s)$.

2 A formula for the twisted L-values

Let $\phi \in \mathcal{A}_0(\pi)$ resp. $E_\chi \in Eis(\sigma(\chi))$ be automorphic forms with Whittaker functions $w = \otimes_\ell w_\ell \in W(\pi, \tau)$ resp. $v_\chi = \otimes_\ell v_{\chi,\ell} \in W(\sigma(\chi), \bar\tau)$. ($W(\pi, \tau) = \otimes_\ell W(\pi_\ell, \tau_\ell)$ and $W(\sigma(\chi), \bar\tau) = \otimes_\ell W(\sigma(\chi)_\ell, \bar\tau_\ell)$ denote the Whittaker modells of π and $\sigma(\chi)$.) Then, the zeta integral decomposes into the product

$$I(\phi, E_\chi, s) = \prod_\ell I(w_\ell, v_{\chi,\ell}, s),$$

where

$$I(w_\ell, v_{\chi,\ell}, s) = \int_{\mathrm{N}_{n-1}(\mathbb{Q}_\ell)\backslash \mathrm{GL}_{n-1}(\mathbb{Q}_\ell)} w_\ell(\begin{pmatrix} g & \\ & 1 \end{pmatrix}) v_{\chi,\ell}(g) |\det g|_\ell^{s-1/2}\, dg$$

denotes the local zeta integral, and we have to choose ϕ , E_χ such that

$$(*) \qquad\qquad I(w_\ell, v_{\chi,\ell}, s) = L(\pi_\ell \times \sigma(\chi)_\ell, s)$$

holds for all places ℓ .

We start by choosing E_χ , i.e. we have to choose a section $\psi_\chi = \otimes_\ell \psi_{\chi,\ell} \in \sigma(\chi) = \otimes_\ell \sigma(\chi)_\ell$.

(1) If $\ell \neq p, \infty$, then $\sigma(\chi)_\ell$ is unramified and we let $\psi_{\chi,\ell}$ be the spherical function normalized by $\psi_{\chi,\ell}(e) = 1$.

(2) If $\ell = p$ we define $\psi_{\chi,p}$ to be the essential vector in $\sigma(\chi)_p$. This implies by definition, that we have

$$\psi_{\chi,p}(gk) = \psi_{\chi,p}(g)\chi_p(k_{n-1,n-1}) \qquad \text{for} \quad g \in \mathrm{GL}_{n-1}(\mathbb{Q}_p),\ k \in K_0(n-1, f).$$

Denote by w the matrix

$$w = \begin{pmatrix} & & & 1 \\ & & p^e & \\ & & & \\ & p^e & & \\ p^e & & & \end{pmatrix}.$$

Then $\psi_{\chi,p}(g)$ reads as follows:

$$\psi_{\chi,p}(g) := \begin{cases} \chi_p(b_{1,1}) \prod_i \alpha_p(b_{i,i})^{k_i} \delta^{1/2}(b)\, \chi_p(k_{n-1,n-1}) & \text{for} \quad , g = bwk,\ b \in B_{n-1}(\mathbb{Q}_p),\ k \in K_0(n-1, f) \\ 0 & \text{else} \end{cases}$$

(3) At infinity we use cohomology to define ψ_∞ .

Next we choose ϕ , i.e. we choose the Whittaker function $w = \otimes_\ell w_\ell \in W(\pi, \tau)$ of ϕ .

(1) If $\ell \neq p, \infty$ we define w_ℓ to be the essential vector in $W(\pi_\ell, \tau_\ell)$.

¿From Théorème (4.1) in [J-P-S 2] we immediately deduce that with this choice (*) is fulfilled for all places $\ell \neq p, \infty$.

(2) If $\ell = p$ the construction of a Whittakerfunction $w_p \in W(\pi_p, \tau_p)$ such that (*) is satisfied now relies on the following Lemma.

Lemma. *Denote by* $w_p^1 \in W(\pi_p, \tau_p)$ *a Whittaker function which is invariant on the right by the action of the Iwahori subgroup* $\mathcal{B} \leq \mathrm{GL}_n(\mathbb{Z}_p)$ .. *Let* $\varepsilon_1, \ldots, \varepsilon_{n-1} \in \mathbb{Z}_p^*$ *be* p *-adic units and set* $\varepsilon_n := 1$. *We define the Whittaker function*

$$w_{\varepsilon_1,\ldots,\varepsilon_{n-1}}(g) := \sum_{i>j} \sum_{u_{i,j} \in f^{-(j-i)}\mathbb{Z}_p/\mathbb{Z}_p} \prod_{t=2,\ldots,n} \tau_p(-\varepsilon_t^{-1}\varepsilon_{t-1}u_{t-1,t})\, w_p^1\left(g \begin{pmatrix} 1 & & u_{i,j} \\ & \ddots & \\ & & 1 \end{pmatrix}\right),$$

where i *and* j *run over* $i \in \{1, \ldots, n-1\}$ *and* $j \in \{2, \ldots, n\}$. *Then,* $w_{\varepsilon_1,\ldots,\varepsilon_{n-1}}$ *satisfies the following properties:*
(a) Denote by $w_{\varepsilon_1,\ldots,\varepsilon_{n-1}}(\begin{pmatrix} g & \\ & 1 \end{pmatrix}), g \in \mathrm{GL}_{n-1}(\mathbb{Q}_p)$ *the restriction of* $w_{\varepsilon_1,\ldots,\varepsilon_{n-1}}$ *to the subgroup* $\mathrm{GL}_{n-1}(\mathbb{Q}_p)$ *of* $\mathrm{GL}_n(\mathbb{Q}_p)$. *Then we have*

$$\text{support}\, w_{\varepsilon_1,\ldots,\varepsilon_{n-1}}(\begin{pmatrix} g & \\ & 1 \end{pmatrix}) \subset N_{n-1}(\mathbb{Q}_p) \cdot K_{\varepsilon_1,\ldots,\varepsilon_{n-1}}(f)$$

where

$$K_{\varepsilon_1,\ldots,\varepsilon_{n-1}}(f) := \{k = (k_{i,j}) \in \mathrm{GL}_{n-1}(\mathbb{Z}_p) :\ k_{i,i} \equiv \varepsilon_i\ (f), \quad k_{i,j} \equiv 0\ (f^{i-j+1}) \quad \text{for } i > j\}.$$

(b)

$$w_{\varepsilon_1,\ldots,\varepsilon_{n-1}}(\begin{pmatrix} k & \\ & 1 \end{pmatrix}) = \prod_{h=2,\ldots,n} f^{h(h-1)/2}\, w_p^1(e) \quad \textit{for} \quad k \in K_{\varepsilon_1,\ldots,\varepsilon_{n-1}}(f).$$

(i.e. $w_{\varepsilon_1,\dots,\varepsilon_{n-1}}(\begin{pmatrix} k & \\ & 1\end{pmatrix})$ *is independent of* $k \in K_{\varepsilon_1,\dots,\varepsilon_{n-1}}(f)$ *.)*

We now define

$$w_p(g) := \sum_{\varepsilon_1,\dots,\varepsilon_{n-1}\in(\mathbb{Z}/f\mathbb{Z})^*} \chi_p^{-1}(\varepsilon_{n-1})\, w_{\varepsilon_1,\dots,\varepsilon_{n-1}}(g).$$

This implies

$$\mathrm{support}(w_p(\begin{pmatrix} g & \\ & 1\end{pmatrix})) \subset N_{n-1}(\mathbb{Q}_p)\cdot \dot{\bigcup}_{\varepsilon_1,\dots,\varepsilon_{n-1}\in(\mathbb{Z}/f\mathbb{Z})^*} K_{\varepsilon_1,\dots,\varepsilon_{n-1}}(f) \subset N_{n-1}(\mathbb{Q}_p)\cdot \mathrm{GL}_{n-1}(\mathbb{Z}_p),$$

and

$$w_p(\begin{pmatrix} k & \\ & 1\end{pmatrix}) = w_p(e)\chi_p(k_{n-1,n-1}) \qquad \text{for} \quad k \in \bigcup_{\varepsilon_1,\dots,\varepsilon_{n-1}\in(\mathbb{Z}/f\mathbb{Z})^*} K_{\varepsilon_1,\dots,\varepsilon_{n-1}}(f).$$

It is immediate by the properties of $\psi_{\chi,p}$ and $w_{\varepsilon_1,\dots,\varepsilon_{n-1}}$ that

$$I(w_p, v_{\chi,p}, s) = \mathrm{vol}(K_{1,\dots,1}(f))\, v_{\chi,p}(e)\, w_p^1(e)\, f^{\cdots}.$$

Since we have

$$L(\pi_p \otimes \sigma(\chi)_p, s) = L(\pi_p \otimes \chi_p, s+k_1) \prod_i L(\pi_p, s+k_i)$$

and $L(\pi_p \otimes \chi_p, s+k_1) = 1$ since π_p is unramified, we see, that (*) is fullfilled for the place p too (modulo some factors).

On the other hand plugging in the definition of $w_{\varepsilon_1,\dots,\varepsilon_{n-1}}$ the expression for w_p becomes

$$w_p(g) = G(\chi_p) f^{\cdots} \sum_{\substack{\varepsilon\in(\mathbb{Z}/f\mathbb{Z})^* \\ u_i\in\mathbb{Z}/p\mathbb{Z}}} \chi_p(\varepsilon)\delta(u_i) \sum_{\substack{i,j \\ j>i+1}} \sum_{u_{i,j}\in f^{-(j-i)}\mathbb{Z}_p/\mathbb{Z}_p} w_p^1\left(g \begin{pmatrix} 1 & u_1/p & & & u_{i,j} \\ & \ddots & \ddots & & \\ & & \ddots & u_{n-2}/p & \\ & & & \ddots & \varepsilon/f \\ & & & & 1 \end{pmatrix}\right)$$

where $\delta(u_i) := \sum_{\gamma\in(\mathbb{Z}/p\mathbb{Z})^*} \tau_p(\gamma u_i/p)$, i.e. $\delta(u_i) = p-1$ if $u_i \in p\mathbb{Z}_p$ and $\delta(u_i) = -1$ if $u_i \notin p\mathbb{Z}_p$. Thus, if we define $\phi^1 \in \mathcal{A}_0(n)$ to be the cusp form belonging to the Whittaker function $w = \otimes_{\ell\neq p} w_\ell \otimes w^1$ and specialize $s = 1/2$ we finally get the formula

Proposition. *For all characters* $\chi \neq \mathrm{id}$ *as above the value* $L(\pi \otimes \chi, k_1 + 1/2)$ *reads*

$$L(\pi \otimes \chi, k_1 + 1/2) = \text{some factors} \cdot P_\infty(1/2) \cdot \sum_{\substack{i,j \\ j>i+1}} \sum_{\substack{u_{i,j} \in f^{-(j-i)}\mathbb{Z}_p/\mathbb{Z}_p \\ u_1,\dots,u_{n-2} \in \mathbb{Z}_p/p\mathbb{Z}_p,\ \varepsilon \in (\mathbb{Z}/f\mathbb{Z})^*}} \chi_p(\varepsilon)\delta(u_i)$$

$$\cdot \int_{\mathrm{GL}_{n-1}(\mathbb{Q})\backslash \mathrm{GL}_{n-1}(\mathbb{A})} \phi^1 \left(\begin{pmatrix} g & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & u_1/p & & & u_{i,j} \\ & \ddots & \ddots & & \\ & & \ddots & u_{n-2}/p & \\ & & & \ddots & \varepsilon/f \\ & & & & 1 \end{pmatrix}_p \right) E_\chi(g)\, dg.$$

Here, $P_\infty \in \mathbb{C}[T]$ is a polynomial, which comes from the choice of the factors of $w = \otimes_\ell w_\ell$ and $v = \otimes_\ell v_{\chi,\ell}$ at infinity (which does not satisfy (*)) and x_p, $x \in \mathrm{GL}_n(\mathbb{Q}_p)$ denotes the embedding of x into the p-component of $\mathrm{GL}_n(\mathbb{A})$

Example. We want to look at the formula of the Proposition in the case of GL_2. Here the crucial identity for the local zeta integral at the prime p, which has to be satisfied, reads

$$I(w_p, \chi_p, s) = \int_{\mathbb{Q}_p^*} w_p(\begin{pmatrix} a & \\ & 1 \end{pmatrix})\chi_p(a)\, d^*a = L(\pi \otimes \ \chi_p, s) = 1.$$

This is fulfilled (modulo certain factors, which are constant in χ) if we set

$$w_p(g) := \sum_\varepsilon \chi_p^{-1}(\varepsilon) w_\varepsilon(g)$$

and again plugging in the definition of w_ε we find

$$w_p(g) = f^{-1} G(\chi_p) \sum_{i \in (\mathbb{Z}/f\mathbb{Z})^*} \chi_p(-i)\, w^1(g \begin{pmatrix} 1 & i/f \\ & 1 \end{pmatrix}).$$

This is now completely analogous to A. Weil's formula for the twist $f \otimes \chi := \sum a_n \chi(n)\, q^n$ of a modular form $f = \sum a_n q^n$, which reads

$$f \otimes \chi = f^{-1} G(\chi) \sum_{u \in \mathbb{Z}/f\mathbb{Z}} \chi^{-1}(u) f(z + u/f).$$

This formula was used by Mazur, to construct p-adic L-functions for classical cusp forms on the upper half plane.

Remark on the Lemma. (a) The existence of a Whittaker function satisfying the above properties follows from Proposition 2 resp. Theorem F in [Gel-Kaz] and we used the idea of their proof to construct $w_{\varepsilon_1,\dots,\varepsilon_{n-1}}$

(b) The proof of the Lemma is by induction. In fact the Lemma follows from the following Lemma', which can be proven by induction on $k \in \{2, \dots, n-1\}$.

Lemma'. *Denote by* $w_p^1 \in W(\pi_p, \tau_p)$ *a Whittaker function which is invariant on the right by the action of the Iwahori subgroup* $\mathcal{B} \leq \mathrm{GL}_n(\mathbb{Z}_p)$ *. Let* $\varepsilon_1, \dots, \varepsilon_{k-1} \in \mathbb{Z}_p^*$ *be* p *-adic units (* $k \leq n-1$ *). We define the Whittaker function*

$$w_{\varepsilon_1,\dots,\varepsilon_{k-1}}(g) := \sum_{i>j} \sum_{u_{i,j} \in f^{-(j-i)}\mathbb{Z}_p/\mathbb{Z}_p} \prod_{t=2,\dots,n} \tau_p(-\varepsilon_t^{-1}\varepsilon_{t-1}u_{t-1,t})\, w^1 \left(g \begin{pmatrix} 1 & & u_{i,j} & \\ & \ddots & & \\ & & 1 & \\ & & & \mathbf{1}_{n-k\times n-k} \end{pmatrix} \right),$$

where i *and* j *run over* $i \in \{1, \dots, k-1\}$ *and* $j \in \{2, \dots, k\}$ *. Then,* $w_{\varepsilon_1,\dots,\varepsilon_{k-1}}$ *satisfies the following properties:*

(a) Denote by $w_{\varepsilon_1,\dots,\varepsilon_{k-1}}(\begin{pmatrix} g & \\ & 1 \end{pmatrix}, g \in \mathrm{GL}_{k-1}(\mathbb{Q}_p)$ *the restriction of* $w_{\varepsilon_1,\dots,\varepsilon_{k-1}}$ *to the subgroup* $\mathrm{GL}_{k-1}(\mathbb{Q}_p)$ *of* $\mathrm{GL}_n(\mathbb{Q}_p)$ *. Then we have*

$$\text{support}\, w_{\varepsilon_1,\dots,\varepsilon_{k-1}}(\begin{pmatrix} g & \\ & 1 \end{pmatrix}) \subset N_{k-1}(\mathbb{Q}_p) \cdot K_{\varepsilon_1,\dots,\varepsilon_{k-1}}(f)$$

where

$$K_{\varepsilon_1,\dots,\varepsilon_{k-1}}(f) := \{k = (k_{i,j}) \in \mathrm{GL}_{k-1}(\mathbb{Z}_p) : \, k_{i,i} \equiv \varepsilon_i \ (f) \quad k_{i,j} \equiv 0 \ (f^{i-j+1}) \quad \text{for } i > j\}.$$

(b)

$$w_{\varepsilon_1,\dots,\varepsilon_{k-1}}(\begin{pmatrix} k & \\ & 1 \end{pmatrix}) = \prod_{h=2,\dots,k} f^{h(h-1)/2}\, w^1(e) \quad \textit{for } k \in K_{\varepsilon_1,\dots,\varepsilon_{k-1}}(f).$$

3 The relation to the cohomology of symmetric spaces

We now explain, why this formula has an interpretation in the cohomology of symmetric space of $\mathrm{GL}_n(\mathbb{R})$. To this end we have to assume, that π_∞ and $\sigma(\chi)_\infty$ have non-trivial cohomology; for $\sigma(\chi)$ this is a condition on the numbers k_i . For $n \in \mathbb{N}$ there is only one generic representation ρ_n of $\mathrm{GL}_n(\mathbb{R})$ with non-trivial cohomology (cf. [Sp]) and from [Sp, Thm. 4.2.2] or [Cl, Lemme 3.14] we deduce

$$H^{i(n)}(\mathrm{gl}_n, \mathrm{SO}_n(\mathbb{R})Z_n^0(\mathbb{R}), \rho_n) \neq 0 \quad \text{for} \quad i(n) = \begin{cases} (n/2)^2 & \text{for } n \text{ even} \\ (n-1)/2 + (n-1)^2/4 & \text{for } n \text{ odd} \end{cases}$$

Let $\omega_\phi \in H^{i(n)}_{cusp}(\tilde{S}_n, \mathbb{C})(\pi_f)$ resp. $\omega_\chi \in H^{i(n-1)}(\tilde{S}_{n-1}, \mathbb{C})(\sigma(\chi)_f)$ be the differential forms attached to w^1, and ψ_χ via the embeddings

$$W(\pi_f, \tau_f) \hookrightarrow H^{i(n)}(\tilde{S}_n, \mathbb{C})$$

and

$$\sigma(\chi)_f \hookrightarrow H^{i(n-1)}(\tilde{S}_{n-1}, \mathbb{C}).$$

where $\tilde{S}_n = \lim S_n(K)$ and $S_n(K) := \mathrm{GL}_n(\mathbb{Q})\backslash\mathrm{GL}_n(\mathbb{A})/\mathrm{SO}_n(\mathbb{R})Z_n^0(\mathbb{R})K$, $K \leq \mathrm{GL}_n(\hat{\mathbb{Z}})$. Since $i(n) + i(n-1) = \dim \mathrm{GL}_{n-1}(\mathbb{R})/\mathrm{SO}_{n-1}(\mathbb{R})$ the integral occuring in the Proposition can be expressed using Poincaré-duality as a sum of terms of the form

$$\int_{\mathrm{GL}_{n-1}(\mathbb{Q})\backslash\mathrm{GL}_{n-1}(\mathbb{A})} \omega_\phi(\begin{pmatrix} g & \\ & 1 \end{pmatrix} u) \wedge \omega_\chi(g), \quad u \in U_n(\mathbb{Q}_p).$$

The algebraicity of the L-values would now follow from the fact that the π_f resp. $\sigma(\chi)_f$ isotypical components of the cohomology groups are defined even over a number field $E/\mathbb{Q}$. Indeed, by [Cl, Théorème 3.13] we know that the representation π_f is defined over a number field $E/\mathbb{Q}$ we get an embedding

$$W_{\bar{\mathbb{Q}}}(\pi_f, \tau_f) \hookrightarrow H^{i(n)}(\tilde{S}_n, \bar{\mathbb{Q}})$$

which is defined over $\bar{\mathbb{Q}}$. For non-cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$ it is proven in [Ha 1, Theorem 2] (cf also [Ha 5] for a discussion for general n) that the above injection is defined over $\bar{\mathbb{Q}}$

$$(+) \qquad\qquad \sigma_{\bar{\mathbb{Q}}}(\chi)_f \hookrightarrow H^{i(n-1)}(\tilde{S}_{n-1}, \bar{\mathbb{Q}}).$$

(cf. [$\mathcal{A}$] for the details in the case GL_3 and the definition of the $\bar{\mathbb{Q}}$-subspaces.)

Thus assuming (+) and $\Omega(\pi) := P_\infty(1/2) \cdot$ some factors $\neq 0$, the formula of the Proposition would imply:

There exists a period $\Omega(\pi) \in \mathbb{C}^*$ *such that*

$$\frac{L(\pi \otimes \chi, k_1 + 1/2)}{\Omega(\pi)} \in \bar{\mathbb{Q}}.$$

is an algebraic number for all characters as above.

Remark. (a) For cuspidal representations of $\mathrm{GL}_3(\mathbb{A})$ with non-trivial cohomology this is proven in [$\mathcal{A}$], Cor. 3.3 using the method of comparison of intertwining operators, which goes back to G. Harder (cf. [Ha 2]) and which does not need an explicit formula as above for the values of the L-Functions. Moreover, the proof given in [$\mathcal{A}$] generalizes in a straight forward manner also to the groups GL_n (even witout assuming π_p unramified).

(b) However, it seems likely, that using (an improvement of) the formula in the proposition one should be able to construct p-adic L-Functions interpolating the automorphic L-Functions on $\mathrm{GL}_n(\mathbb{A})$ (in the case $\mathrm{GL}_2(\mathbb{A})$ Weil's formula has been used by Mazur to contruct p-adic L-Functions; see below for the case $\mathrm{GL}_3(\mathbb{A})$).

We finish by suggesting the choice of $\sigma(\chi)$, distinguished by the cases n is even or odd. By D_i we understand the discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ of lowest weight $i+1$. We have $D_i \subset \mathrm{Ind}(\alpha_\infty^{i/2}, \alpha_\infty^{-i/2})$.

3.a π a cuspidal representation of GL_n with n even. In this case we assume that π has infinity part

$$\pi_\infty = \mathrm{Ind}(D_1, D_3, \ldots, D_{n-1}),$$

i.e. π has non-trivial cohomology (cf. [Sp] or [Cl]). We fix the infinity part of χ to be $\chi_\infty = \mathrm{id}$.
We choose $\sigma(\chi)$ to be

$$\sigma(\chi) := \mathrm{Ind}(\chi, \alpha^1, \alpha^{-1}, ldots, \alpha^{\frac{n-2}{2}}, \alpha^{-\frac{n-2}{2}}).$$

Then $\sigma(\chi)_\infty$ contains the representation

$$\mathrm{Ind}(\mathrm{id}, D_2, D_4, \ldots, D_{n-2}) \subset \sigma(\chi)_\infty,$$

which has non trivial cohomology. The Rankin-Selberg convolution then factors as

$$L(\pi\otimes\sigma(\chi), 1/2) = L(\pi\otimes\chi, 1/2)L(\pi, 3/2)L(\pi, -1/2)\cdot\ldots\cdot L(\pi, (n-1)/2)L(\pi, (3-n)/2)$$

and the Proposition would yield the algebraicity of the value

$$\frac{L(\pi\otimes\chi, 1/2)}{\Omega(\pi)} \in \bar{\mathbb{Q}}$$

as χ runs over all characters with infinity component $\chi_\infty = \mathrm{id}$.

Remark. (a) $s = 1/2$ is critical for $\pi\otimes\mathrm{sgn}$ in the sense, that neither $L(\pi_\infty, s)$ nor $L(\pi_\infty, 1-s)$ have a pole at $s = 1/2$.

3.b π a cuspidal representation of GL_n with n odd. Let $\eta : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ denote an idele class character with conductor $f_\eta = p$ and infinity component $\eta_\infty = \mathrm{sgn}$, i.e. η corresponds to an odd Dirichlet character of $\mathbb{Z}/p\mathbb{Z}$.

In this case we assume that π has infinity part

$$\pi_\infty = \mathrm{Ind}(\mathrm{id}, D_2, D_4, \ldots, D_{n-1}),$$

i.e. π has non-trivial cohomology (cf. [Sp] or [Cl]). We fix the infinity part of χ to be $\chi_\infty = \mathrm{id}$.
We choose $\sigma(\chi)$ to be

$$\sigma(\chi) := \eta\otimes\mathrm{Ind}(\chi\alpha^{1/2}, \alpha^{-1/2}, \ldots, \alpha^{\frac{n-2}{2}}, \alpha^{-\frac{n-2}{2}}).$$

Then $\sigma(\chi)_\infty$ contains the representation

$$\mathrm{Ind}(D_1, D_3, \dots, D_{n-2}) \subset \sigma(\chi)_\infty,$$

(note that $\mathrm{sgn}\otimes D_i \cong D_i$) which has non trivial cohomology. The Rankin-Selberg convolution then factors as

$$L(\pi\otimes\sigma(\chi),1/2) = L(\pi\otimes\eta\chi,1)L(\pi\otimes\eta,0)L(\pi\otimes\eta,2)\cdot\ldots\cdot L(\pi\otimes\eta,(n-1)/2)L(\pi\otimes\eta,(3-n)/2)$$

and the Proposition would yield the algebraicity of the value

$$\frac{L(\pi\otimes\eta\chi,0)}{\Omega(\pi)} \in \bar{\mathbb{Q}}$$

as χ runs over all characters with infinity component $\chi_\infty = \mathrm{id}$.

Remark. (a) $s=0$ and $s=1$ are critical for $\pi\otimes\mathrm{sgn}$ in the sense, that neither $L(\pi_\infty\otimes\mathrm{sgn}, s)$ nor $L(\check{\pi}_\infty\otimes\mathrm{sgn}, 1-s)$ have a pole at $s=1/2$. They are related to each other by the functional equation $s\mapsto 1-s$.

Finally we want to give an application to p -adic interpolation on $\mathrm{GL}_3(\mathbb{A})$. Let π ba a cuspidal representation of $\mathrm{GL}_3(\mathbb{A})$ with infinity component $\pi_\infty \cong \mathrm{Ind}(\mathrm{id}, D_l)$, where $l\in 2\mathbb{Z}$. Let $\eta : \mathbb{Q}^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ be the trivial character if $l/2$ is even and the above defined character η if $l/2$ is odd. We set

$$E_{\varepsilon,p^e}(g) := \frac{2}{\phi(p^e)} \sum_{\substack{\nu:\mathbb{Q}^*\backslash\mathbb{A}^*\to\mathbb{C}^* \\ a_\nu\le e,\nu_\infty=\mathrm{id}}} \nu_p^{-1}(\varepsilon)p^{-(e-a_\nu)l/2}\, E_\nu(g\begin{pmatrix} p^{-1} & \\ & 1\end{pmatrix}^{e-a_\nu}).$$

Here, $f_\nu = p^{a_\nu}$ denotes the conductor of ν . We define

$$\mu_\pi(\varepsilon+f\mathbb{Z}_p) := \int_{\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})} \phi^1\left(\begin{pmatrix} g & \\ & 1\end{pmatrix}\begin{pmatrix} 1 & p/f & 0 \\ & 1 & \varepsilon/f \\ & & 1\end{pmatrix}_p\right) E_{1,p^e}\, dg.$$

Then μ takes values in a fixed number field and satisfies the distribution relation

$$\mu(\varepsilon+f\mathbb{Z}_p) = \alpha\, p^{4+l/2} \sum_{x=0,\dots,p-1} \mu(\varepsilon+xf+pf\mathbb{Z}_p).$$

($\alpha\in\bar{\mathbb{Q}}$ depends on π_p) and the (critical) value $L(\pi\otimes\chi\eta, 1-l/2)$ has the following expression as an integral, which is an improvement on the above proposition.

$$L(\pi\otimes\chi\eta, 1-l/2) = \text{some factors}\cdot\int_{\mathbb{Z}_p}\chi_p\eta_p^2\, d\mu.$$

To prove the boundedness of the distribution, we have to assume that π is p -ordinary, which implies that α is a p -adic unit. Only under this assumption it is possible that the following can hold.

The values $\mu_\pi(\varepsilon + f\mathbb{Z}_p) := p^{4+l/2}\,\mu(\varepsilon + f\mathbb{Z}_p)$ *are contained in a finitely generated* $\mathbb{Z}$ *-submodule of* $\mathbb{C}$ *.*

This would imply the existence of the p -adic L -Function, which interpolates the automorphic L -Function $L(\pi \otimes \chi\eta, 1 - l/2)$.

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