L^p-estimates of solutions to the singular Helmholtz equation in the half-space

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Abstract. Solutions to the Dirichlet problem for the singular Helmholtz equation

$$\Big(\Delta_{x,t} - \frac{2\nu - 1}{t}\frac{d}{dt} + k^2\Big)u(x,t) = 0$$

in the half-space $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, \ t > 0\}$ are represented by the oscillatory integrals of the Poisson type. We obtain some L^p-estimates of these integrals by using known results for the Bochner-Riesz multipliers.

1. INTRODUCTION

For $t > 0, \ x \in \mathbb{R}^n, \ k > 0,$ let

$$p_t(x) = c_n t \left(\frac{k}{\sqrt{t^2 + |x|^2}}\right)^{(n+1)/2} H^{(1)}_{(n+1)/2}(k\sqrt{t^2 + |x|^2}) \tag{1.1}$$

where $c_n = i\pi^{(1-n)/2}2^{-(n+1)/2}$, $H^{(1)}_{(n+1)/2}(z)$ is the Hankel function of the first kind. The convolution

$$u(x,t) \equiv (P_t f)(x) = (p_t * f)(x) \tag{1.2}$$

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represents a methaharmonic continuation of f into the half-space $\mathbb{R}_+^{n+1}$, i.e.

$$\Delta_{x,t}u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}_+^{n+1} \quad \text{and} \quad \lim_{t\to 0} u(x,t) = f(x)$$

for sufficiently good f. The properties of methaharmonic functions were studied in [6]. The Fourier transform of $p_t(x)$ has the form

$$\hat{p}_t(\xi) = \begin{cases} \exp(-t\sqrt{|\xi|^2 - k^2} & \text{if } |\xi| > k, \\ \exp\,(it\sqrt{k^2 - |\xi|^2}) & \text{if } |\xi| < k. \end{cases} \tag{1.3}$$

More detailed information about convolutions (1.2) with arbitrary wave number $k \in \mathbb{C}$ can be found in [3], Sec. 22.1. The case of real k is the most difficult. The following statement was announced in [3] (p. 300).

Theorem 1.1. *For $k = 1$ and $t = 1$ the operator P_t is bounded in $L^p(\mathbb{R}^n)$, $n \geq 2$, if and only if $|1/p - 1/2| < 1/n$.*

Below we prove this theorem and obtain L^p-estimates for more general analytic family of convolutions

$$u_\nu(x,t) \equiv (P_t^{(\nu)} f)(x) = (p_t^{(\nu)} * f)(x) \tag{1.4}$$

with the kernel

$$p_t^{(\nu)}(x) = \frac{i\pi^{(1-n)/2} t^{2\nu}}{2^{(n+1)/2}} \left(\frac{k}{\sqrt{t^2 + |x|^2}} \right)^{\nu + n/2} H^{(1)}_{\nu+n/2}(k\sqrt{t^2 + |x|^2}). \tag{1.5}$$

The convolutions (1.4) can be associated with the Dirichlet problem for the singular Helmholtz equation

$$\Big(\Delta_{x,t} - \frac{2\nu - 1}{t}\frac{d}{dt} + k^2\Big) u(x,t) = 0, \quad x \in \mathbb{R}^n, \quad t > 0. \tag{1.6}$$

The case $\nu = 1/2$ corresponds to Theorem 1.1. The argument, presented below, is based on the use of known results for the Bochner-Riesz multiplier

$$T_\nu f = F^{-1}[(1 - |\xi|^2)_+^\nu F f] \tag{1.7}$$

(see, e.g., [2, 4] and references therein).

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2. Preliminaries

Let $\tilde{x}=(x,t)\in\mathbb{R}^{n+1}_+$, $|\tilde{x}|=(|x|^2+t^2)^{1/2}$. By the formula 7.13.1(1) from [1],

$$p_t^{(\nu)}(x)=\frac{it^{2\nu}}{(2\pi)^{n/2}}\left(\frac{k}{|\tilde{x}|}\right)^{\nu+n/2}\sqrt{\frac{1}{k|\tilde{x}|}}\,e^{i(k|\tilde{x}|-(2\nu+n+1)\pi/4)}(1+o(|\tilde{x}|^{-1}))=$$
$$=O(|\tilde{x}|^{-\nu-(n+1)/2}),\qquad |\tilde{x}|\to\infty,\tag{2.1}$$

and therefore $p_t^{(\nu)}\in L^1$ for $Re\,\nu>(n-1)/2$. In particular, for (1.1) $(\nu=1/2)$ we have $p_t\in L^1$ for $n<2$, i.e. for $n=1$ only. Below we evaluate the Fourier transform of $p_t^{(\nu)}$.

Lemma 2.1. *Let $Re\,\nu>-1$. Then for $|\xi|\neq k$,*

$$\hat{p}_t^{(\nu)}(\xi)\stackrel{\text{def}}{=}\lim_{N\to\infty}\int\limits_{|x|<N}p_t^{(\nu)}(x)e^{ix\xi}dx=$$
$$=(t\omega)^\nu\begin{cases} i(\pi/2)^{1/2}H_\nu^{(1)}(t\omega) & \text{if } |\xi|<k,\\ (2/\pi)^{1/2}K_\nu(t\omega) & \text{if } |\xi|>k,\end{cases}\qquad \omega=|k^2-|\xi|^2|,$$

K_ν being the McDonald function. For $\nu=1/2$ this formula coincides with (1.3).

Proof. By Theorem 3.3 from [5, Chapter 4],

$$\hat{p}_t^{(\nu)}(\xi)=\frac{it^{2\nu}k^{\nu+n/2}}{\rho^{n/2-1}}\sqrt{\frac{\pi}{2}}\int\limits_0^\infty J_{n/2-1}(\rho s)\frac{H^{(1)}_{\nu+n/2}(k\sqrt{t^2+|x|^2})}{(\sqrt{t^2+|x|^2})^{\nu+n/2}}s^{n/2}ds,\qquad \rho=|\xi|.$$

This integral can be evaluated with the aid of the formula 7.14.2(48) from [1]. □

The function $\hat{p}_t^{(\nu)}(\xi)$ has an exponential decay at infinity. In order to study a local behaviour of $\hat{p}_t^{(\nu)}(\xi)$ let

$$c_\nu^{(1)}=\frac{2^\nu\Gamma(\nu)}{\pi i};\qquad c_\nu^{(2)}=\frac{e^{-i\nu\pi}2^{-\nu}\Gamma(-\nu)}{\pi i}\qquad(\nu\neq0,-1,-2,\dots).$$

According to definitions of the functions $H_\nu^{(1)}$, K_ν ([1]), for $r\to0$ we have

$$r^\nu H_\nu^{(1)}(r)=\begin{cases} c_\nu^{(1)}+o(1), & Re\,\nu>0,\\ c_\nu^{(1)}+c_\nu^{(2)}r^{2\nu}+o(1), & Re\,\nu=0,\quad \nu\neq0,\\ \dfrac{2}{\pi i}\log\dfrac{1}{r}\,(1+o(1)), & \nu=0,\\ r^{2\nu}(c_\nu^{(2)}+o(1)), & Re\,\nu<0,\quad \nu\neq-1,-2,\dots,\\ (-1)^{|\nu|}c_{|\nu|}^{(1)}r^{2\nu}(1+o(1)), & \nu=-1,-2,\dots;\end{cases}\tag{2.2}$$

$$r^\nu K_\nu(r) = \frac{\pi i}{2}\begin{cases} c_\nu^{(1)} + o(1), \quad Re\ \nu > 0, \\ c_\nu^{(1)} + c_\nu^{(2)} e^{i\nu\pi} r^{2\nu} + o(1), \quad Re\ \nu = 0, \quad \nu \neq 0, \\ \frac{2}{\pi i} \log \frac{1}{r}\ (1 + o(1)), \quad \nu = 0, \\ r^{2\nu} e^{i\nu\pi}(c_\nu^{(2)} + o(1)), \quad Re\ \nu < 0, \quad \nu \neq -1, -2, \dots, \\ (-1)^{|\nu|} c_{|\nu|}^{(1)} e^{i\nu\pi} r^{2\nu}(1 + o(1)), \quad \nu = -1, -2, \dots\ . \end{cases} \tag{2.3}$$

Owing to these relations, $\hat{p}_t^{(\nu)}(\xi)$ is a bounded function in the case $Re\ \nu \geq 0,\ \nu \neq 0$, and has a singularity on the sphere $|\xi| = k$ otherwise. Moreover,

$$\lim_{t\to 0} \hat{p}_t^{(\nu)}(\xi) = i c_\nu^{(1)} \sqrt{\pi/2} = \frac{2^{\nu - 1/2}\Gamma(\nu)}{\pi^{1/2}} \quad \text{for} \quad Re\ \nu > 0,$$

and $\lim\limits_{t\to 0} \hat{p}_t^{(\nu)}(\xi)$ does not exist (or it is infinite) for $Re\ \nu \leq 0$.

Thus we arrive at the following

Proposition 2.2 (cf. Lemma 22.5 from [3]).

(i) *For t fixed, the operator $P_t^{(\nu)}$, defined by* (1.4), *is bounded in L^2 if and only if $Re\ \nu \geq 0,\ \nu \neq 0$.*

(ii) *If $Re\ \nu > 0$, then the operator $\tilde{P}_t^{(\nu)} = (\pi^{1/2} 2^{1/2-\nu}/\Gamma(\nu)) P_t^{(\nu)}$ is an approximate identity in L^2 for $t \to 0$, i.e.*

$$\lim_{t\to 0} \|\tilde{P}_t^{(\nu)} f - f\|_2 = 0.$$

Remark 2.3. For fixed $\xi,\ |\xi| \neq k$, the function $\phi : t \to \hat{p}_t^{(\nu)}(\xi)$ satisfies the differential equation

$$\left(\frac{d^2}{dt^2} - \frac{2\nu - 1}{t} \frac{d}{dt} + k^2 - |\xi|^2 \right) \phi(t) = 0, \quad t > 0.$$

Hence in the case $Re\ \nu > 0$ the function $u(x,t) = (\tilde{P}_t^{(\nu)} f)(x)$ is a solution to the Dirichlet problem for the equation (1.6) with the boundary condition $\lim\limits_{t\to 0} u(x,t) = f(x)$. By Proposition 2.2 one can assume $f \in L^2$ and interprete this limit relation in the L^2-norm (here we are not concerned with the uniqueness problem related to the Sommerfeld condition at infinity).

Problems. For $f \in L^p$ it would be interesting to characterize the set of all p, ν, n, for which the following statements hold:

1. $\|\tilde{P}_t^{(\nu)} f\|_p \leq c(t)\|f\|_p, \quad t$ is fixed.

2. $\lim\limits_{t\to 0} \|\tilde{P}_t^{(\nu)} f - f\|_p = 0.$

3. $\|\sup_{t>0}|\tilde{P}_t^{(\nu)} f|\|_p \leq \text{const } \|f\|_p.$

4. $\lim\limits_{t\to 0} (\tilde{P}_t^{(\nu)} f)(x) = f(x)$ a.e. on $\mathbb{R}^n$.

What is the behaviour of $\|\tilde{P}_t^{(\nu)} f\|_p$ as $t \to \infty$?

A number of relevant open problems arises in studying oscillatory potentials

$$\int\limits_{\Omega} \frac{e^{ik|x-y|}}{|x-y|} f(y)dy, \quad x \in \Omega \subset \mathbb{R}^2,$$

and their fractional and multidimensional modifications (see [3], Sec. 22 for further details).

3. L^p-ESTIMATES FOR $\tilde{P}_t^{(\nu)}$.

Let for simplicity $k = 1$. Then

$$p_t^{(\nu)}(x) = \frac{c_\nu(t)}{|\tilde{x}|^{\nu+n/2}} H^{(1)}_{\nu+n/2}(|\tilde{x}|), \quad c_\nu(t) = \frac{i\pi^{(1-n)/2} t^{2\nu}}{2^{(1+n)/2}}, \quad |\tilde{x}| = \sqrt{t^2 + |x|^2}, \quad t > 0.$$

We take a function $\psi_t(r) \in C^\infty(\mathbb{R}_+)$ such that $\psi_t(r) \equiv 1$ for $r \geq 2t$ and $\psi_t(r) \equiv 0$ for $r \leq t$. Then $p_t^{(\nu)}(x) = u_t(x) + v_t(x)$ where

$$u_t(x) = \psi_t(|x|)\, p_t^{(\nu)}(x), \qquad v_t(x) = (1 - \psi_t(|x|))\, p_t^{(\nu)}(x) \in L^1. \tag{3.1}$$

By using the asymptotic relation ([1])

$$p_t^{(\nu)}(x) = c_\nu(t)\sqrt{2/\pi}\; e^{-i(2\nu+n+1)\pi/4} |\tilde{x}|^{-\nu-(n+1)/2} e^{i|\tilde{x}|} \times$$
$$\times \sum_{m=0}^{M-1} \Big(\nu + \frac{n}{2}, m\Big)(-2i|\tilde{x}|)^{-m} + c_\nu(t) O(|\tilde{x}|^{-M-\nu-(n+1)/2},$$

we have

$$u_t(x) = \sum_{m=0}^{M-1} c_m k_{t,m}(x) + R_{M,t}(x) \tag{3.2}$$

where c_m are certain coefficients, $c_0 \neq 0$,

$$k_{t,m}(x) = \psi_t(|x|)\, \frac{e^{i|\tilde{x}|}}{|\tilde{x}|^{\nu+(n+1)/2+m}}, \tag{3.3}$$

$R_{M,t}(x) = t^{2\nu}\psi_t(|x|)\, O(|\tilde{x}|^{-M-\nu-(n+1)/2})$ ($\in L^1$ if M is sufficiently large).

The function (3.3) can be transformed as follows. Put

$$\lambda = t^2/|x|^2 \ (< 1), \qquad \beta = \nu + m + (n+1)/2.$$

By the Taylor formula,

$$k_{t,m}(x) = |x|^{-\beta}(1+\lambda)^{-\beta/2} e^{i|x|\sqrt{1+\lambda}}\psi_t(|x|) =$$
$$= |x|^{-\beta} e^{i|x|}\psi_t(|x|)\Big[\sum_{l=0}^{L-1} \tilde{c}_l \lambda^l + O(\lambda^L)\Big] = \sum_{l=0}^{L-1} \tilde{c}_l k_{t,l,m}(x) + R_{t,L,m}(x)$$

where $\tilde{c}_0 = 1$,

$$k_{t,l,m}(x) = t^{2l}\psi_t(|x|)\, \frac{e^{i|x|}}{|x|^{\beta+2l}},$$

$$|R_{t,L,m}(x)| \leq \text{const}\ t^{2L}\, \frac{\psi_t(|x|)}{|x|^{\beta+L}}\Big(1 + \frac{1}{|x|} + \ldots + \frac{1}{|x|^{L-1}}\Big).$$

Hence

$$u_t(x) = \sum_{m=0}^{M-1}\sum_{l=0}^{L-1} c_{l,m}\, t^{2l+2\nu}\, \psi_t(|x|)\, \frac{e^{i|x|}}{|x|^{\nu+(n+1)/2+m+2l}}$$

(up to negligible terms), and therefore $P_t^{(\nu)}$ admits the following representation

$$P_t^{(\nu)} = t^{2\nu} \sum_{j=0}^{J-1} p_j(t) U_t^{\nu+j} + R_{J,t}. \tag{3.4}$$

Here $p_j(t)$ are polynomials of degree $\leq j$, $p_0(t) = \text{const} \neq 0$, U_t^ν is a convolution operator with the kernel

$$\psi_t(|x|)\, \frac{e^{i|x|}}{|x|^{\nu+(n+1)/2}}, \tag{3.5}$$

and $R_{J,t}$ is a certain operator which is bounded in L^p for all $p \in [1,\infty]$, $Re\ \nu > 0$, $t > 0$.

Operators U_t^ν were investigated for fixed t (e.g. $t = 1$) by many authors in connection with the Bochner-Riesz multiplier T_ν (see (1.7)). We recall, that up to a constant multiple, T_ν is represented by the convolution with the kernel

$$|x|^{-\nu-n/2} J_{\nu+n/2}(|x|) = \sqrt{\frac{2}{\pi}}\, \frac{\cos\left(|x| - (2\nu+n+1)\pi/4\right)}{|x|^{\nu+(n+1)/2}} + O(|x|^{-\nu-(n+3)/2}), \quad |x| \to \infty.$$

The asymptotics of this kernel is close to that of the expression (3.5) (but the character of the oscillation is different !). If $\nu > (n-1)/2$, then T_ν is bounded in L^p for all $p \in [1, \infty]$. In the case $0 < \nu \le (n-1)/2$ the following results are known (see, e.g. [2]).

Theorem 3.1. *Let* $1 < p < \infty, \quad n \ge 2, \quad \Delta_p = |1/p - 1/2|, \quad p_\nu = \nu/n + 1/2n.$

(i) *If* $\Delta_p \ge p_\nu$, *then* T_ν *is not bounded in* L^p.

(ii) *In the cases*

$(a) \quad n = 2$

and

$(b) \quad n \ge 3, \quad \nu > (n-1)/2(n+1),$

T_ν *is bounded in* L^p *if and only if* $\Delta_p < p_\nu$.

If $n \ge 3$, then the boundedness of T_ν in L^p for $0 < \nu \le (n-1)/2(n+1), \quad \Delta_p < p_\nu$ represents an open problem (see references in [4] concerning some progress in this case).

We observe, that although the statements (i), (ii) are usually formulated for T_ν, in fact, they were proved for the operators U_t^ν (with $t = 1$). Note also, that in the case $\nu = 0$, the operator T_ν is bounded in L^2 whereas U_t^ν is unbounded (owing to (3.4) and unboundedness of $P_t^{(\nu)}$).

Conclusion. By (3.4), Theorem 3.1 remains true if T_ν is replaced by $P_t^{(\nu)}$ (with $t > 0$ fixed). Theorem 1.1 for the methaharmonic continuation (the case $\nu = 1/2$) lies within the scope of the statement (ii) above.

An analog of the statement (i) for $P_t^{(\nu)}$ can be proved directly by considering the action of $P_t^{(\nu)}$ on the characteristic function of a sufficiently small ball.

These observations bring a small light to the problems mentioned in Section 2. It is natural to expect that the behaviour of the norm $\|P_t^{(\nu)} f\|_p$ for large t can be determined from (3.4) provided the behaviour of $\|U_t^\nu f\|_p$ for $t \to \infty$ is known.

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