Applications of Michael's Continuous Selection Theorem to Operator Extension Problems

by

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Abstract

A global approach and Michael's continuous selection theorem are used to prove a slightly improved version of the Lindenstrauss - Pełczyński extension theorem for operators from subspaces of c_0 into $C(K)$ spaces.

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1. Introduction. J. Lindenstrauss and A. Pełczyński proved in [L-P] the following result: *Let E be a subspace of c_0, let K be a compact Hausdorff space and let T be an operator from E into $C(K)$. Then for every $\varepsilon > 0$, there is an extension $\hat{T} : c_0 \to C(K)$ of T with $\|\hat{T}\| \le (1+\varepsilon)\|T\|$.*

The original proof [**?**] is based on a construction of a sequence $\{T_n\}_{n=1}^{\infty}$ of extensions $T_n : E_n \to C(K)$ of T, where $E = E_0 \subset E_1 \subset \cdots, \overline{\cup_{n=1}^{\infty} E_n} = c_0$ and $\dim(E_n/E_{n-1}) = 1$.

The following simple observation (see e.g. [Z] Proposition 2) suggests a different approach to the above extension theorem.

The Extension Criterion: Let $\varepsilon > 0$ and let E be a subspace of X. Then the following are equivalent:

(1.1) *For every compact Hausdorff space K and every operator $T : E \to C(K)$ there is an extension $\hat{T} : X \to C(K)$ with $\|\hat{T}\| \le (1+\varepsilon)\|T\|$.*

(1.2) *there is a ω^* continuous function $\varphi : Ball(E^*) \to (1+\varepsilon) Ball(X^*)$ such that $\varphi(e^*)(e) = e^*(e)$ for all $e \in E$ and $e^* \in$* $\mathrm{Ball}(E^*)$.

We include the simple proof for the sake of completeness:

Assume (1.1) and let $K = \mathrm{Ball}(E^*)$ under the ω^* topology. Let $J : E \to C(K)$ be the natural isometric embedding, i.e., $(Je)(e^*) = e^*(e)$ for every $e \in E$ and $e^* \in \mathrm{Ball}(E^*)$. By (1.1), given $\varepsilon > 0$, J admits an extension $\hat{J} : X \to C(K)$ with $\|\hat{J}\| < 1 + \varepsilon$. Identifying $e^* \in \mathrm{Ball}(E^*)$ with the corresponding point evaluation functional $\delta(e^*) \in C(K)^*$ (determined by $\delta(e^*)(f) = f(e^*)$ for all $f \in C(K)$) we obtain the desired w^* continuous function φ : $\mathrm{Ball}(E^*) \to (1+\varepsilon)\mathrm{Ball}(X^*)$ defined by $\varphi(e^*) = \hat{J}^*\delta(e^*)$. Conversely, assume (1.2) and let $T : E \to C(K)$ be an operator of norm $\|T\| = 1$. Let $\varphi_T : K \to \mathrm{Ball}(E^*)$ be defined by $\varphi_T(k)(e) = (Te)(k)$ for every $k \in K$. Then, clearly, φ_T is w^* continuous, and because $\varphi : \mathrm{Ball}(E^*) \to (1+\varepsilon)\mathrm{Ball}(X^*)$ is w^* continuous, so is the composite mapping $\psi_T = \varphi \circ \varphi_T$: $K \to (1+\varepsilon)\mathrm{Ball}(X^*)$. Now define $\hat{T} : X \to C(K)$ by $(\hat{T}x)(k) = \psi_T(k)(x)$. Then $\hat{T}$ is linear because $\psi_T(k)$ is a linear functional, $\|\hat{T}\| = \sup\{|(\hat{T}x)(k)| : \|x\| \le 1,\ k \in K\} \le \sup\{\|\psi_T(k)\|\|x\| : \|x\| \le 1,\ k \in K\| \le 1 + \varepsilon$ and $\hat{T}$ extends T because $\varphi(e^*)(e) = e^*(e)$ for all $e^* \in \mathrm{Ball}(E^*)$ and $e \in E$. □

The purpose of this note is to show that Michael's continuous selection theorem [**?**] implies the following

Theorem 1.1 *Let E be a subspace of c_0. Then for every $\varepsilon > 0$ there is a ω^* continuous function $\varphi : Ball(E^*) \to (1+\varepsilon)Ball(c_0^*)$ such that*

$$\varphi(e^*)(e) = e^*(e) \quad \textit{for every} \quad e \in E \quad \textit{and} \quad e^* \in Ball(E^*) \tag{1.3}$$

and

$$\|\varphi(e^*)\| \leq (1+\varepsilon)\|e^*\| \quad \textit{for all} \quad e^* \in Ball(E^*). \tag{1.4}$$

In view of the Extension Criterion, Theorem 1.1 clearly yields the Lindenstrauss-Pełczyński extension theorem and condition (1.4) adds the following two features to it:

Corollary 1.2 *Let T be an operator from a subspace E of c_0 into a $C(K)$ space Y. Let $G \subset K$ and assume that, for every $k \in G$ and $e \in Ball(E)$, $|(Te)(k)| \leq \eta$. Then, for every $\varepsilon > 0$, there is an extension $\hat{T} : c_0 \to Y$ of T such that $\|\hat{T}\| \leq (1+\varepsilon)\|T\|$ and $|(\hat{T}x)(k)| < (1+\varepsilon)\eta$ for all $x \in Ball(X)$ and $k \in G$.*

Corollary 1.3 *Let $X_i = c_0$ for $i = 1, 2, \ldots$ and put $X = \left(\sum_{i=1}^{\infty} \oplus X_i\right)_{l_p}$, where $1 < p < \infty$. For each $i \geq 1$ let E_i be a subspace of X_i and let $E = \left(\sum_{i=1}^{\infty} \oplus E_i\right)_{l_p}$. Let $T : E \to C(K)$ be any operator of norm 1. Then, for every $\varepsilon > 0$, T admits an extension $\hat{T} : X \to C(K)$ with $\|\hat{T}\| \leq 1 + \varepsilon$.*

Proof of Corollary 1.2.: Assume $\|T\| \leq 1$, define $\psi_T : K \to \mathrm{Ball}(E^*)$ by $\psi_T(k)(e) = (Te)(k)$ and let $\varphi : \mathrm{Ball}(E^*) \to (1+\varepsilon)\mathrm{Ball}(c_0^*)$ be the function the existence of which is guaranteed by the Theorem. Put $\psi = \varphi \circ \psi_T$ and define $\hat{T}$ by $(\hat{T}x)(k) = \psi(k)(x)$. It is easy to check that $\hat{T}$ extends T and $\|\hat{T}\| \leq (1+\varepsilon)\|T\|$. Suppose that $|(Te)(k)| \leq \eta$ for all $k \in G$ and $e \in \mathrm{Ball}(E)$ then $\|\psi_T(k)\| \leq \eta$ for $k \in G$ hence $\|\psi(k)\| = \|(\varphi \circ \psi_T)(k)\| \leq (1+\varepsilon)\|\psi_T(k)\| \leq (1+\varepsilon)\eta$ and so, for any $x \in \mathrm{Ball}(c_0)$ and $k \in G$, $|(Tx)(k)| = |\psi(k)(x)| \leq \|\psi(k)\| \leq (1+\varepsilon)\eta$. □

Proof of Corollary 1.3.: By Theorem 1.1, given $\varepsilon > 0$, for each $i \geq 1$ there is a w^* continuous function $\varphi_i : \mathrm{Ball}(E_i^*) \to (1+\varepsilon)\mathrm{Ball}(X_i^*)$ with $\varphi_i(e_i^*)(e_i) = e_i^*(e_i)$ for all $e_i^* \in \mathrm{Ball}(E_i^*)$ and $e_i \in E_i$ and $\|\varphi_i(e_i^*)\| \leq (1+\varepsilon)\|e_i^*\|$.

Since $E^* = (\sum_{i=1}^{\infty} \oplus E_i^*)_{l_q}$ and $X^* = (\sum_{i=1}^{\infty} \oplus X_i^*)_{l_q}$, where $p^{-1} + q^{-1} = 1$, we define $\varphi : \mathrm{Ball}(E^*) \to (1+\varepsilon)\mathrm{Ball}(X^*)$ as follows: If $e^* = \sum_{i=1}^{\infty} e_i^*$, where $e_i^* \in \mathrm{Ball}(E_i^*)$ and

$\sum_{i=1}^{\infty}\|e_i^*\|^q \leq 1$, put $\varphi(e^*) = \sum_{i=1}^{\infty}\varphi_i(e_i^*)$. The function φ is well defined because $\|e^*\|^q = \sum_{i=1}^{\infty}\|e_i^*\|^q$ and, for each $i \geq 1$, $\|\varphi_i(e_i^*)\|^q \leq (1+\varepsilon)^q\|e_i^*\|^q$.

It is easy to check that $\varphi(e^*)$ is w^* continuous and that $\varphi(e^*)(e) = e^*(e)$ for all $e \in E$ and $e^* \in \text{Ball}(E^*)$. Hence, the Extension Criterion ensures the existence of an extension $\hat{T} : X \to C(K)$ of T with $\|\hat{T}\| \leq 1+\varepsilon$. □

Remark 1.4. The above Corollary 1.3 can be extended to more general direct sums of c_0 in the obvious way.

The proof of Theorem 1.1. is based on E. Michael's continuous selection theorem. We consider the set function $\psi : \text{Ball}(E^*) \to (1+\varepsilon)\text{Ball}(c_0^*)$ defined by

$$(1.5) \qquad \psi(e^*) = \begin{cases} \{0\} & \text{if } e^* = 0 \\ \{x^* \in (1+\varepsilon)\text{Ball}(c_0^*) : x^* \text{ extends } e^* & \\ \text{and } \|x^*\| < (1+\varepsilon)\|e^*\|\}, & \text{if } e^* \neq 0 \end{cases}$$

Under the ω^* topology, $\psi(e^*)$ is a convex subset of the ω^* compact and metrizable set $(1+\varepsilon)\text{Ball}(c_0^*)$. Consequently, in order to prove Theorem 1.1. it suffices to prove the following

Theorem 1.5 *The carrier ψ is ω^* l.s.c.*

Indeed, if we prove that the carrier $\psi(e^*)$ is ω^* l.s.c. it then easily follows that $\bar{\psi}(e^*)$ is ω^* l.s.c., where $\bar{\psi}(e^*)$ denotes the ω^* closure of $\psi(e^*)$ for every $e^* \in \text{Ball}(E^*)$. Hence, by Michael's continuous selection theorem [?] there exists a ω^* continuous function φ : $\text{Ball}(E^*) \to (1+\varepsilon)\text{Ball}(c_0^*)$ such that, for every $e^* \in \text{Ball}(E^*)$, $\varphi(e^*) \in \bar{\psi}(e^*)$. It follows that $\varphi(e^*)$ is an extension of e^* and $\|\varphi(e^*)\| \leq (1+\varepsilon)\|e^*\|$.

Theorem 1.5 will be proved in Section 3. Let us remark that carrier ψ seems to be the first example of its kind which is ω^* l.s.c.

NOTATION: We use standard Banach space theory notation as can be found in [?].

2. Preliminaries. We start with an easy fact which allows us to reduce the extension problem to subspaces $F \subset c_0$ which are positioned in c_0 in a convenient way.

Lemma 2.1 *Let X be a Banach space with a basis $\{x_n\}_{n=1}^{\infty}$ and let F be a subspace of X. Then, for every $1 > \varepsilon > 0$, there is an automorphism J on X so that $\|I - J^{-1}\| < \varepsilon$ and so that the vectors in $J(F)$ with finite expansion with respect to the basis are dense in $J(F)$.*

Proof: We may clearly assume that $\|x_n\| = 1$ for all $n \geq 1$. Let $\{f_k\}_{k=1}^{\infty}$ be a dense sequence in Ball(F) and let $\{\delta_n\}_{n=1}^{\infty}$ be a sequence of positive numbers with $\sum\limits_{n=1}^{\infty} \delta_n < \mu^{-1}\varepsilon$, where μ is the basis' constant. We will construct by induction an increasing sequence $\{i(k)\}_{k=1}^{\infty}$ of integers and a sequence $\{y_n\}_{n=1}^{\infty} \subset X$ such that $\|y_n - x_n\| \leq \delta_n$ for all $n \geq 1$ and, for each $k \geq 1$, $f_k \in [y_n]_{n=1}^{i(k)}$. This is done as follows: Suppose that $\{i(k)\}_{k=1}^{m}$ and $\{y_n\}_{n=1}^{i(m)}$ have been chosen so that $\|y_n - x_n\| \leq \delta_n$ for $1 \leq n \leq i(m)$ and, for each $1 \leq k \leq m$, $f_k \in [y_n]_{n=1}^{i(k)}$. Clearly, if ε is small enough then $\{y_n\}_{n=1}^{i(m)} \cup \{x_n\}_{n=i(m)+1}^{\infty}$ is a basis of X. Consider f_{m+1} : if, for some N, $f_{m+1} \in \left[\{y_n\}_{n=1}^{i(m)} \cup \{x_n\}_{n=i(m)+1}^{N}\right]$ then put $i(m+1) = N$ and $y_n = x_n$ for all $i(m) < n \leq i(m+1)$. On the other hand, if $f_{m+1} = \sum\limits_{j=1}^{i(m)} a_j y_j + \sum\limits_{j=1}^{\infty} b_j x_j$ is an infinite expansion, pick $j(m+1) > i(m)$ for which $b_{j(m+1)} \neq 0$ and let $i(m+1) > j(m+1)$ be so large that $|b_{j(m+1)}|^{-1} \cdot \|\sum\limits_{h=i(m+1)}^{\infty} b_h x_h\| < \delta_{j(m+1)}$. Now define

$$y_{j(m+1)} = x_{j(m+1)} + (\sum_{h=i(m+1)+1}^{\infty} b_h x_h) b_{j(m+1)}^{-1}$$

and, for all $i(m) < n < j(m+1)$ and $j(m+1) < n < i(m+1)$ let $y_n = x_n$. It follows that $\|y_n - x_n\| \leq \delta_n$ for all $1 \leq n \leq i(m+1)$ and $f_{m+1} = \sum\limits_{j=1}^{i(m)} a_j y_j + \sum\limits_{j=i(m)+1}^{i(m+1)} b_j y_j$. This completes the induction step and the construction of y_n. Clearly, if ε is small enough, then the map J defined by $J y_n = x_n$ is an automorphism on X with $\|I - J^{-1}\| \leq \varepsilon$. Let F_0 denote the linear span of $\{f_k\}$. The construction of $\{y_n\}_{n=1}^{\infty}$ ensures that $F_0 \subset$ linear span of $\{y_n\}_{n=1}^{\infty}$ hence $J(F_0) \subset$ linear span of $\{x_n\}_{n=1}^{\infty}$. □

NOTATION: Throughout the paper $\{u_n\}_{n=1}^{\infty}$ will denote the unit vector basis of c_0, $U =$ linear span of $\{u_n\}_{n=1}^{\infty}$, and, for each $n \geq 1$, $P_n =$ the n-th basis projection and $U_n = P_n(c_0)$.

Our next step is the following

Proposition 2.2 *Let E_0 be a subspace of U and let $E_n = E_0 \cap U_n$. Then, for every $\delta > 0$ and every integer $M \geq 1$ there is an integer $N > M$ such that whenever $e \in E_0$ and $\|e\| = 1$ there is a $v \in E_0$ satisfying the following four conditions:*

$$v \in E_N \tag{2.1}$$

$$\|v\| < 1 + \delta \tag{2.2}$$

$$\|P_M(e - v)\| < \delta \tag{2.3}$$

and

$$(2.4) \qquad \|e - v\| < 1 + \delta.$$

Proof: Let $n(0)$ be the smallest integer $\geq M$ for which $E_{n(0)} \neq \{0\}$. We construct the integers $n(0) < n(1) < n(2) < \cdots$ inductively as follows: having defined $n(k)$ we let $0 < \theta < \frac{1}{4}\min\{\delta, 1\}$ and pick a $\frac{\theta}{2}$-net

$$P_{n(k)}y_1, P_{n(k)}y_2, \cdots, P_{n(k)}y_{m(k)} \quad \text{in} \quad P_{n(k)}(\mathrm{Ball}E_0).$$

For each $1 \leq j \leq m(k)$ let $c_j = \inf\{\|z\| : z \in E_0 \text{ and } \|P_{n(k)}z - P_{n(k)}y_j\| < \frac{\theta}{2}\}$.

Clearly, because $\|y_j\| \leq 1$ also $c_j \leq 1$. Choose $z_j \in E_0$ such that $\|z_j\| < (1+\frac{\theta}{2})c_j \leq 1+\frac{\theta}{2}$ and $\|P_{n(k)}(y_j - z_j)\| \leq \frac{\theta}{2}$. Then, clearly, $\{P_{n(k)}z_j\}_{j=1}^{m(k)}$ is a θ-net in $P_{n(k)}(\mathrm{Ball}E_0)$. Since each z_j is a finite linear combination of $\{u_n\}_{n=1}^{\infty}$, we can find $n(k+1) > n(k)$ so large that, for all $1 \leq j \leq m(k), (I - P_{n(k+1)})z_j = 0$. It follows that if $e \in E_0$ and $\|e\| = 1$ then there is an integer $1 \leq j(k) \leq m(k)$ so that $\|P_{n(k)}z_{j(k)} - P_{n(k)}e\| < \theta$ while the definition of c_j ensures that $\|z_{j(k)}\| \leq (1+\frac{\theta}{2})\|e\| = 1+\frac{\theta}{2}$.

We are now ready to select N and v which appear in the statement of the Proposition. Let $m > 4\delta^{-1}$, $m > k$ and put $N = n(m+1)$ and $v = m^{-1}\sum_{k=1}^{m} z_{j(k)}$. Because $N \geq n(k+1)$ and because $z_{j(k)} \in E_{n(k+1)}$ we have that $v \in E_N$. Also, each $\|z_{j(k)}\| < 1+\frac{\theta}{2}$ hence $\|v\| \leq m^{-1}\sum_{k=1}^{m}\|z_{j(k)}\| \leq 1+\frac{\theta}{2} < 1+\delta$. This establishes (2.1) and (2.2). Since $\|P_{n(k)}z_{j(k)} - P_{n(k)}e\| < \theta$ and $n(1) > M$

$$\|P_M z_{j(k)} - P_M e\| < \theta \;\text{ for all }\; 1 \leq k \leq m \;\text{ and therefore}$$

$$\|P_M(v - e)\| \leq m^{-1}\sum_{k=1}^{m}\|P_M(z_{j(k)} - e)\| < \theta < \delta, \;\text{ proving (2.3)}$$

Finally, to prove (2.4), we use an argument of E. Odell: Note that each $z_{j(k)}$ can be expressed by $P_{n(k)}z_{j(k)} + (P_{n(k+1)} - P_{n(k)})z_{j(k)}$ where

$$\|(P_{n(k+1)} - P_{n(k)})z_{j(k)}\| < 1 + \frac{\theta}{2} \;\text{ and }\; \|P_{n(k)}z_{j(k)} - P_{n(k)}e\| < \theta.$$

The elements $\{(P_{n(k+1)} - P_{n(k)})z_{j(k)}\}_{k=1}^{m}$ are disjointly supported with respect to the natural basis of c_0 therefore $\|\sum_{k=1}^{m}(P_{n(k+1)} - P_{n(k)})z_{j(k)}\| = \max\limits_{1\leq k\leq m}\|(P_{n(k+1)} - P_{n(k)})z_{j(k)}\|$. It follows that

$$\|e - v\| \;=\; \|m^{-1}\sum_{k=1}^{m}(e - z_{j(k)})\| \leq$$

$$
\begin{aligned}
&\leq \|m^{-1}\sum_{k=1}^{m}(e - P_{n(k)}z_{j(k)})\| + \|m^{-1}\sum_{k=1}^{m}(P_{n(k+1)} - P_{n(k)})z_{j(k)}\| \\
&\leq m^{-1}\sum_{k=1}^{m}\|P_{n(k)}(e - z_{j(k)})\| + m^{-1}\|\sum_{k=1}^{m}(I - P_{n(k)})e\| + m^{-1}\max_{1\leq k\leq m}\|z_{j(k)}\| \\
&\leq \theta + 1 + m^{-1}(1+\theta) < 1+\delta
\end{aligned}
$$

□

An immediate consequence of Proposition 2.2 is the following

Corollary 2.3 *Under the assumptions of Proposition* 2.2, *given* $\eta > 0$ *and an integer* M *there is an integer* N *such that whenever* $e^* \in E_0^*, \|e^*\| = 1$ *and* $\|e^*|_{E_N}\| < \frac{\eta}{3}$, *any Hahn Banach extension* $y^* \in c_0^*$ *of* e^* *satisfies the inequality*

$$
\|P_M^* y^*\| < \eta. \tag{2.7}
$$

Proof: Let $\delta = \frac{\eta}{6}$ and choose an integer $N > M$ so that the conclusion of Proposition 2.2 holds. Let $e^* \in E_0^*$ be a functional of norm 1 and assume that $\|e^*|_{E_N}\| < \frac{\eta}{3}$. Pick $e \in \mathrm{Ball}E_0$ for which $e^*(e) > 1 - \frac{\eta}{3}$. By Proposition 2.2, there is a $v \in E_N$ so that conditions (2.1) - (2.4) hold. It follows from (2.1) and (2.2) that $e^*(e-v) = e^*(e) - e^*(v) \geq 1 - \frac{\eta}{3} - \frac{\eta}{3}(1+\delta) > 1 - \frac{2\eta}{3}(1+\frac{\eta}{6})$. Let $y^* \in c_0^*$ be any Hahn-Banach extension of e^*, then $y^*(e-v) = e^*(e-v) > 1-\delta$. Since $\|P_M^* y^*\| + \|(I-P_M^*)y^*\| = \|y^*\| = 1$, if $\|P_M^* y^*\| > \eta$ then $\|(I-P_M^*)y^*\| < 1-\eta$ and we get by (2.3) the contradiction

$$
\begin{aligned}
1 - 2\left(1+\frac{\eta}{6}\right)\frac{\eta}{3} &\leq y^*(e-v) \leq |P_M y^*(e-v)| + |(I-P_M^*)y^*(e-v)| \\
&\leq \|P_M(e-v)\| + \|(I-P_M^*)(y^*)\|\|e-v\| \\
&\leq \frac{\eta}{6} + (1-\eta)\|e-v\| \\
&\leq \frac{\eta}{6} + (1-\eta)(1+\frac{\eta}{6}) \\
&\leq 1 + \frac{\eta}{3} - \eta(1+\frac{\eta}{6}).
\end{aligned}
$$

□

3. Proof of Theorem 1.5. In view of Lemma 2.1 we may assume that $E_0 = E \cap$ (linear span of $\{u_n\}_{n=1}^{\infty}$) is dense in E. We must prove that the carrier ψ is ω^* l.s.c. Let us start by establishing the ω^* lower semicontinuity of ψ at 0. Recall that $\psi(0) = \{0\}$ and let V be any ω^* neighborhood of 0 in $(1+\varepsilon)\mathrm{Ball}(c_0^*)$. Then, there exist $\eta > 0$ and M so that

$(1+\varepsilon)\mathrm{Ball}(c_0^*) \cap \left[\eta(1+\varepsilon)\mathrm{Ball}(c_0^*) + ([u_i]_{i=1}^M)^\perp\right] \subset V$. Choose N so large that the conclusion of Corollary 2.3 holds. Now let $e^* \varepsilon (\mathrm{Ball}(E_0^*)) \cap (\frac{\eta}{3}\mathrm{Ball}(E_0^*) + E_N^\perp)$ and choose any Hahn-Banach extension $y^* \in c_0^*$ of e^*. By Corollary 2.3, $y^* \in V \cap \psi(e^*)$. This establishes the ω^* lower semi continuity of ψ at 0.

Let us now show that ψ is ω^* l.s.c. at $e_0^* \neq 0$. Given a ω^* neighborhood V of 0 in $\lambda\mathrm{Ball}(c_0^*)$, where $\lambda = 1+\varepsilon$, we may assume that there is a $\beta > 0$ so that

$$V = (\lambda\,\mathrm{Ball}(c_0^*)) \cap \left(\beta\,\lambda\,\mathrm{Ball}(c_0^*) + ([u_i]_{i=1}^M)^\perp\right).$$

Pick $x_0^* \in \psi(e_0^*)$, then $\|x_0^*\| < \lambda\|e_0^*\|$. Let $0 < \delta < 20^{-1}\min\{\lambda\|e_0^*\| - \|x_0^*\|, \beta, 1\}$ and pick $e_0 \in \mathrm{Ball}(E_0)$ so that $e_0^*(e_0) > (1-\delta)\|e_0^*\|$. Choose M so large that $P_M e_0 = e_0$ and $\sum_{i=M+1}^\infty |x_0^*(u_i)| = \|(I - P_M^*)x_0^*\| < \delta$. Choose $N > M$ so large that the conclusions of Proposition 2.2 and Corollary 2.3 with $\delta\|e_0^*\| = \eta$ hold. Let

$$e^* \in (\mathrm{Ball}(E_0^*)) \cap \left(\frac{\eta}{3}\mathrm{Ball}(E_0^*) + E_N^\perp\right)$$

and assume that $\|e_0^* + e^*\| \leq 1$. Pick $e \in \mathrm{Ball}(E_0)$ so that $e^*(e) > (1-\eta)\|e^*\|$. Using Proposition 2.3 we find $v \in E_N$ so that conditions (2.1) – (2.4) are satisfied and let $x^* \in c_0^*$ be any Hahn-Banach extension of e^*. By Corollary 2.3, $\|P_M^* x^*\| < \delta\|e_0^*\|$. We use here the fact that e_0 and $e - v$ are almost disjointly supported as elements of c_0 and x_0^* and x^* are almost disjointly supported as elements of l_1. We have that $\|e_0 + e - v\| \leq 1 + 2\delta$ and $(1+2\delta)\|e_0^* + e^*\| \geq (e_0^* + e^*)(e_0 + e - v)) \geq e_0^*(e) + e^*(e - v) - \delta\|e_0^*\| - \delta\|e^*\| \geq \|e_0^*\|(1-2\delta) + \|e^*\|(1-2\delta)$. This yields

$$\|e^*\| \leq (1-2\delta)^{-1}((1+2\delta)\|e_0^* + e^*\| - (1-2\delta)\|e_0^*\|)\,. \tag{3.1}$$

We also have

$$\|e_0^* + e^*\| \geq (e_0^* + e^*)(e_0) \geq (1-\delta)\|e_0^*\| - \delta\|e_0^*\| = (1-2\delta)\|e_0^*\|\,. \tag{3.2}$$

Clearly, $x_0^* + x^*$ extends $e_0^* + e^*$ and, by Corollary 2.3, $x^* \in V$. Moreover, by (3.1) and (3.2),

$$\begin{aligned}
\|x_0^*\| + \|x^*\| &< \lambda\|e_0^*\| - 20\delta + \|e^*\| \\
&\leq (\lambda - 20\delta)\|e_0^*\| + (1+4\delta)[(1+4\delta)\|e_0^* + e^*\| - (1-2\delta)\|e_0^*\|] \\
&\leq (1+4\delta)^2\|e_0^* + e^*\| + [\lambda - 20\delta - (1 + 2\delta - 8\delta^2)]\,\|e_0^*\| \\
&\leq [(1+4\delta)^2 + (\lambda - 1 - 20\delta)(1+4\delta)]\,\|e_0^* + e^*\| < \lambda\|e_0^* + e^*\|\,.
\end{aligned}$$

It follows that $x_0^* + x^* \in \psi(e_0^* + e^*)$ and hence ψ is w^* l.s.c. at e_0^*. □

4. Concluding remarks.

Remark 4.1: This work was done a long time ago but has not been prepared for publication until now. Theorem 1.1 was used, unfortunately without any explanation or reference, in the proof of Theorem 4 of [**?**].

Remark 4.2: Let E be a subspace of c_0 and let X be any separable space containing E. Our proof can be slightly modified to show that, for every $\varepsilon > 0$, the carrier $\psi : \mathrm{Ball}(E^*) \to (2+\varepsilon)\,\mathrm{Ball}(X^*)$ defined by $\psi(0) = \{0\}$ and, for $e^* \neq 0$, $\psi(e^*) = \{x^* \in (2+\varepsilon)\,\mathrm{Ball}(X^*) : x^*$ extends e^* and $\|x^*\| < (2+\varepsilon)\|e^*\|\}$ is ω^* l.s.c. However, this does not provide us with any new information beyond Corollary 2 of [**?**] and the corresponding additional features analogous to Corollary 1.2 and Corollary 1.3.

The proof is similar to that of Theorem 1.5 but uses the following corollary of Proposition 2.2.

Corollary 4.3 *Under the assumptions of Proposition* 2.2*, for any* $1 > \mu$*,* $\varepsilon > 0$ *and every* $0 \neq e_0^* \in Ball(E_0^*)$ *there is a* $\delta > 0$ *and an integer* N *so that if* $e^* \in Ball(E_0^*) \cap (\delta Ball(E_0^*) + E_N^\perp)$ *and* $\|e_0^* + e^*\| \leq 1$ *then*

$$\|e^*\| \leq (1+\varepsilon)(\|e_0^* + e^*\| - (1-\mu)\|e_0^*\|) \tag{2.5}$$

and

$$\|e_0^*\| \leq (1-\varepsilon)^{-1}\|e_0^* + e^*\| \,. \tag{2.6}$$

Proof: Let $\delta > 0$ be so small that $(1+\delta)(1-\delta)^{-1} < 1+\varepsilon, \delta < \frac{1}{2}\varepsilon\|e_0^*\|$ and $(1+\delta)^{-1}((1-\delta)\|e_0^*\| - 5\delta) > (1-\mu)\|e_0^*\|$ and find $e_0 \in \mathrm{Ball}E_0$ so that $e_0^*(e_0) > \|e_0^*\|(1-\delta)$. Pick M so large that $e_0 \in E_M$. Now choose $N > M$ so large that the conclusion of Proposition 2.2 holds. Since $e^* \in (\mathrm{Ball}E_0^*) \cap (\delta\mathrm{Ball}E_0^* + E_N^\perp)$ we have that $|e^*(e_0)| < \delta$ and so

$$\|e_0^* + e^*\| \geq |(e_0^* + e^*)(e_0)| \geq e_0^*(e_0) - \delta \geq \|e_0^*\| - 2\delta > \|e_0^*\|(1-\varepsilon)$$

which proves (2.6). To prove (2.5), let $e \in \mathrm{Ball}E_0$ be an element satisfying $e^*(e) > \|e^*\|(1-\delta)$. Let $v \in E_N$ be an element satisfying conditions (2.1)- (2.4). It follows that

$$e^*(e-v) = e^*(e) - e^*(v) \geq \|e^*\|(1-\delta) - \delta(1+\delta)\,,$$

$$|e_0^*(e-v)| < \delta + \delta(1+\delta)\,,$$

$$\|e_0 + e - v\| \leq 1 + \delta$$

and hence

$$\begin{aligned}(1+\delta)\|e_0^* + e^*\| &> (e_0^* + e^*)(e_0 + e - v) \geq \\ &\geq e_0^*(e_0) + e^*(e - v) - \delta(3+\delta) \\ &\geq \|e_0^*\|(1-\delta) + \|e^*\|(1-\delta) - 5\delta.\end{aligned}$$

The choice of δ ensures that

$$(1+\delta)\|e_0^* + e^*\| > (1-\mu)(1+\delta)\|e_0^*\| + \|e^*\|(1-\delta)$$

and therefore,

$$\begin{aligned}\|e^*\| &\leq (1-\delta)^{-1}(1+\delta)(\|e_0^* + e^*\| - (1-\mu)\|e_0^*\|) \\ &\leq (1+\varepsilon)(\|e_0^* + e^*\| - (1-\mu)\|e_0^*\|)\end{aligned}$$

□

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