THE MINIMAL CARDINALITY WHERE THE REZNICHENKO PROPERTY FAILS

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Abstract. A topological space X has the Fréchet-Urysohn property if for each subset A of X and each element x in $\overline{A}$, there exists a countable sequence of elements of A which converges to x. Reznichenko introduced a natural generalization of this property, where the converging sequence of elements is replaced by a sequence of disjoint finite sets which eventually intersect each neighborhood of x. In [5], Kočinac and Scheepers conjecture:

> The minimal cardinality of a set X of real numbers such that $C_p(X)$ does not have the weak Fréchet-Urysohn property is equal to $\mathfrak{b}$.

($\mathfrak{b}$ is the minimal cardinality of an unbounded family in the Baire space ${}^{\mathbb{N}}\mathbb{N}$). We prove the Kočinac-Scheepers conjecture by showing that if $C_p(X)$ has the Reznichenko property, then a continuous image of X cannot be a subbase for a non-feeble filter on $\mathbb{N}$.

1. Introduction

A topological space X has the Fréchet-Urysohn property if for each subset A of X and each $x \in \overline{A}$, there exists a sequence $\{a_n\}_{n\in\mathbb{N}}$ of elements of A which converges to x. If $x \notin A$ then we may assume that the elements a_n, $n \in \mathbb{N}$, are distinct. The following natural generalization of this property was introduced by Reznichenko [7]:

> For each subset A of X and each element x in $\overline{A}\setminus A$, there exists a countably infinite pairwise disjoint collection $\mathcal{F}$ of finite subsets of A such that for each neighborhood U of x, $U \cap F \neq \emptyset$ for all but finitely many $F \in \mathcal{F}$.

In [7] this is called the *weak Fréchet-Urysohn* property. In other works [5, 6, 10] this also appears as the *Reznichenko* proeprty.

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For a topological space X denote by $C_p(X)$ the space of continuous real-valued functions with the topology of pointwise convergence. A comprehensive duality theory was developed by Arkhangel'skiĭ and others (see, e.g., [2, 9, 5, 6] and references therein) which characterizes topological properties of $C_p(X)$ for a Tychonoff space X in terms of covering properties of X. In [5, 6] this is done for a conjunction of the Reznichenko property and some other classical property (countable strong fan tightness in [5] and countable fan tightness in [6]). According to Sakai [9], a space X has countable fan tightness if for each $x \in X$ and each sequence $\{A_n\}_{n\in\mathbb{N}}$ of subsets of X with $x \in \overline{A_n} \setminus A_n$ for each n, there exist finite sets $F_n \subseteq A_n$, $n \in \mathbb{N}$, such that $x \in \overline{\bigcup_n F_n}$. In Theorem 19 of [6], Kočinac and Scheepers prove that for a Tychonoff space X, $C_p(X)$ has countable fan tightness as well as Reznichenko's property if, and only if, each finite power of X has the Hurewicz covering property.

The *Baire space* ${}^{\mathbb{N}}\mathbb{N}$ of infinite sequences of natural numbers is equipped with the product topology (where the topology of $\mathbb{N}$ is discrete). A quasiordering $\leq^*$ is defined on the Baire space ${}^{\mathbb{N}}\mathbb{N}$ by eventual dominance:

$$f \leq^* g \quad \text{if} \qquad f(n) \leq g(n) \text{ for all but finitely many } n.$$

We say that a subset Y of ${}^{\mathbb{N}}\mathbb{N}$ is *bounded* if there exists g in ${}^{\mathbb{N}}\mathbb{N}$ such that for each $f \in Y$, $f \leq^* g$. Otherwise, we say that Y is *unbounded*. $\mathfrak{b}$ denotes the minimal cardinality of an unbounded family in ${}^{\mathbb{N}}\mathbb{N}$. According to a theorem of Hurewicz [3], a set of reals X has the Hurewicz property if, and only if, each continuous image of X in ${}^{\mathbb{N}}\mathbb{N}$ is bounded. This and the preceding discussion imply that for each set of reals X of cardinality smaller than $\mathfrak{b}$, $C_p(X)$ has the Reznichenko property. Kočinac and Scheepers conclude their paper [5] with the following.

Conjecture 1. $\mathfrak{b}$ *is the minimal cardinality of a set X of real numbers such that $C_p(X)$ does not have the Reznichenko property.*

We prove that this conjecture is true.

2. A proof of the Kočinac-Scheepers conjecture

Throughout the paper, when we say that $\mathcal{U}$ is a *cover* of X we mean that $X \subseteq \cup\mathcal{U}$ but X is not contained in any member of $\mathcal{U}$. A cover $\mathcal{U}$ of a space X is an *ω-cover* of X if each finite subset F of X is contained in some member of $\mathcal{U}$. This notion is due to Gerlits and Nagy [2], and is starring in [5, 6]. According to [5, 6], a cover $\mathcal{U}$ of X is *ω-groupable* if there exists a partition $\mathcal{P}$ of $\mathcal{U}$ into finite sets such that for each finite $F \subseteq X$ and all but finitely many $\mathcal{F} \in \mathcal{P}$, there exists $U \in \mathcal{F}$ such that

$F \subseteq U$. Thus, each *ω-groupable* cover is an ω-cover and contains a countable ω-groupable cover.

In [6] it is proved that if each open ω-cover of a set of reals X is ω-groupable and $C_p(X)$ has countable fan tightness, then $C_p(X)$ has the Reznichenko property. Recently, Sakai [10] proved that the assumption of countable fan tightness is not needed here. More precisely, say that an open ω-cover $\mathcal{U}$ of X is *ω-shrinkable* if for each $U \in \mathcal{U}$ there exists a closed subset $C_U \subseteq U$ such that $\{C_U : U \in \mathcal{U}\}$ is an ω-cover of X. Then the following duality result holds.

Theorem 2 (Sakai [10]). *For a Tychonoff space X, the following are equivalent:*

(1) *$C_p(X)$ has the Reznichenko property;*
(2) *Each ω-shrinkable open ω-cover of X is ω-groupable.*

It is the other direction of this result that we are interested in here. Observe that any *clopen* ω-cover is trivially ω-shrinkable.

Corollary 3. *Assume that X is a Tychonoff space such that $C_p(X)$ has the Reznichenko property. Then each clopen ω-cover of X is ω-groupable.*

From now on X will always denote a set of reals. As all powers of sets of reals are Lindelöf, we may assume that all covers we consider are countable [2]. For conciseness, we introduce some notation. For collections of covers of X $\mathfrak{U}$ and $\mathfrak{V}$, we say that X satisfies $\binom{\mathfrak{U}}{\mathfrak{V}}$ (read: $\mathfrak{U}$ choose $\mathfrak{V}$) if each element of $\mathfrak{U}$ contains an element of $\mathfrak{V}$ [14]. Let C_Ω and $C_{\Omega^{gp}}$ denote the collections of clopen ω-covers and clopen ω-groupable covers of X, respectively. Corollary 3 says that the Reznichenko property for $C_p(X)$ implies $\binom{C_\Omega}{C_{\Omega^{gp}}}$.

As a warm up towards the real solution, we make the following observation. According to [11], a space X satisfies $\mathsf{Split}(\mathfrak{U}, \mathfrak{V})$ if every cover $\mathcal{U} \in \mathfrak{U}$ can be split into two disjoint subcovers $\mathcal{V}$ and $\mathcal{W}$ which contain elements of $\mathfrak{V}$. Observe that $\binom{C_\Omega}{C_{\Omega^{gp}}}$ implies $\mathsf{Split}(C_\Omega, C_\Omega)$. The *critical cardinality* of a property $\mathbf{P}$ (or collection) of sets of reals, $\mathsf{non}(\mathbf{P})$, is the minimal cardinality of a set of reals which does not satisfy this property. Write

$$\mathfrak{rez} = \mathsf{non}(\{X : C_p(X) \text{ has the Reznichenko property}\}).$$

Then we know that $\mathfrak{b} \leq \mathfrak{rez}$, and the Kočinac-Scheepers conjecture asserts that $\mathfrak{rez} = \mathfrak{b}$. By Corollary 3, $\mathfrak{rez} \leq \mathsf{non}(\mathsf{Split}(C_\Omega, C_\Omega))$. In [4] it is proved that $\mathsf{non}(\mathsf{Split}(C_\Omega, C_\Omega)) = \mathfrak{u}$, where $\mathfrak{u}$ is the *ultrafilter number* denoting the minimal size of a base for a nonprincipal ultrafilter on $\mathbb{N}$. Consequently, $\mathfrak{rez} \leq \mathfrak{u}$. It is well known that $\mathfrak{b} \leq \mathfrak{u}$, but it is consistent that $\mathfrak{b} < \mathfrak{u}$. Thus this does not prove the conjecture. However, this is the approach that we will use: We will use the language of filters to prove that $\mathsf{non}(\binom{C_\Omega}{C_{\Omega^{gp}}}) = \mathfrak{b}$. By Corollary 3, $\mathfrak{b} \le \mathfrak{rez} \le \mathsf{non}(\binom{C_\Omega}{C_{\Omega^{gp}}})$, so this will suffice.

A *nonprincipal filter* on $\mathbb{N}$ is a family $\mathcal{F} \subseteq P(\mathbb{N})$ that contains all cofinite sets but not the empty set, is closed under supersets, and is closed under finite intersections (in particular, all elements of a nonprincipal filter are infinite). A *base* $\mathcal{B}$ for a nonprincipal filter $\mathcal{F}$ is a subfamily of $\mathcal{F}$ such that for each $A \in \mathcal{F}$ there exists $B \in \mathcal{B}$ such that $B \subseteq A$. If the closure of $\mathcal{B}$ under finite intersections is a base for a nonprincipal filter $\mathcal{F}$, then we say that $\mathcal{B}$ is a *subbase* for $\mathcal{F}$. A family $\mathcal{Y} \subseteq P(\mathbb{N})$ is *centered* if for each finite subset $\mathcal{A}$ of $\mathcal{Y}$, $\cap\mathcal{A}$ is infinite. Thus a subbase $\mathcal{B}$ for a nonprincipal filter is a centered family such that for each n there exists $B \in \mathcal{B}$ with $n \notin B$. For a nonprincipal filter $\mathcal{F}$ on $\mathbb{N}$ and a finite-to-one function $f : \mathbb{N} \to \mathbb{N}$, $f(\mathcal{F}) := \{A \subseteq \mathbb{N} : f^{-1}[A] \in \mathcal{F}\}$ is again a nonprincipal filter on $\mathbb{N}$.

A filter $\mathcal{F}$ is *feeble* if there exists a finite-to-one function f such that $f(\mathcal{F})$ consists of only the cofinite sets. $\mathcal{F}$ is feeble if, and only if, there exists a partition $\{F_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ into finite sets such that for each $A \in \mathcal{F}$, $A \cap F_n \neq \emptyset$ for all but finitely many n (take $F_n = f^{-1}[\{n\}]$). Thus $\mathcal{B}$ is a subbase for a feeble filter if, and only if:

(1) $\mathcal{B}$ is centered,
(2) For each n there exists $B \in \mathcal{B}$ such that $n \notin B$; and
(3) There exists a partition $\{F_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ into finite sets such that for each k and each $A_1, \dots, A_k \in \mathcal{B}$, $A_1 \cap \cdots \cap A_k \cap F_n \neq \emptyset$ for all but finitely many n.

Define a topology on $P(\mathbb{N})$ by identifying it with *Cantor's space* ${}^{\mathbb{N}}\{0,1\}$ (which is equipped with the product topology).

Theorem 4. *For a set of reals X, the following are equivalent:*

(1) *X satisfies $\binom{C_\Omega}{C_{\Omega^{gp}}}$;*
(2) *For each continuous function $\Psi : X \to P(\mathbb{N})$, $\Psi[X]$ is not a subbase for a non-feeble filter on $\mathbb{N}$.*

Proof. ($1 \Rightarrow 2$) Assume that $\Psi : X \to P(\mathbb{N})$ is continuous and $\mathcal{B} = \Psi[X]$ is a subbase for a nonprincipal filter $\mathcal{F}$ on $\mathbb{N}$. Consider the (clopen!) subsets $O_n = \{A \subseteq \mathbb{N} : n \in A\}$, $n \in \mathbb{N}$, of $P(\mathbb{N})$. For each n, there exists $B \in \mathcal{B}$ such that $B \notin O_n$ ($n \notin B$), thus $X \not\subseteq \Psi^{-1}[O_n]$.

As $\mathcal{B}$ is centered, $\{O_n\}_{n\in\mathbb{N}}$ is an ω-cover of $\mathcal{B}$, and therefore $\{\Psi^{-1}[O_n]\}_{n\in\mathbb{N}}$ is a clopen ω-cover of X. Let $A \subseteq \mathbb{N}$ be such that the enumeration $\{\Psi^{-1}[O_n]\}_{n\in A}$ is bijective. Apply $\binom{C_\Omega}{C_{\Omega^{gp}}}$ to obtain a partition $\{F_n\}_{n\in\mathbb{N}}$ of A into finite sets such that for each finite $F \subseteq X$, and all but finitely many n, there exists $m \in F_n$ such that $F \subseteq \Psi^{-1}[O_m]$ (that is, $\Psi[F] \subseteq O_m$, or $\bigcap_{x\in F}\Psi(x)\cap F_n \neq \emptyset$). Add to each F_n an element from $\mathbb{N}\setminus A$ so that $\{F_n\}_{n\in\mathbb{N}}$ becomes a partition of $\mathbb{N}$. Then the sequence $\{F_n\}_{n\in\mathbb{N}}$ witnesses that $\mathcal{B}$ is a subbase for a *feeble* filter.

($2 \Rightarrow 1$) Assume that $\mathcal{U} = \{U_n\}_{n\in\mathbb{N}}$ is a clopen ω-cover of X. Define $\Psi : X \to P(\mathbb{N})$ by

$$\Psi(x) = \{n : x \in U_n\}.$$

As $\mathcal{U}$ is clopen, Ψ is continuous. As $\mathcal{U}$ is an ω-cover of X, $\mathcal{B} = \Psi[X]$ is centered (see Lemma 2.2 in [13]). For each n there exists $x \in X \setminus U_n$, thus $n \notin \Psi(x)$. Therefore $\mathcal{B}$ is a subbase for a feeble filter. Fix a partition $\{F_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ into finite sets such that for each $\Psi(x_1),\dots,\Psi(x_k) \in \mathcal{B}$, $\Psi(x_1)\cap\cdots\cap\Psi(x_k)\cap F_n \neq \emptyset$ (that is, there exists $m \in F_n$ such that $x_1,\dots,x_k \in U_m$) for all but finitely many n. This shows that $\mathcal{U}$ is groupable. $\square$

Corollary 5. $\mathsf{non}(\binom{C_\Omega}{C_{\Omega^{gp}}}) = \mathfrak{b}$.

Proof. Every nonprincipal filter on $\mathbb{N}$ with a (sub)base of cardinality smaller than $\mathfrak{b}$ is feeble (essentially, [12]), and by an unpublished result of Petr Simon, there exists a non-feeble filter with a (sub)base of cardinality $\mathfrak{b}$ – see [1] for the proofs. Use Theorem 4. $\square$

This completes the proof of the Kočinac-Scheepers conjecture.

3. Consequences and open problems

Let $\mathcal{B}_\Omega$ and $\mathcal{B}_{\Omega^{gp}}$ denote the collections of *countable Borel* ω-covers and ω-groupable covers of X, respectively. The same proof as in Theorem 4 shows that the analogue theorem where "continuous" is replaced by "Borel" holds.

$\mathcal{U}$ is a *large* cover of a space X if each member of X is contained in infinitely many members of $\mathcal{U}$. Let $\mathcal{B}_\Lambda$, Λ, and C_Λ denote the collections of countable large Borel, open, and clopen covers of X, respectively. According to [6], a large cover $\mathcal{U}$ of X is *groupable* if there exists a partition $\mathcal{P}$ of $\mathcal{U}$ into finite sets such that for each $x \in X$ and all but finitely many $\mathcal{F} \in \mathcal{P}$, $x \in \cup\mathcal{F}$. Let $\mathcal{B}_{\Lambda^{gp}}$, Λ^{gp}, and $C_{\Lambda^{gp}}$ denote the collections of countable groupable Borel, open, and clopen covers of X, respectively.

Corollary 6. *The critical cardinalities of the classes* $\binom{\mathcal{B}_\Lambda}{\mathcal{B}_{\Lambda^{gp}}}$, $\binom{\mathcal{B}_\Omega}{\mathcal{B}_{\Omega^{gp}}}$, $\binom{\mathcal{B}_\Omega}{\mathcal{B}_{\Lambda^{gp}}}$, $\binom{\Lambda}{\Lambda^{gp}}$, $\binom{\Omega}{\Omega^{gp}}$, $\binom{\Omega}{\Lambda^{gp}}$, $\binom{C_\Lambda}{C_{\Lambda^{gp}}}$, $\binom{C_\Omega}{C_{\Omega^{gp}}}$, *and* $\binom{C_\Omega}{C_{\Lambda^{gp}}}$ *are all equal to* $\mathfrak{b}$.

Proof. By the Borel version of Theorem 4, $\mathsf{non}(\binom{\mathcal{B}_\Omega}{\mathcal{B}_{\Omega^{gp}}}) = \mathfrak{b}$. In [15] it is proved that $\mathsf{non}(\binom{\mathcal{B}_\Lambda}{\mathcal{B}_{\Lambda^{gp}}}) = \mathfrak{b}$. These two properties imply all other properties in the list. Now, all properties in the list imply either $\binom{C_\Lambda}{C_{\Lambda^{gp}}}$ or $\binom{C_\Omega}{C_{\Lambda^{gp}}}$, whose critical cardinality is $\mathfrak{b}$ by Theorem 4 and [15]. $\square$

If we forget about the topology and consider *arbitrary* countable covers, we get the following characterization of $\mathfrak{b}$, which extends Theorem 15 of [6] and Corollary 2.7 of [15]. For a cardinal κ, denote by Λ_κ, Ω_κ, Λ_κ^{gp}, and Ω_κ^{gp} the collections of countable large covers, ω-covers, groupable covers, and ω-groupable covers of κ, respectively.

Corollary 7. *For an infinite cardinal κ, the following are equivalent:*

1. $\kappa < \mathfrak{b}$,
2. $\binom{\Lambda_\kappa}{\Lambda_\kappa^{gp}}$,
3. $\binom{\Omega_\kappa}{\Lambda_\kappa^{gp}}$; *and*
4. $\binom{\Omega_\kappa}{\Omega_\kappa^{gp}}$.

It is an open problem [10] whether item (2) in Sakai's Theorem 2 can be replaced with $\binom{\Omega}{\Omega^{gp}}$ (by the theorem, if X satisfies $\binom{\Omega}{\Omega^{gp}}$, then $C_p(X)$ has the Reznichenko property; the other direction is the unclear one).

For collections $\mathfrak{U}$ and $\mathfrak{V}$ of covers of X, we say that X satisfies $\mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V})$ if:

> For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$, there is a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ such that each $\mathcal{F}_n$ is a finite subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \mathfrak{V}$.

In [15] it is proved that $\binom{\Lambda}{\Lambda^{gp}} = \mathsf{S}_{fin}(\Lambda, \Lambda^{gp})$ (which is the same as the Hurewicz covering property [6]). We do not know whether the analogue result for $\binom{\Omega}{\Omega^{gp}}$ is true.

Problem 8. *Does* $\binom{\Omega}{\Omega^{gp}} = \mathsf{S}_{fin}(\Omega, \Omega^{gp})$*?*

In [6] it is proved that X satisfies $\mathsf{S}_{fin}(\Omega, \Omega^{gp})$ if, and only if, all finite powers of X satisfy the Hurewicz covering property $\mathsf{S}_{fin}(\Lambda, \Lambda^{gp})$, which we now know is the same as $\binom{\Lambda}{\Lambda^{gp}}$.

Added after publication. The answer to Problem 8 is "No", in the following strong sense: Masami Sakai proves in: Weak Fréchet-Urysohn property in function spaces *(preprint), that every analytic set of reals (and, in particular, the Baire space* ${}^{\mathbb{N}}\mathbb{N}$*) satisfies* $\binom{\mathcal{B}_\Omega}{\mathcal{B}_{\Omega^{gp}}}$*. But we know that* ${}^{\mathbb{N}}\mathbb{N}$ *does not satisfy the Hurewicz covering property.*

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