TOPOLOGICAL DIAGONALIZATIONS AND HAUSDORFF DIMENSION

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Abstract. The Hausdorff dimension of a product $X\times Y$ can be strictly greater than that of Y, even when the Hausdorff dimension of X is zero. But when X is countable, the Hausdorff dimensions of Y and $X \times Y$ are the same. Diagonalizations of covers define a natural hierarchy of properties which are weaker than "being countable" and stronger than "having Hausdorff dimension zero". Fremlin asked whether it is enough for X to have the strongest property in this hierarchy (namely, being a γ-set) in order to assure that the Hausdorff dimensions of Y and $X \times Y$ are the same.

We give a negative answer: Assuming the Continuum Hypothesis, there exists a γ-set $X \subseteq \mathbb{R}$ and a set $Y \subseteq \mathbb{R}$ with Hausdorff dimension zero, such that the Hausdorff dimension of $X+Y$ (a Lipschitz image of $X\times Y$) is maximal, that is, 1. However, we show that for the notion of a *strong* γ-set the answer is positive. Some related problems remain open.

1. Introduction

The Hausdorff dimension of a subset of $\mathbb{R}^k$ is a derivative of the notion of Hausdorff *measures* [4]. However, for our purposes it will be more convenient to use the following equivalent definition. Denote the diameter of a subset A of $\mathbb{R}^k$ by $\operatorname{diam}(A)$. The *Hausdorff dimension* of a set $X \subseteq \mathbb{R}^k$, $\dim(X)$, is the infimum of all positive δ such that for each positive ϵ there exists a cover $\{I_n\}_{n\in\mathbb{N}}$ of X with

$$\sum_{n\in\mathbb{N}} \operatorname{diam}(I_n)^\delta < \epsilon.$$

From the many properties of Hausdorff dimension, we will need the following easy ones.

Lemma 1.

1. *If* $X \subseteq Y \subseteq \mathbb{R}^k$, *then* $\dim(X) \leq \dim(Y)$.
2. *Assume that* $X_1, X_2, \ldots$ *are subsets of* $\mathbb{R}^k$ *such that* $\dim(X_n) = \delta$ *for each* n. *Then* $\dim(\bigcup_n X_n) = \delta$.
3. *Assume that* $X \subseteq \mathbb{R}^k$ *and* $Y \subseteq \mathbb{R}^m$ *is such that there exists a Lipschitz surjection* $\phi : X \to Y$. *Then* $\dim(X) \geq \dim(Y)$.
4. *For each* $X \subseteq \mathbb{R}^k$ *and* $Y \subseteq \mathbb{R}^m$, $\dim(X \times Y) \geq \dim(X) + \dim(Y)$.

Equality need not hold in item (4) of the last lemma. In particular, one can construct a set X with Hausdorff dimension zero and a set Y such that $\dim(X \times$

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$Y) > \dim(Y)$. On the other hand, when X is countable, $X \times Y$ is a union of countably many copies of Y, and therefore

$$\dim(X \times Y) = \dim(Y). \tag{1}$$

Having Hausdorff dimension zero can be thought of as a notion of smallness. Being countable is another notion of smallness, and we know that the first notion is not enough restrictive in order to have Equation 1 hold, but the second is.

Notions of smallness for sets of real numbers have a long history and many applications – see, e.g., [11]. We will consider some notions which are weaker than being countable and stronger than having Hausdorff dimension zero.

According to Borel [3], a set $X \subseteq \mathbb{R}^k$ has *strong measure zero* if for each sequence of positive reals $\{\epsilon_n\}_{n\in\mathbb{N}}$, there exists a cover $\{I_n\}_{n\in\mathbb{N}}$ of X such that $\mathrm{diam}(I_n) < \epsilon_n$ for all n. Clearly strong measure zero implies Hausdorff dimension zero. It does not require any special assumptions in order to see that the converse is false. A perfect set can be mapped onto the unit interval by a uniformly continuous function and therefore cannot have strong measure zero.

Proposition 2 (folklore)**.** *There exists a perfect set of reals X with Hausdorff dimension zero.*

Proof. For $0 < \lambda < 1$, denote by $C(\lambda)$ the Cantor set obtained by starting with the unit interval, and at each step removing from the middle of each interval a subinterval of size λ times the size of the interval (So that $C(1/3)$ is the canonical middle-third Cantor set, which has Hausdorff dimension $\log 2/\log 3$.) It is easy to see that if $\lambda_n \nearrow 1$, then $\dim(C(\lambda_n)) \searrow 0$.

Thus, define a special Cantor set $C(\{\lambda_n\}_{n\in\mathbb{N}})$ by starting with the unit interval, and at step n removing from the middle of each interval a subinterval of size λ_n times the size of the interval. For each n, $C(\{\lambda_n\}_{n\in\mathbb{N}})$ is contained in a union of 2^n (shrunk) copies of $C(\lambda_n)$, and therefore $\dim(C(\{\lambda_n\}_{n\in\mathbb{N}})) \leq \dim(C(\lambda_n))$. □

As every countable set has strong measure zero, the latter notion can be thought of an "approximation" of countability. In fact, Borel conjectured in [3] that every strong measure zero set is countable, and it turns out that the usual axioms of mathematics (ZFC) are not strong enough to prove or disprove this conjecture: Assuming the Continuum Hypothesis there exists an uncountable strong measure zero set (namely, a Luzin set), but Laver [10] proved that one cannot prove the existence of such an object from the usual axioms of mathematics.

The property of strong measure zero (which depends on the metric) has a natural topological counterpart. A topological space X has *Rothberger's property C''* [13] if for each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of covers of X there is a sequence $\{U_n\}_{n\in\mathbb{N}}$ such that for each n $U_n \in \mathcal{U}_n$, and $\{U_n\}_{n\in\mathbb{N}}$ is a cover of X. Using Scheepers' notation [15], this property is a particular instance of the following selection hypothesis (where $\mathfrak{U}$ and $\mathfrak{V}$ are any collections of covers of X):

$\mathsf{S}_1(\mathfrak{U},\mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$, there is a sequence $\{U_n\}_{n\in\mathbb{N}}$ such that $U_n \in \mathcal{U}_n$ for each n, and $\{U_n\}_{n\in\mathbb{N}} \in \mathfrak{V}$.

Let $\mathcal{O}$ denote the collection of all open covers of X. Then the property considered by Rothberger is $\mathsf{S}_1(\mathcal{O},\mathcal{O})$. Fremlin and Miller [5] proved that a set $X \subseteq \mathbb{R}^k$ satisfies $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ if, and only if, X has strong measure zero with respect to each metric which generates the standard topology on $\mathbb{R}^k$.

But even Rothberger's property for X is not strong enough to have Equation 1 hold: It is well-known that every Luzin set satisfies Rothberger's property (and, in particular, has Hausdorff dimension zero).

Lemma 3. *The mapping $(x, y) \mapsto x + y$ from $\mathbb{R}^2$ to $\mathbb{R}$ is Lipschitz.*

Proof. Observe that for nonnegative reals a and b, $(a - b)^2 \geq 0$ and therefore $a^2 + b^2 \geq 2ab$. Consequently,

$$a + b = \sqrt{a^2 + 2ab + b^2} \leq \sqrt{2(a^2 + b^2)} = \sqrt{2}\sqrt{a^2 + b^2}.$$

Thus, $|(x_1+y_1)-(x_2+y_2)| \leq \sqrt{2}\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. $\square$

Assuming the Continuum Hypothesis, there exists a Luzin set $L \subseteq \mathbb{R}$ such that $L + L$, a Lipschitz image of $L \times L$, is equal to $\mathbb{R}$ [9].

We therefore consider some stronger properties. An open cover $\mathcal{U}$ of X is an *ω-cover* of X if each finite subset of X is contained in some member of the cover, but X is not contained in any member of $\mathcal{U}$. $\mathcal{U}$ is a γ-cover of X if it is infinite, and each element of X belongs to all but finitely many members of $\mathcal{U}$. Let Ω and Γ denote the collections of open ω-covers and γ-covers of X, respectively. Then $\Gamma \subseteq \Omega \subseteq \mathcal{O}$, and these three classes of covers introduce 9 properties of the form $\mathsf{S}_1(\mathfrak{U}, \mathfrak{V})$. If we remove the trivial ones and check for equivalences [9, 19], then it turns out that only six of these properties are really distinct, and only three of them imply Hausdorff dimension zero:

$$\mathsf{S}_1(\Omega, \Gamma) \to \mathsf{S}_1(\Omega, \Omega) \to \mathsf{S}_1(\mathcal{O}, \mathcal{O}).$$

The properties $\mathsf{S}_1(\Omega, \Gamma)$ and $\mathsf{S}_1(\Omega, \Omega)$ were also studied before. $\mathsf{S}_1(\Omega, \Omega)$ was studied by Sakai [14], and $\mathsf{S}_1(\Omega, \Gamma)$ was studied by Gerlits and Nagy in [8]: A topological space X is a *γ-set* if each ω-cover of X contains a γ-cover of X. Gerlits and Nagy proved that X is a γ-set if, and only if, X satisfies $\mathsf{S}_1(\Omega, \Gamma)$. It is not difficult to see that every countable space is a γ-set. But this property is not trivial: Assuming the Continuum Hypothesis, there exist uncountable γ-sets [7].

$\mathsf{S}_1(\Omega, \Omega)$ is closed under taking finite powers [9], thus the Luzin set we used to see that Equation 1 need not hold when X satisfies $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ does not rule out that possibility that this Equation holds when X satisfies $\mathsf{S}_1(\Omega, \Omega)$. However, in [2] it is shown that assuming the Continuum Hypothesis, there exist Luzin sets L_0 and L_1 satisfying $\mathsf{S}_1(\Omega, \Omega)$, such that $L_0 + L_1 = \mathbb{R}$. Thus, the only remaining candidate for a nontrivial property of X where Equation 1 holds is $\mathsf{S}_1(\Omega, \Gamma)$ (γ-sets). Fremlin (personal communication) asked whether Equation 1 is indeed provable in this case. We give a negative answer, but show that for a yet stricter (but nontrivial) property which was considered in the literature, the answer is positive.

The notion of a strong γ-set was introduced in [7]. However, we will adopt the following simple characterization from [19] as our formal definition. Assume that $\{\mathfrak{U}_n\}_{n\in\mathbb{N}}$ is a sequence of collections of covers of a space X, and that $\mathfrak{V}$ is a collection of covers of X. Define the following selection hypothesis.

$\mathsf{S}_1(\{\mathfrak{U}_n\}_{n\in\mathbb{N}}, \mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ where $\mathcal{U}_n \in \mathfrak{U}_n$ for each n, there is a sequence $\{U_n\}_{n\in\mathbb{N}}$ such that $U_n \in \mathcal{U}_n$ for each n, and $\{U_n\}_{n\in\mathbb{N}} \in \mathfrak{V}$.

A cover $\mathcal{U}$ of a space X is an *n-cover* if each n-element subset of X is contained in some member of $\mathcal{U}$. For each n denote by $\mathcal{O}_n$ the collection of all open n-covers of a space X. Then X is a *strong γ-set* if X satisfies $\mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \Gamma)$.

In most cases $\mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}},\mathfrak{V})$ is equivalent to $\mathsf{S}_1(\Omega,\mathfrak{V})$ [19], but not in the case $\mathfrak{V}=\Gamma$: It is known that for a strong γ-set $G\subseteq\{0,1\}^{\mathbb{N}}$ and each $A\subseteq\{0,1\}^{\mathbb{N}}$ of measure zero, $G\oplus A$ has measure zero too [7]; this can be contrasted with Theorem 5 below. In Section 3 we show that Equation 1 is provable in the case that X is a strong γ-set, establishing another difference between the notions of γ-sets and strong γ-sets, and giving a positive answer to Fremlin's question under a stronger assumption on X.

2. The product of a γ-set and a set of Hausdorff dimension zero

Theorem 4. *Assuming the Continuum Hypothesis (or just $\mathfrak{p}=\mathfrak{c}$), there exist a γ-set $X\subseteq\mathbb{R}$ and a set $Y\subseteq\mathbb{R}$ with Hausdorff dimension zero such that the Hausdorff dimension of the algebraic sum*

$$X+Y=\{x+y:x\in X,y\in Y\}$$

(a Lipschitz image of $X\times Y$ in $\mathbb{R}$) is 1. In particular, $\dim(X\times Y)\geq 1$.

Our theorem will follow from the following related theorem. This theorem involves the *Cantor space* $\{0,1\}^{\mathbb{N}}$ of infinite binary sequences. The Cantor space is equipped with the product topology and with the product measure.

Theorem 5 (Bartoszyński and Recław [1])**.** *Assume the Continuum Hypothesis (or just $\mathfrak{p}=\mathfrak{c}$). Fix an increasing sequence $\{k_n\}_{n\in\mathbb{N}}$ of natural numbers, and for each n define*

$$A_n=\{f\in\{0,1\}^{\mathbb{N}}:f\restriction[k_n,k_{n+1})\equiv 0\}.$$

If the set

$$A=\bigcap_{m\in\mathbb{N}}\bigcup_{n\geq m}A_n$$

has measure zero, then there exists a γ-set $G\subseteq\{0,1\}^{\mathbb{N}}$ such that the algebraic sum $G\oplus A$ is equal to $\{0,1\}^{\mathbb{N}}$ (where where $\oplus$ denotes the modulo 2 coordinatewise addition).

Observe that the assumption in Theorem 5 holds whenever $\sum_n 2^{-(k_{n+1}-k_n)}$ converges.

Lemma 6. *There exists an increasing sequence of natural numbers $\{k_n\}_{n\in\mathbb{N}}$ such that $\sum_n 2^{-(k_{n+1}-k_n)}$ converges, and such that for the sequence $\{B_n\}_{n\in\mathbb{N}}$ defined by*

$$B_n=\left\{\sum_{i\in\mathbb{N}}\frac{f(i)}{2^{i+1}}:f\in\{-1,0,1\}^{\mathbb{N}}\text{ and }f\restriction[k_n,k_{n+1})\equiv 0\right\}$$

for each n, the set

$$Y=\bigcap_{m\in\omega}\bigcup_{n\geq m}B_n$$

has Hausdorff dimension zero.

Proof. Fix a sequence p_n of positive reals which converges to 0. Let $k_0=0$. Given k_n find k_{n+1} satisfying

$$3^{k_n}\cdot\frac{1}{2^{p_n(k_{n+1}-2)}}\leq\frac{1}{2^n}.$$

Clearly, every B_n is contained in a union of 3^{k_n} intervals such that each of the intervals has diameter $1/2^{k_{n+1}-2}$. For each positive δ and ϵ, choose m such that

$\sum_{n\geq m} 1/2^n < \epsilon$ and such that $p_n < \delta$ for all $n \geq m$. Now, Y is a subset of $\bigcup_{n\geq m} B_n$, and

$$\sum_{n\geq m} 3^{k_n}\left(\frac{1}{2^{k_{n+1}-2}}\right)^{\delta} < \sum_{n\geq m} 3^{k_n}\left(\frac{1}{2^{k_{n+1}-2}}\right)^{p_n} < \sum_{n\geq m}\frac{1}{2^n} < \epsilon.$$

Thus, the Hausdorff dimension of Y is zero. □

The following lemma concludes the proof of Theorem 4.

Lemma 7. *There exists a γ-set $X \subseteq \mathbb{R}$ and set $Y \subseteq \mathbb{R}$ with Hausdorff dimension zero such that $X + Y = \mathbb{R}$. In particular,* $\dim(X + Y) = 1$.

Proof. Choose a sequence $\{k_n\}_{n\in\mathbb{N}}$ and a set Y as in Lemma 6. Then $\sum_n 2^{-(k_{n+1}-k_n)}$ converges, and the corresponding set A defined in Theorem 5 has measure zero. Thus, there exists a γ-set G such that $G \oplus A = \{0,1\}^{\mathbb{N}}$.

Define $\Phi : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ by

$$\Phi(f) = \sum_{i\in\mathbb{N}} \frac{f(i)}{2^{i+1}}.$$

As Φ is continuous, $X = \Phi[G]$ is a γ-set of reals. Assume that z is a member of the interval $[0,1]$, let $f \in \{0,1\}^{\mathbb{N}}$ be such that $z = \sum_i f(i)/2^{i+1}$. Then $f = g \oplus a$ for appropriate $g \in G$ and $a \in A$. Define $h \in \{-1,0,1\}^{\mathbb{N}}$ by $h(i) = f(i) - g(i)$. For infinitely many n, $a \restriction [k_n, k_{n+1}) \equiv 0$ and therefore $f \restriction [k_n, k_{n+1}) \equiv g \restriction [k_n, k_{n+1})$, that is, $h \restriction [k_n, k_{n+1}) \equiv 0$ for infinitely many n. Thus, $y = \sum_i h(i)/2^{i+1} \in Y$, and for $x = \Phi(g)$,

$$x + y = \sum_{i\in\mathbb{N}} \frac{g(i)}{2^{i+1}} + \sum_{i\in\mathbb{N}} \frac{h(i)}{2^{i+1}} = \sum_{i\in\mathbb{N}} \frac{g(i)+h(i)}{2^{i+1}} = \sum_{i\in\mathbb{N}} \frac{f(i)}{2^{i+1}} = z.$$

This shows that $[0,1] \subseteq X + Y$. Consequently, $X + (Y + \mathbb{Q}) = (X + Y) + \mathbb{Q} = \mathbb{R}$. Now, observe that $Y + \mathbb{Q}$ has Hausdorff dimension zero since Y has. □

3. The product of a strong γ-set and a set of Hausdorff dimension zero

Theorem 8. *Assume that $X \subseteq \mathbb{R}^k$ is a strong γ-set. Then for each $Y \subseteq \mathbb{R}^l$,* $\dim(X \times Y) = \dim(Y)$.

Proof. The proof for this is similar to that of Theorem 7 in [7].

It is enough to show that $\dim(X \times Y) \leq \dim(Y)$.

Lemma 9. *Assume that $Y \subseteq \mathbb{R}^l$ is such that* $\dim(Y) < \delta$. *Then for each positive ϵ there exists a* large *cover $\{I_n\}_{n\in\mathbb{N}}$ of Y (i.e., such that each $y \in Y$ is a member of infinitely many sets I_n) such that* $\sum_n \operatorname{diam}(I_n)^\delta < \epsilon$.

Proof. For each m choose a cover $\{I^m_n\}_{n\in\mathbb{N}}$ of Y such that $\sum_n \operatorname{diam}(I^m_n)^\delta < \epsilon/2^m$. Then $\{I^m_n : m,n \in \mathbb{N}\}$ is a large cover of Y, and $\sum_{m,n} \operatorname{diam}(I^m_n)^\delta < \sum_n \epsilon/2^m = \epsilon$. □

Lemma 10. *Assume that $Y \subseteq \mathbb{R}^l$ is such that* $\dim(Y) < \delta$. *Then for each sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive reals there exists a large cover $\{A_n\}_{n\in\mathbb{N}}$ of Y such that for each n A_n is a union of finitely many sets, $I^n_1, \ldots, I^n_{m_n}$, such that* $\sum_j \operatorname{diam}(I^n_j)^\delta < \epsilon_n$.

Proof. Assume that $\{\epsilon_n\}_{n\in\mathbb{N}}$ is a sequence of positive reals. By Lemma 9, there exists a large cover $\{I_n\}_{n\in\mathbb{N}}$ of Y such that $\sum_n \operatorname{diam}(I_n)^\delta < \epsilon_1$. For each n let $k_n = \min\{m : \sum_{j\geq m} \operatorname{diam}(I_j)^\delta < \epsilon_n\}$. Take

$$A_n = \bigcup_{j=k_n}^{k_{n+1}-1} I_j.$$

□

Fix $\delta > \dim(Y)$ and $\epsilon > 0$. Choose a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive reals such that $\sum_n 2n\epsilon_n < \epsilon$, and use Lemma 10 to get the corresponding large cover $\{A_n\}_{n\in\mathbb{N}}$.

For each n we define an n-cover $\mathcal{U}_n$ of X as follows. Let F be an n-element subset of X. For each $x \in F$, find an open interval I_x such that $x \in I_x$ and

$$\sum_{j=1}^{m_n} \operatorname{diam}(I_x \times I_j^n)^\delta < 2\epsilon_n.$$

Let $U_F = \bigcup_{x\in F} I_x$. Set

$$\mathcal{U}_n = \{U_F : F \text{ is an } n\text{-element subset of } X\}.$$

As X is a strong γ-set, there exist elements $U_{F_n} \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_{F_n}\}_{n\in\mathbb{N}}$ is a γ-cover of X. Consequently,

$$X \times Y \subseteq \bigcup_{n\in\mathbb{N}} (U_{F_n} \times A_n) \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{x\in F_n} \bigcup_{j=1}^{m_n} I_x \times I_j^n$$

and

$$\sum_{n\in\mathbb{N}} \sum_{x\in F_n} \sum_{j=1}^{m_n} \operatorname{diam}(I_x \times I_j^n)^\delta < \sum_n n \cdot 2\epsilon_n < \epsilon.$$

□

4. Open problems

There are ways to strengthen the notion of γ-sets other than moving to strong γ-sets. Let $\mathcal{B}_\Omega$ and $\mathcal{B}_\Gamma$ denote the collections of *countable Borel* ω-covers and γ-covers of X, respectively. As every open ω-cover of a set of reals contains a countable ω-subcover [9], we have that $\Omega \subseteq \mathcal{B}_\Omega$ and therefore $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ implies $\mathsf{S}_1(\Omega, \Gamma)$. The converse is not true [16].

Problem 11. *Assume that $X \subseteq \mathbb{R}$ satisfies $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$. Is it true that for each $Y \subseteq \mathbb{R}$, $\dim(X \times Y) = \dim(Y)$?*

We conjecture that assuming the Continuum Hypothesis, the answer to this problem is negative. We therefore introduce the following problem. For infinite sets of natural numbers A, B, we write $A \subseteq^* B$ if $A \setminus B$ is finite. Assume that $\mathcal{F}$ is a family of infinite sets of natural numbers. A set P is a *pseudointersection* of $\mathcal{F}$ if it is infinite, and for each $B \in \mathcal{F}$, $A \subseteq^* B$. $\mathcal{F}$ is *centered* if each finite subcollection of $\mathcal{F}$ has a pseudointersection. Let $\mathfrak{p}$ denote the minimal cardinality of a centered family which does not have a pseudointersection. In [16] it is proved that $\mathfrak{p}$ is also the minimal cardinality of a set of reals which does not satisfy $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.

Problem 12. *Assume that the cardinality of X is smaller than $\mathfrak{p}$. Is it true that for each $Y \subseteq \mathbb{R}$, $\dim(X \times Y) = \dim(Y)$?*

Another interesting open problem involves the following notion [17, 18]. A cover $\mathcal{U}$ of X is a *τ-cover* of X if it is a large cover, and for each $x, y \in X$, one of the sets $\{U \in \mathcal{U} : x \in U \text{ and } y \notin U\}$ or $\{U \in \mathcal{U} : y \in U \text{ and } x \notin U\}$ is finite. Let T denote the collection of open τ-covers of X. Then $\Gamma \subseteq \mathrm{T} \subseteq \Omega$, therefore $\mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \Gamma)$ implies $\mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \mathrm{T})$.

Problem 13. *Assume that $X \subseteq \mathbb{R}$ satisfies $\mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \mathrm{T})$. Is it true that for each $Y \subseteq \mathbb{R}$, $\dim(X \times Y) = \dim(Y)$?*

It is conjectured that $\mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \mathrm{T})$ is strictly stronger than $\mathsf{S}_1(\Omega, \mathrm{T})$ [19]. If this conjecture is false, then the results in this paper imply a negative answer to Problem 13.

Another type of problems is the following: We have seen that the assumption that X is a γ-set and Y has Hausdorff dimension zero is not enough in order to prove that $X \times Y$ has Hausdorff dimension zero. We also saw that if X satisfies a stronger property (strong γ-set), then $\dim(X \times Y) = \dim(Y)$ for all Y. Another approach to get a positive answer would be to strengthen the assumption on Y rather than X.

Assume that X, Y are γ-sets. Does it necessarily follow that $\dim(X \times Y) = 0$? What about the case that (X is a γ-set, and) Y satisfies the Sakai property $\mathsf{S}_1(\Omega, \Omega)$, or Rothberger's property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$, or just strong measure zero? An argument similar to that of Theorem 8 shows that if X is a strong γ-set we get a positive answer.

Theorem 14. *Assume that $X \subseteq \mathbb{R}^k$ is a strong γ-set. Then for each strong measure zero set $Y \subseteq \mathbb{R}^l$, $X \times Y$ has strong measure zero. In particular, the Hausdorff dimension of $X \times Y$ is zero.*

Proof. Assume that $\{\epsilon_n\}_{n\in\mathbb{N}}$ is a sequence of positive reals. For each n let $k_n = 1 + \cdots + (n-1)$, and define $\tilde{\epsilon}_n = \min\{\epsilon_i : i < k_{n+1}\}/2$. As Y has strong measure zero, there exists a large cover $\{I_n\}_{n\in\mathbb{N}}$ of Y such that for each n, $\operatorname{diam}(I_n) < \tilde{\epsilon}_n$.

For each n define an n-cover $\mathcal{U}_n$ of X as follows. Let F be an n-element subset of X. For each $x \in F$, find an open interval I_x such that $x \in I_x$ and $\operatorname{diam}(I_x \times I_n) < 2\tilde{\epsilon}_n$. Let $U_F = \bigcup_{x\in F} I_x$. Set

$$\mathcal{U}_n = \{U_F : F \text{ is an } n\text{-element subset of } X\}$$

and choose elements $U_{F_n} \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_{F_n}\}_{n\in\mathbb{N}}$ is a γ-cover of X. Then

$$X \times Y \subseteq \bigcup_{n\in\mathbb{N}} (U_{F_n} \times I_n) \subseteq \bigcup_{n\in\mathbb{N}} \bigcup_{x\in F_n} I_x \times I_n$$

Enumerate each F_n as $\{x_0^n, \dots, x_{n-1}^n\}$, and define a cover $\{J_n\}_{n\in\mathbb{N}}$ of $X \times Y$ by $J_{k_n+i} = I_{x_i^n} \times I_n$ for each n and $i < n$. For each natural number N, let n be such that $N = k_n + i$ for some $i < n$. Then

$$\operatorname{diam}(J_N) = \operatorname{diam}(I_{x_i^n} \times I_n) < 2\tilde{\epsilon}_n \le \epsilon_{k_n+i} = \epsilon_N.$$

□

In fact, the corresponding result for γ-sets also holds; we plan to treat this problem in [20].

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