MULTISCALED WAVELET TRANSFORMS, RIDGELET TRANSFORMS, AND RADON TRANSFORMS ON THE SPACE OF MATRICES

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Abstract. Let $\mathfrak{M}_{n,m}$ be the space of real $n \times m$ matrices which can be identified with the Euclidean space $\mathbb{R}^{nm}$. We introduce continuous wavelet transforms on $\mathfrak{M}_{n,m}$ with a multivalued scaling parameter represented by a positive definite symmetric matrix. These transforms agree with the polar decomposition on $\mathfrak{M}_{n,m}$ and coincide with classical ones in the rank-one case $m = 1$. We prove an analog of Calderón's reproducing formula for L^2-functions and obtain explicit inversion formulas for the Riesz potentials and Radon transforms on $\mathfrak{M}_{n,m}$. We also introduce continuous ridgelet transforms associated to matrix planes in $\mathfrak{M}_{n,m}$. An inversion formula for these transforms follows from that for the Radon transform.

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1. Introduction

It is known that diverse wavelet-like transforms can be generated by operators of fractional integration and used to invert these operators. On the other hand, numerous problems in integral geometry, for instance, reconstruction of functions from their integrals over planes in $\mathbb{R}^n$, reduce to inversion of fractional integrals. The following example illustrates these statements and explains how wavelet transforms arise in the context of integral-geometrical problems; see also [?] for the more detailed exposition.

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Consider the Riesz potential

$$(I^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int_{\mathbb{R}^n} |x-y|^{\alpha-n} f(y) dy, \qquad x \in \mathbb{R}^n, \tag{1.1}$$

$$\gamma_n(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)}, \qquad Re\, \alpha > 0, \quad \alpha - n \neq 0, 2, \dots,$$

and replace the kernel $|x-y|^{\alpha-n}$ by the integral

$$|x-y|^{\alpha-n} = c_{\alpha,w}^{-1} \int_0^\infty w\left(\frac{|x-y|}{a}\right) \frac{da}{a^{n-\alpha+1}}, \tag{1.2}$$

where $w(\cdot)$ is good enough and $c_{\alpha,w} = \int_0^\infty w(s)\, s^{n-\alpha-1} ds \neq 0$. Changing the order of integration, we obtain

$$(I^\alpha f)(x) = \frac{c_{\alpha,w}^{-1}}{\gamma_n(\alpha)} \int_0^\infty \frac{(\mathcal{W}_a f)(x)}{a^{1-\alpha}} da, \tag{1.3}$$

where

$$(\mathcal{W}_a f)(x) = \frac{1}{a^n} \int_{\mathbb{R}^n} f(y) w\left(\frac{|x-y|}{a}\right) dy. \tag{1.4}$$

If w obeys some cancellation conditions, then (**??**) represents the classical continuous wavelet transform with the scaling parameter $a > 0$ [?], [?], [?]. If we start with a fractional integral different from $I^\alpha f$, say, with the Bessel potential or whatever (see [?], Section 10.7), we arrive at a wavelet transform, which differs from (**??**).

Since the Fourier transform of $I^\alpha f$ is $|y|^{-\alpha}(\mathcal{F}f)(y)$ in a certain sense, then, formally, $(I^\alpha)^{-1} = I^{-\alpha}$, and it is natural to expect that the inverse operator $(I^\alpha)^{-1}$ can be represented in the form (**??**) with α replaced by $-\alpha$, namely,

$$(I^\alpha)^{-1} f = d_{\alpha,w} \int_0^\infty \frac{\mathcal{W}_a f}{a^{1+\alpha}} da, \tag{1.5}$$

$d_{\alpha,w}$ being a normalizing factor. In particular, for $\alpha = 0$,

$$f = d_{0,w} \int_0^\infty \frac{\mathcal{W}_a f}{a} da. \tag{1.6}$$

The equality (**??**) is a modification of Calderón's reproducing formula [?], [?], [?], [?]. If w is normalized and has the form $w = u * v$, then

(**??**) turns into the classical Calderón identity

$$f = \int_0^\infty \frac{f * u_a * v_a}{a} \, da \tag{1.7}$$

where $u_a(x) = a^{-n} u(x/a), \ v_a(x) = a^{-n} v(x/a)$.

Of course, this argument is purely heuristic and formulas (**??**)-(**??**) require justification in the framework of a suitable class of functions f under certain cancellation conditions for the wavelet function w.

What is the connection between this argument and the Radon transform in integral geometry? Suppose that $\mathfrak{T}$ is the manifold of all k-dimensional planes τ in $\mathbb{R}^n$. For functions $f : \mathbb{R}^n \to \mathbb{C}$ and $\varphi : \mathfrak{T} \to \mathbb{C}$, the Radon transform and its dual are defined by

$$\hat{f}(\tau) = \int_{x \in \tau} f(x), \qquad \check{\varphi}(x) = \int_{\tau \ni x} \varphi(\tau), \tag{1.8}$$

respectively, and obey the Fuglede equality

$$(\hat{f})^\vee = c \, I^k f, \qquad c = \text{const}, \tag{1.9}$$

see [**?**], [**?**], [**?**] for details. Combining (**??**) with (**??**), we obtain an inversion formula for $\hat{f}$ in the "wavelet form"

$$f = \text{const} \int_0^\infty \frac{\mathcal{W}_a (\hat{f})^\vee}{a^{1+k}} \, da, \tag{1.10}$$

where $\int_0^\infty = \lim_{\varepsilon \to 0} \int_\varepsilon^\infty$ in a certain sense.

This formalism can be applied in a wider context, when, instead of the Euclidean distance $|x - y|$ between two points, one deals with the distance $|x - \tau|$ between the point $x \in \mathbb{R}^n$ and the k-dimensional plane $\tau \subset \mathbb{R}^n$. Starting with the intertwining operator

$$(P^\alpha f)(\tau) = \frac{1}{\gamma_{n-k}(\alpha)} \int_{\mathbb{R}^n} f(x) |x - \tau|^{\alpha + k - n} \, dx \tag{1.11}$$

which is called the *generalized Semyanistyi fractional integral* (cf. [**?**] for $k = n - 1$), one arrives at the corresponding wavelet-like transform, recently called the *continuous k-plane ridgelet transform*; see [**?**], [**?**], and references therein. These transforms have proved to be useful in applications [**?**], [**?**], [**?**], and are of independent theoretical interest.

A common feature of these examples is that the scaling parameter $a > 0$ is one-dimensional no matter what the dimension of the ambient space $\mathbb{R}^n$ is. The situation changes drastically if we replace n by nm, and regard $\mathbb{R}^{nm}$ as the space of $n \times m$ real matrices $x = (x_{i,j})$. Then a similar procedure, starting with the properly defined Riesz potential, yields a new wavelet-like transform which is applicable to functions of matrix argument and depends on the matrix-valued scaling parameter. This parameter is represented by a positive definite symmetric matrix of size $m \times m$. Apart from extra flexibility that might be useful in applications, such transforms have a rich theory which relies on diverse higher rank phenomena.

In the present paper, we focus on the L^2 theory of the new wavelet transforms mentioned above, and give some applications. The paper is organized as follows. Section 2 contains necessary prerequisites. We fix our notation and recall basic facts related to Riesz potentials and Radon transforms on the space of rectangular matrices. In Section 3, we introduce continuous wavelet transforms for functions of matrix argument and prove the corresponding reproducing formula of the Calderón type. In Section 4, we show how wavelet transforms can be used for inversion of Riesz potentials on matrix spaces. Unlike the rank-one case $m = 1$, for which numerous inversion formulas are known [**?**], [**?**], the corresponding higher rank problem is very difficult; see [**?**], [**?**] for the discussion. Wavelet transforms prove to be a convenient tool to resolve this problem in the L^2-case. In Section 5, we apply our wavelet transforms to inversion of the Radon transform associated to the so-called matrix k-planes. These Radon transforms were studied in detail in [**?**], [**?**], and [**?**], where it was shown that the inversion problem for them has the same difficulties as for the Riesz potentials on the space of rectangular matrices. In Section 5, we introduce continuous ridgelet transforms of functions of matrix argument, generalizing those in [**?**] and [**?**], and prove a reproducing formula for these transforms. This result is a consequence of the inversion formula for the Radon transform.

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2. Preliminaries

In this section, we fix our notation and recall some basic facts, that will be used throughout the paper. The main references are [**?**], [**?**], [**?**].

2.1. **Notation and some auxiliary facts.** Let $\mathfrak{M}_{n,m}$ be the space of real matrices $x = (x_{i,j})$ having n rows and m columns. We identify $\mathfrak{M}_{n,m}$ with the real Euclidean space $\mathbb{R}^{nm}$ and set $dx = \prod_{i=1}^{n} \prod_{j=1}^{m} dx_{i,j}$

for the Lebesgue measure on $\mathfrak{M}_{n,m}$. In the following, x' denotes the transpose of x, I_m is the identity $m \times m$ matrix, 0 stands for zero entries. Given a square matrix a, we denote by $\mathrm{tr}(a)$ the trace of a, and by $|a|$ the absolute value of the determinant of a, respectively. We hope the reader will not confuse $|a|$ with the similar notation for the absolute value of a number because the meaning of a will be clear each time from the context. For $x \in \mathfrak{M}_{n,m}$, $n \geq m$, we set

$$|x|_m = \det(x'x)^{1/2} = |x'x|^{1/2}. \tag{2.1}$$

If $m = 1$, this is the usual Euclidean norm in $\mathbb{R}^n$. If $m > 1$, then $|x|_m$ is the volume of the parallelepiped spanned by the column-vectors of the matrix x, cf. [**?**, p. 251].

Let $\mathcal{P}_m$ be the cone of positive definite symmetric matrices $r = (r_{i,j})_{m \times m}$ with the elementary volume $dr = \prod_{i \leq j} dr_{i,j}$, and let $\overline{\mathcal{P}}_m$ be the closure of $\mathcal{P}_m$, that is the set of all positive semi-definite $m \times m$ matrices. We write $r > 0$ if $r \in \mathcal{P}_m$, and $r \geq 0$ if $r \in \overline{\mathcal{P}}_m$, respectively. Given s_1 and s_2 in $\overline{\mathcal{P}}_m$, we write $s_1 > s_2$ for $s_1 - s_2 \in \mathcal{P}_m$. If $a \in \overline{\mathcal{P}}_m$ and $b \in \mathcal{P}_m$, then $\int_a^b f(s)ds$ denotes the integral over the compact set

$$(a, b) = \{s : s - a \in \mathcal{P}_m, \, b - s \in \mathcal{P}_m\}.$$

The group $G = GL(m, \mathbb{R})$ of real non-singular $m \times m$ matrices g acts transitively on $\mathcal{P}_m$ by the rule $r \to grg'$. The corresponding G-invariant measure is

$$d_*r = |r|^{-d} dr, \qquad |r| = \det(r), \qquad d = (m+1)/2 \tag{2.2}$$

[**?**, p. 18].

A function w_0 on $\mathcal{P}_m$ is called *symmetric* if

$$w_0(s^{1/2} r s^{1/2}) = w_0(r^{1/2} s r^{1/2}), \qquad \forall r, s \in \mathcal{P}_m. \tag{2.3}$$

For example, any function of the form $w_0(r) = w_1(\mathrm{tr}(r))$ is symmetric. Instead of the trace, one can take any function of the form $\psi(\sigma_1, \ldots, \sigma_m)$, where $\sigma_1, \ldots, \sigma_m$ are elementary symmetric functions of the eigenvalues $\lambda_1, \ldots, \lambda_m$ of r. This follows from the general fact that if A and B are nonsingular square matrices then AB and BA have the same eigenvalues; see, e.g., [**?**, pp. 584, 585].

We use a standard notation $O(n)$ for the group of real orthogonal $n \times n$ matrices; $SO(n) = \{\gamma \in O(n) : \det(\gamma) = 1\}$. The corresponding invariant measures on $O(n)$ and $SO(n)$ are normalized to be of total mass 1. The Lebesgue space $L^p = L^p(\mathfrak{M}_{n,m})$ and the Schwartz space $\mathcal{S} = \mathcal{S}(\mathfrak{M}_{n,m})$ are identified with respective spaces on $\mathbb{R}^{nm}$. We denote by $C_c(\mathfrak{M}_{n,m})$ the space of compactly supported continuous functions on $\mathfrak{M}_{n,m}$.

The Fourier transform of a function $f \in L^1(\mathfrak{M}_{n,m})$ is defined by

$$(\mathcal{F}f)(y) = \int\limits_{\mathfrak{M}_{n,m}} \exp(\mathrm{tr}(iy'x))f(x)dx, \qquad y \in \mathfrak{M}_{n,m} \, . \tag{2.4}$$

This is the usual Fourier transform on $\mathbb{R}^{nm}$ so that the relevant Parseval formula reads

$$(\mathcal{F}f, \mathcal{F}\varphi) = (2\pi)^{nm} \, (f, \varphi), \tag{2.5}$$

where

$$(f, \varphi) = \int\limits_{\mathfrak{M}_{n,m}} f(x) \, \overline{\varphi(x)} \, dx.$$

We write c, c_1, $c_2, \dots$ for different constants the meaning of which is clear from the context.

Lemma 2.1. *[(see, e.g., [Mu, pp. 57–59])]*
(i) If $x = ayb$, where $y \in \mathfrak{M}_{n,m}$, $a \in GL(n, \mathbb{R})$, and $b \in GL(m, \mathbb{R})$, then $dx = |a|^m |b|^n dy$.
(ii) If $r = qsq'$, where $s \in \mathcal{P}_m$ and $q \in GL(m, \mathbb{R})$, then $dr = |q|^{m+1} ds$.
(iii) If $r = s^{-1}$, $s \in \mathcal{P}_m$, then $r \in \mathcal{P}_m$ and $dr = |s|^{-m-1} ds$.

The Siegel gamma function associated to the cone $\mathcal{P}_m$ is defined by

$$\Gamma_m(\alpha) = \int\limits_{\mathcal{P}_m} \exp(-\mathrm{tr}(r))|r|^{\alpha - d} dr, \qquad d = (m+1)/2, \tag{2.6}$$

[?], [?], [?], [?], [?]. This integral converges absolutely if and only if $Re \, \alpha > d - 1$, and can be written as a product of ordinary Γ-functions:

$$\Gamma_m(\alpha) = \pi^{m(m-1)/4} \Gamma(\alpha) \Gamma(\alpha - \frac{1}{2}) \dots \Gamma(\alpha - \frac{m-1}{2}). \tag{2.7}$$

For $n \geq m$, let $V_{n,m} = \{v \in \mathfrak{M}_{n,m} : v'v = I_m\}$ be the Stiefel manifold of orthonormal m-frames in $\mathbb{R}^n$. We fix the invariant measure dv on $V_{n,m}$ [Mu, p. 70] normalized by

$$\sigma_{n,m} \equiv \int\limits_{V_{n,m}} dv = \frac{2^m \pi^{nm/2}}{\Gamma_m(n/2)} \tag{2.8}$$

and denote $d_* v = \sigma_{n,m}^{-1} dv$. A polar decomposition on $\mathfrak{M}_{n,m}$ is defined according to the following lemma; see, e.g., [?, pp. 66, 591], [?].

Lemma 2.2. *Let* $x \in \mathfrak{M}_{n,m}$, $n \geq m$. *If* $\mathrm{rank}(x) = m$, *then*

$$x = vr^{1/2}, \qquad v \in V_{n,m}, \qquad r = x'x \in \mathcal{P}_m,$$

and $dx = 2^{-m} |r|^{(n-m-1)/2} dr dv$.

The following statement is new and suggestive. It contains a matrix generalization of the relevant formula by Smith and Solmon [**?**, Lemma 2.2] corresponding to the case $m = 1$.

Lemma 2.3. *Let* $1 \le k \le n - m$. *Then*

$$\int\limits_{\mathfrak{M}_{n,m}} f(x)dx = \frac{\sigma_{n,m}}{\sigma_{n-k,m}} \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} f(\xi z)|z|_m^k \, dz. \tag{2.9}$$

Proof. Let $I = \int_{\mathfrak{M}_{n,m}} f(x)dx$, $d = (m+1)/2$. By Lemma **??**,

$$\begin{aligned} I &= 2^{-m} \int\limits_{V_{n,m}} dv \int\limits_{\mathcal{P}_m} f(vr^{1/2})|r|^{n/2-d} dr \\ &= \frac{\sigma_{n,m}}{2^m} \int\limits_{SO(n)} d\gamma \int\limits_{\mathcal{P}_m} f(\gamma v_0 r^{1/2})|r|^{n/2-d} dr, \qquad \forall v_0 \in V_{n,m}. \end{aligned}$$

Choose $v_0 = \xi_0 u_0$, where

$$\xi_0 = \left[\begin{array}{c} 0 \\ I_{n-k} \end{array} \right] \in V_{n,n-k}, \qquad u_0 = \left[\begin{array}{c} 0 \\ I_m \end{array} \right] \in V_{n-k,m}.$$

By setting $\xi = \gamma\xi_0$, we obtain

$$\begin{aligned} I &= \frac{\sigma_{n,m}}{2^m} \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathcal{P}_m} f(\xi u_0 r^{1/2})|r|^{n/2-d} dr \qquad (\xi \to \xi\beta) \\ &= \frac{\sigma_{n,m}}{2^m} \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{O(n-k)} d\beta \int\limits_{\mathcal{P}_m} f(\xi\beta u_0 r^{1/2})|r|^{n/2-d} dr \\ &= \frac{\sigma_{n,m}}{2^m \sigma_{n-k,m}} \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{V_{n-k,m}} du \int\limits_{\mathcal{P}_m} f(\xi u r^{1/2})|r|^{n/2-d} dr \\ &= \frac{\sigma_{n,m}}{\sigma_{n-k,m}} \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} f(\xi z)|z|_m^k \, dz. \end{aligned}$$

□

2.2. **Riesz potentials.** We recall basic facts from [?] and [?] related to Riesz potentials of functions of matrix argument. These potentials arise in different aspects of analysis [?], [?], [?]. They have a number of specific higher rank features and coincide for $m = 1$ with classical integrals of Marcel Riesz [?], [?], [?]. In the following, we assume $m \ge 2$.

The Riesz potential of order $\alpha \in \mathbb{C}$ of a function $f \in \mathcal{S}(\mathfrak{M}_{n,m})$ is defined as analytic continuation of the integral

$$(I^\alpha f)(x) = \frac{1}{\gamma_{n,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} f(x-y)|y|_m^{\alpha-n} dy, \tag{2.10}$$

where $|y|_m = \det(y'y)^{1/2}$,

$$\gamma_{n,m}(\alpha) = \frac{2^{\alpha m}\, \pi^{nm/2}\, \Gamma_m(\alpha/2)}{\Gamma_m((n-\alpha)/2)}, \quad \alpha \neq n-m+1,\, n-m+2, \dots, \tag{2.11}$$

cf. (**??**). This integral converges absolutely if and only if $Re\,\alpha > m-1$ and extends to all $\alpha \in \mathbb{C}$ as a meromorphic function whose only poles are at the points $\alpha = n-m+1, n-m+2, \dots$. The order of these poles is the same as in $\Gamma_m((n-\alpha)/2)$.

Theorem 2.4. *If f and ϕ are Schwartz functions on $\mathfrak{M}_{n,m}$, then for all complex $\alpha \neq n-m+1,\, n-m+2, \dots$,*

$$(I^\alpha f, \phi) = (2\pi)^{-nm}(|y|_m^{-\alpha}(\mathcal{F}f)(y),\ (\mathcal{F}\phi)(y)), \tag{2.12}$$

the expression on each side being understood in the sense of analytic continuation.

This statement is a consequence of the relevant functional equation for the corresponding zeta distributions, see [**?**], [**?**].

If $\alpha = k,\ k = 0,1,2,\dots,m-1$ and $k \neq n-m+1, n-m+2, \dots$, then $I^\alpha f$ is a convolution with a positive measure supported by the manifold of all matrices x of rank $\le k$. Combining this fact with the case $Re\,\alpha > m-1$, we introduce the Wallach-like set

$$\mathbf{W}_{n,m} = \mathbf{W}_1 \cap \mathbf{W}_2, \tag{2.13}$$

where

$$\mathbf{W}_1 = \{0,1,2,\dots,k_0\}, \qquad k_0 = \min(m-1, n-m);$$

$$\mathbf{W}_2 = \{\alpha : Re\,\alpha > m-1;\ \alpha \neq n-m+1, n-m+2, \dots\}.$$

An analog of (**??**) is defined in [**?**, p. 137] for distributions of different type. For $\alpha \in \mathbf{W}_{n,m}$, one can write

$$(I^\alpha f)(x) = (f * \mu_\alpha)(x) = \int\limits_{\mathfrak{M}_{n,m}} f(x-y) d\mu_\alpha(y), \tag{2.14}$$

[**?**, Theorem 3.14], [**?**, Theorem 5.1], where the measure μ_α is defined by

$$(2.15)\quad (\mu_\alpha, \psi) = \begin{cases} \dfrac{1}{\gamma_{n,m}(\alpha)} \displaystyle\int\limits_{\mathfrak{M}_{n,m}} |y|_m^{\alpha-n}\overline{\psi(y)}dy & \text{if } Re\,\alpha > m-1, \\ \\ c_k \displaystyle\int\limits_{\mathfrak{M}_{k,m}} dy \int\limits_{O(n)} \overline{\psi\left(\gamma\begin{bmatrix} y \\ 0 \end{bmatrix}\right)}\, d\gamma & \text{if } \alpha = k. \end{cases}$$

Here, $k = 1, 2, \ldots, n-m$, ψ is a compactly supported continuous function, and

$$(2.16)\qquad c_k = 2^{-km}\,\pi^{-km/2}\,\Gamma_m\Big(\frac{n-k}{2}\Big)/\Gamma_m\Big(\frac{n}{2}\Big).$$

Owing to (**??**), for $\alpha = 0$, μ_α is the usual delta function, and we set $I^0 f = f$. Note that the sets of α in both lines of (**??**) may overlap. In this case we have two different representations of (μ_α, ψ).

One can use (**??**) as a definition of $I^\alpha f$, $\alpha \in \mathbf{W}_{n,m}$, for arbitrary locally integrable function provided the integral $f * \mu_\alpha$ converges absolutely.

Theorem 2.5 ([?], Section 5.3)**.** *Let* $f \in L^p(\mathfrak{M}_{n,m})$, $1 \le p < n/(Re\,\alpha + m - 1)$.
(i) If $Re\,\alpha > m-1$, then

$$(2.17)\qquad \Big|\int\limits_{\mathfrak{M}_{n,m}} \exp(-\mathrm{tr}(x'x))\,(I^\alpha f)(x)\,dx\Big| \le c_1\,||f||_p\,.$$

(ii) If $\alpha = k$, $k = 1, 2, \ldots, n-m$, then

$$(2.18)\qquad \int\limits_{\mathfrak{M}_{n,m}} \frac{|(I^k f)(x)|}{|I_m + x'x|^{\lambda/2}}\,dx \le c_2||f||_p\,,$$

provided

$$\lambda > k + \max\left(m-1, \frac{n+m-1}{p'}\right), \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

This statement shows that for $\alpha \in \mathbf{W}_{n,m}$ and $f \in L^p$, the definition (**??**) is meaningful, i.e., $(I^\alpha f)(x)$ is finite for almost all $x \in \mathfrak{M}_{n,m}$ provided $1 \le p < n/(Re\,\alpha + m - 1)$. The last equality agrees with the classical one $1 \le p < n/Re\,\alpha$ for $m = 1$ [?] which is sharp. We do not know whether the restriction $p < n/(Re\,\alpha + m - 1)$ is necessary if $m > 1$.

2.3. **Radon transforms on the space of matrices.** The main references for this subsection are [?], [?], [?]. We fix positive integers k, n, and m, $0 < k < n$, and let $V_{n,n-k}$ be the Stiefel manifold of orthonormal $(n-k)$-frames in $\mathbb{R}^n$. For $\xi \in V_{n,n-k}$ and $t \in \mathfrak{M}_{n-k,m}$, the linear manifold

$$\tau = \tau(\xi, t) = \{x \in \mathfrak{M}_{n,m} : \xi' x = t\} \tag{2.19}$$

will be called a *matrix k-plane* in $\mathfrak{M}_{n,m}$. We denote by $\mathfrak{T}$ the set of all such planes. Each $\tau \in \mathfrak{T}$ is an ordinary km-dimensional plane in $\mathbb{R}^{nm}$, but the set $\mathfrak{T}$ has measure zero in the manifold of all such planes.

Note that $\tau(\xi, t) = \tau(\xi\theta', \theta t)$ for all $\theta \in O(n-k)$. We identify functions $\varphi(\tau)$ on $\mathfrak{T}$ with functions $\varphi(\xi, t)$ on $V_{n,n-k} \times \mathfrak{M}_{n-k,m}$ satisfying $\varphi(\xi\theta', \theta t) = \varphi(\xi, t)$ for all $\theta \in O(n-k)$, and supply $\mathfrak{T}$ with the measure $d\tau$ so that

$$\int\limits_{\mathfrak{T}} \varphi(\tau)\, d\tau = \int\limits_{V_{n,n-k} \times \mathfrak{M}_{n-k,m}} \varphi(\xi, t)\, d\xi dt. \tag{2.20}$$

The *matrix k-plane Radon transform* $f(x) \to \hat{f}(\tau)$ assigns to a function $f(x)$ on $\mathfrak{M}_{n,m}$ a collection of integrals of f over all matrix planes $\tau \in \mathfrak{T}$. Namely,

$$\hat{f}(\tau) = \int_{x \in \tau} f(x).$$

Precise meaning of this integral is the following:

$$\hat{f}(\tau) \equiv \hat{f}(\xi, t) = \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ t \end{bmatrix}\right) d\omega, \tag{2.21}$$

where $g_\xi \in SO(n)$ is a rotation satisfying

$$g_\xi \xi_0 = \xi, \qquad \xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}. \tag{2.22}$$

The corresponding dual Radon transform $\varphi(\tau) \to \check{\varphi}(x)$ assigns to a function $\varphi(\tau)$ on $\mathfrak{T}$ its mean value over all matrix planes τ through x:

$$\check{\varphi}(x) = \int\limits_{\tau \ni x} \varphi(\tau), \qquad x \in \mathfrak{M}_{n,m}.$$

This means that

$$\check{\varphi}(x) = \int\limits_{V_{n,n-k}} \varphi(\xi, \xi' x) d_*\xi. \tag{2.23}$$

The corresponding duality relation reads

$$\int_{\mathfrak{M}_{n,m}} f(x)\check{\varphi}(x)dx = \int_{V_{n,n-k}} d_*\xi \int_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\hat{f}(\xi,t)dt. \tag{2.24}$$

Theorem 2.6 ([?])**.**
(i) *The Radon transform $\hat{f}(\xi,t)$, $f \in L^p(\mathfrak{M}_{n,m})$, is finite for almost all $(\xi,t) \in V_{n,n-k} \times \mathfrak{M}_{n-k,m}$ if and only if*

$$1 \le p < p_0 = \frac{n+m-1}{k+m-1}. \tag{2.25}$$

(ii) *If $\varphi(\xi,t)$ is a locally integrable function on the set $V_{n,n-k} \times \mathfrak{M}_{n-k,m}$, $1 \le k \le n-m$, then the dual Radon transform $\check{\varphi}(x)$ is finite for almost all $x \in \mathfrak{M}_{n,m}$.*

The following statement is a matrix generalization of the so-called projection-slice theorem. It links together the Fourier transform (**??**) and the Radon transform (**??**). In the case $m = 1$, this theorem can be found in [Na, p. 11] (for $k = n-1$) and [Ke, p. 283] (for any $0 < k < n$).

For $y = [y_1 \dots y_m] \in \mathfrak{M}_{n,m}$, let $\mathcal{L}(y) = \mathrm{span}(y_1, \dots, y_m)$ be the span of the n-vectors $y_1, \dots, y_m$. Suppose that $\mathrm{rank}(y) = \ell$. Then $\dim \mathcal{L}(y) = \ell \le m$.

Theorem 2.7 ([?], [?], [?])**.** *Let $f \in L^1(\mathfrak{M}_{n,m})$, $1 \le k \le n-m$. If $y \in \mathfrak{M}_{n,m}$, and ζ is an $(n-k)$-dimensional plane in $\mathbb{R}^n$ containing $\mathcal{L}(y)$, then for any orthonormal frame $\xi \in V_{n,n-k}$ spanning ζ, there exists $b \in \mathfrak{M}_{n-k,m}$ so that $y = \xi b$. In this case*

$$(\mathcal{F}f)(\xi b) = \tilde{\mathcal{F}}[\hat{f}(\xi,\cdot)](b), \quad \xi \in V_{n,n-k}, \quad b \in \mathfrak{M}_{n-k,m}, \tag{2.26}$$

where $\tilde{\mathcal{F}}$ stands for the Fourier transform on $\mathfrak{M}_{n-k,m}$ in the $(\cdot)$-variable.

Corollary 2.8 ([?])**.** The Radon transform $f \to \hat{f}$ is injective on the Schwartz space $\mathcal{S}(\mathfrak{M}_{n,m})$ if and only if $1 \le k \le n-m$.

3. Continuous wavelet transforms

3.1. **Some heuristics.** Following the philosophy which was described in Introduction for the rank-one case, we will introduce continuous wavelet transforms on $\mathfrak{M}_{n,m}$ associated to the Riesz potential (**??**). The heuristic argument presented below shows that these "higher rank" wavelet transforms are essentially multiscaled, with the scaling parameter represented by a positive definite matrix, rather then a positive number as in the rank-one case.

We recall the notation (**??**) for the invariant measure d_*r on $\mathcal{P}_m$ and start with the following simple observation.

Lemma 3.1. *Let w_0 be a symmetric function on $\mathcal{P}_m$ satisfying*

$$\int\limits_{\mathcal{P}_m} \frac{|w_0(r)|}{|r|^{(\alpha-n)/2}}\, d_*r < \infty \quad \textit{and} \quad c_\alpha = \int\limits_{\mathcal{P}_m} \frac{w_0(r)}{|r|^{(\alpha-n)/2}}\, d_*r \neq 0. \tag{3.1}$$

Then for $s \in \mathcal{P}_m$,

$$|s|^{(\alpha-n)/2} = c_\alpha^{-1} \int\limits_{\mathcal{P}_m} \frac{w_0(a^{-1/2} s a^{-1/2})}{|a|^{(n-\alpha)/2}}\, d_*a\ . \tag{3.2}$$

Proof. Using the symmetry (**??**) and changing variable $a = \rho^{-1}$, $d_*a = d_*\rho$, we rewrite (**??**) as

$$|s|^{(\alpha-n)/2} = c_\alpha^{-1} \int\limits_{\mathcal{P}_m} \frac{w_0(s^{1/2} \rho s^{1/2})}{|\rho|^{(\alpha-n)/2}}\, d_*\rho\, .$$

It remains to set $s^{1/2}\rho s^{1/2} = r$. $\square$

According to (**??**), for $Re\, \alpha > m - 1$, the Riesz potential (**??**) is represented as

$$\begin{aligned} (I^\alpha f)(x) &= \frac{c_\alpha^{-1}}{\gamma_{n,m}(\alpha)} \int\limits_{\mathcal{P}_m} |a|^{(\alpha-n)/2} d_*a \int\limits_{\mathfrak{M}_{n,m}} f(x-y) w_0(a^{-1/2} y' y a^{-1/2}) dy \\ &= \frac{c_\alpha^{-1}}{\gamma_{n,m}(\alpha)} \int\limits_{\mathcal{P}_m} |a|^{\alpha/2} d_*a \int\limits_{\mathfrak{M}_{n,m}} f(x - y a^{1/2}) w_0(y'y) dy. \qquad (3.3) \end{aligned}$$

The inner integrals in these expressions resemble the wavelet transform (**??**) and inspire the following.

Definition 3.2. Let $w(y) = w_0(y'y)$, $y \in \mathfrak{M}_{n,m}$, be a radial function satisfying certain cancellation conditions (which depend on the context). Let $w_a(y) = |a|^{-n/2} w(y a^{-1/2})$, $a \in \mathcal{P}_m$. We call

$$\begin{aligned} (\mathcal{W}_a f)(x) &= \int\limits_{\mathfrak{M}_{n,m}} f(x - y a^{1/2})\, w(y) dy \qquad (3.4) \\ &= (f * w_a)(x), \qquad x \in \mathfrak{M}_{n,m}, \end{aligned}$$

the *continuous wavelet transform of f* generated by the wavelet function w and the $\mathcal{P}_m$-valued scaling parameter a.

Note that the symmetry condition for w_0 indicated in Lemma **??** plays an auxiliary role. It was imposed only for technical reasons and not included in Definition **??**. In the following, this condition will appear on the Fourier transform side for $\mathcal{F}w$.

If the function w_0 in Definition **??** is symmetric, then one can write (**??**) as

$$(I^\alpha f)(x) = c_{n,m}(\alpha, w) \int\limits_{\mathcal{P}_m} (\mathcal{W}_a f)(x) |a|^{\alpha/2}\, d_* a, \tag{3.5}$$

$$Re\, \alpha > m-1; \quad \alpha \neq n-m+1, n-m+2, \ldots,$$

$c_{n,m}(\alpha, w) \equiv$ const. This formula is expected to be true for other values of α (e.g., for $\alpha = 0$) if w obeys certain cancellation conditions. Of course, if $Re\, \alpha \leq m-1$, then the integral on the right-hand side of (**??**) diverges in general, and must be interpreted in a suitable way depending on a class of functions f and a choice of the wavelet w.

One can replace the function w in (**??**) by a finite radial Borel measure ν on $\mathfrak{M}_{n,m}$ so that

$$\int\limits_{\mathfrak{M}_{n,m}} \psi(\gamma x)\, d\nu(x) = \int\limits_{\mathfrak{M}_{n,m}} \psi(x)\, d\nu(x)$$

for all $\gamma \in SO(n)$ and $\psi \in C_c(\mathfrak{M}_{n,m})$. If ν obeys some cancellation (for instance, $(\mathcal{F}\nu)(y) \equiv 0$ on matrices y of rank $< m$) we call

$$(\mathcal{W}_{\nu,\, a} f)(x) \equiv (f * \nu_a)(x) = \int\limits_{\mathfrak{M}_{n,m}} f(x - ya^{1/2}) d\nu(y) \tag{3.6}$$

the wavelet transform of f generated by the wavelet measure ν.

3.2. **Calderón's reproducing formula.** Denote formally

$$I(\nu, f)(x) = \int\limits_{\mathcal{P}_m} (\mathcal{W}_{\nu,\, a} f)(x)\, d_* a, \tag{3.7}$$

where, as above, $d_* a = |a|^{-d} da$, $d = (m+1)/2$. The following statement justifies (**??**) for $\alpha = 0$ and generalizes the classical Calderón reproducing formula (cf. Theorem 1 in [?]) to functions of matrix argument.

Theorem 3.3. *Let ν be a radial finite Borel measure on $\mathfrak{M}_{n,m}$ so that for all $y \in \mathfrak{M}_{n,m}$ of rank m, we have $(\mathcal{F}\nu)(y) = u_0(y'y)$, where $u_0(r)$ is* *a symmetric function on $\mathcal{P}_m$. Suppose that the integral*

$$c_\nu = \lim_{\substack{A\to 0\\ B\to\infty}} \frac{2^m}{\sigma_{n,m}} \int\limits_{\{y\in\mathfrak{M}_{n,m}:\, A<y'y<B\}} \frac{(\mathcal{F}\nu)(z)}{|z|_m^n}\, dz \tag{3.8}$$

$(A, B \in \mathcal{P}_m)$ is finite. Then for $f \in L^2(\mathfrak{M}_{n,m})$,

$$c_\nu f(x) = I(\nu, f)(x) \equiv \overset{(L^2)}{\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}} \int\limits_{\varepsilon I_m}^{\rho I_m} (\mathcal{W}_{\nu,\,a} f)(x)\, d_*a. \tag{3.9}$$

Proof. For $0 < \varepsilon < \rho < \infty$, let

$$I_{\varepsilon,\rho}(\nu, f)(x) = \int\limits_{\varepsilon I_m}^{\rho I_m} (\mathcal{W}_{\nu,\,a} f)(x)\, d_*a \tag{3.10}$$

and assume first that $f \in L^1 \cap L^2$. Then, by the generalized Minkowski inequality, $I_{\varepsilon,\rho}(\nu, f) \in L^1 \cap L^2$, and we have

$$\mathcal{F}[I_{\varepsilon,\rho}(\nu, f)](y) = \int\limits_{\varepsilon I_m}^{\rho I_m} \mathcal{F}[\mathcal{W}_{\nu,\,a} f](y)\, d_*a. \tag{3.11}$$

By taking into account that $\mathcal{F}[\mathcal{W}_{\nu,a} f](y) = (\mathcal{F}f)(y)(\mathcal{F}\nu_a)(y)$, and

$$(\mathcal{F}\nu_a)(y) = \int\limits_{\mathfrak{M}_{n,m}} \exp(\mathrm{tr}(ia^{1/2}y'x))d\nu(x) = (\mathcal{F}\nu)(ya^{1/2}),$$

we obtain

$$\mathcal{F}[I_{\varepsilon,\rho}(\nu, f)](y) = k_{\varepsilon,\rho}(y)(\mathcal{F}f)(y), \tag{3.12}$$

where

$$k_{\varepsilon,\rho}(y) = \int\limits_{\varepsilon I_m}^{\rho I_m} (\mathcal{F}\nu)(ya^{1/2})\, d_*a = \int\limits_{\varepsilon I_m}^{\rho I_m} u_0(a^{1/2}y'ya^{1/2})\, d_*a.$$

Suppose that $\mathrm{rank}(y) = m$ (the set of all such y has a full measure in $\mathfrak{M}_{n,m}$). Since u_0 is symmetric, then $u_0(a^{1/2}ra^{1/2}) = u_0(r^{1/2}ar^{1/2})$,

$r = y'y$, and the change of variable $s = r^{1/2} a r^{1/2}$, yields

$$\begin{aligned} k_{\varepsilon,\rho}(y) &= \int\limits_{\varepsilon I_m}^{\rho I_m} u_0(r^{1/2} a r^{1/2})\, d_* a = \int\limits_{\varepsilon r}^{\rho r} u_0(s)\, d_* s \qquad \text{(use Lemma ??)} \\ &= \frac{2^m}{\sigma_{n,m}} \int\limits_{\{z \in \mathfrak{M}_{n,m} :\, \varepsilon r < z'z < \rho r\}} \frac{(\mathcal{F}\nu)(z)}{|z|_m^n}\, dz, \qquad r = y'y. \end{aligned}$$

Since $k_{\varepsilon,\rho}(y)$ is bounded uniformly in ε, ρ, and y, then by the Lebesgue theorem on dominated convergence,

$$\|I_{\varepsilon,\rho}(\nu, f) - c_\nu f\|_2 = (2\pi)^{-nm/2} \|(k_{\varepsilon,\rho} - c_\nu)\mathcal{F} f\|_2 \to 0 \tag{3.13}$$

as $\varepsilon \to 0$, $\rho \to \infty$. This proves the statement for $f \in L^1 \cap L^2$. A standard procedure allows us to extend the result to all $f \in L^2$. We recall this argument for convenience of the reader. For any $f \in L^2$, we have

$$\|I_{\varepsilon,\rho}(\nu, f)\|_2 \le \int\limits_{\varepsilon I_m}^{\rho I_m} \|\mathcal{W}_{\nu,\,a} f\|_2\, d_* a \le c_{\varepsilon,\rho} \|f\|_2 \|\nu\| = c_{\varepsilon,\rho,\nu} \|f\|_2$$

where $||\nu||$ stands for the total variation of $|\nu|$, $c_{\varepsilon,\rho} = \text{const}$, $c_{\varepsilon,\rho,\nu} = c_{\varepsilon,\rho}\, ||\nu||$. Given a small $\delta > 0$, we choose $g \in L^1 \cap L^2$ so that $\|f - g\|_2 < \delta$. Since $k_{\varepsilon,\rho}$ is uniformly bounded, then (**??**) (with f replaced by g) implies the uniform estimate

$$\|I_{\varepsilon,\rho}(\nu, g)\|_2 \le c\|g\|_2 \le c\|g - f\|_2 + c\|f\|_2,$$

and, therefore,

$$\begin{aligned} \|I_{\varepsilon,\rho}(\nu, f)\|_2 &\le \|I_{\varepsilon,\rho}(\nu, f - g)\|_2 + \|I_{\varepsilon,\rho}(\nu, g)\|_2 \\ &\le \delta c_{\varepsilon,\rho,\nu} + c\|f - g\|_2 + c\|f\|_2 \\ &\le \delta(c_{\varepsilon,\rho,\nu} + c) + c\|f\|_2. \end{aligned}$$

Assuming $\delta \to 0$, we obtain $\|I_{\varepsilon,\rho}(\nu, f)\|_2 \le c\|f\|_2$. This gives

$$\begin{aligned} \|I_{\varepsilon,\rho}(\nu, f) - c_\nu f\|_2 &\le \|I_{\varepsilon,\rho}(\nu, f - g)\|_2 + \|I_{\varepsilon,\rho}(\nu, g) - c_\nu g\|_2 \\ &+ c_\nu \|g - f\|_2 \\ &\le (c + c_\nu)\delta + \|I_{\varepsilon,\rho}(\nu, g) - c_\nu g\|_2. \end{aligned}$$

Owing to (**??**) (with f replaced by g),

$$\|I_{\varepsilon,\rho}(\nu, f) - c_\nu f\|_2 \to (c + c_\nu)\delta \quad \text{as} \quad \varepsilon \to 0,\ \rho \to \infty.$$

Since δ is arbitrarily small, we are done. $\square$

4. Inversion of Riesz potentials

We recall that the wavelet transform of a function f on $\mathfrak{M}_{n,m}$ is defined by

$$(\mathcal{W}_a f)(x) = \int\limits_{\mathfrak{M}_{n,m}} f(x - ya^{1/2})\, w(y)\, dy = (f * w_a)(x)$$

where

$$w_a(x) = |a|^{-n/2} w(xa^{-1/2}), \qquad x \in \mathfrak{M}_{n,m}, \quad a \in \mathcal{P}_m. \tag{4.1}$$

Owing to (**??**), it is natural to expect, that the inverse of the Riesz potential (**??**) can be obtained if we formally replace α by $-\alpha$ in (**??**); cf. (**??**). This gives

$$f(x) = c_{n,m}(-\alpha, w) \int\limits_{\mathcal{P}_m} \frac{(\mathcal{W}_a I^\alpha f)(x)}{|a|^{\alpha/2}}\, d_* a. \tag{4.2}$$

Below we give this formula precise meaning.

Theorem 4.1. *Let $\alpha \in \mathbf{W}_{n,m}$ and $f \in L^2 \cap L^p$ for some p satisfying*

$$1 \le p < \frac{n}{Re\, \alpha + m - 1}\ .$$

Suppose that $w \in \mathcal{S}(\mathfrak{M}_{n,m})$ is a radial function such that
(a) *$(\mathcal{F}w)(y) = u_0(y'y)$, where $u_0(r)$ is a symmetric function on $\mathcal{P}_m$ vanishing identically in a neighborhood of the boundary $\partial\mathcal{P}_m$;*
(b) *The integral*

$$\begin{aligned} d_w(\alpha) &= \frac{2^m}{\sigma_{n,m}} \int\limits_{\mathfrak{M}_{n,m}} \frac{(\mathcal{F}w)(z)}{|z|_m^{n+\alpha}}\, dz \\ &= \lim_{B \to \infty} \frac{2^m}{\sigma_{n,m}} \int\limits_{\{z \in \mathfrak{M}_{n,m}:\, z'z < B\}} \frac{(\mathcal{F}w)(z)}{|z|_m^{n+\alpha}}\, dz \end{aligned} \tag{4.3}$$

($B \in \mathcal{P}_m$) is finite. Then

$$d_w(\alpha) f = \int\limits_{\mathcal{P}_m} \frac{\mathcal{W}_a I^\alpha f}{|a|^{\alpha/2}}\, d_* a = \overset{(L^2)}{\lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}}} \int\limits_{\varepsilon I_m}^{\rho I_m} \frac{\mathcal{W}_a I^\alpha f}{|a|^{\alpha/2}}\, d_* a. \tag{4.4}$$

Proof. We observe that the integrand in (**??**) has no singularity at $|z| = 0$ thanks to the assumption (a) above. Moreover, for $Re\, \alpha > m - 1$, this integral is finite automatically, because

$$\int\limits_{\mathfrak{M}_{n,m}} \frac{(\mathcal{F}w)(y)}{|y|_m^{n+\alpha}}\, dy \le c \int\limits_{\mathfrak{M}_{n,m}} |I_m + y'y|^{-(n+\alpha)/2} dy < \infty, \quad c \equiv c(w),$$

see formula (A.6) in [?].

To prove the theorem, we set

$$(4.5)\qquad (T^\alpha_{\varepsilon,\rho}\varphi)(x)=\int\limits_{\varepsilon I_m}^{\rho I_m}\frac{(\mathcal{W}_a\varphi)(x)}{|a|^{\alpha/2}}\,d_*a,\qquad 0<\varepsilon<\rho<\infty$$

(if $Re\,\alpha>m-1$ one can assume $\rho=\infty$), and show that

$$(4.6)\qquad T^\alpha_{\varepsilon,\rho}I^\alpha f=\mathcal{F}^{-1}[\psi^\alpha_{\varepsilon,\rho}\mathcal{F}f],$$

$$(4.7)\qquad \psi^\alpha_{\varepsilon,\rho}(y)=\frac{2^m}{\sigma_{n,m}}\int\limits_{\{z\in\mathfrak{M}_{n,m}:\,\varepsilon(y'y)<z'z<\rho(y'y)\}}\frac{(\mathcal{F}w)(z)}{|z|_m^{n+\alpha}}\,dz.$$

As we have shown this, by the Lebesgue dominated convergence theorem, owing to (**??**) and the uniform boundedness of $\psi^\alpha_{\varepsilon,\rho}$, we obtain the desired result:

$$\|T^\alpha_{\varepsilon,\rho}I^\alpha f-d_w(\alpha)f\|_2=(2\pi)^{-mn/2}\|(\psi^\alpha_{\varepsilon,\rho}-d_w(\alpha))\mathcal{F}f\|_2\to 0$$

as $\varepsilon\to 0,\ \rho\to\infty$.

We observe that for any $f\in L^p$ and $w\in L^1$,

$$(4.8)\qquad T^\alpha_{\varepsilon,\rho}I^\alpha f=I^\alpha g,\qquad g=T^\alpha_{\varepsilon,\rho}f.$$

The validity of interchange of integrals follows by Theorem **??**, according to which, the integral $I^{|\alpha|}[|f|*|w_a|](x)$ is finite for almost all x because $|f|*|w_a|\in L^p$.

We first prove (**??**) for f belonging to the Schwartz space $\mathcal{S}=\mathcal{S}(\mathfrak{M}_{n,m})$. By (**??**),

$$\mathcal{F}(W_af)(y)=\mathcal{F}(f*w_a)(y)=(\mathcal{F}f)(y)(\mathcal{F}w_a)(y),$$

where

$$\begin{aligned}(\mathcal{F}w_a)(y)&=|a|^{-n/2}\int\limits_{\mathfrak{M}_{n,m}}\exp(\mathrm{tr}(iy'x))w(xa^{-1/2})dx\\&=\int\limits_{\mathfrak{M}_{n,m}}\exp(\mathrm{tr}(ia^{1/2}y'z))w(z)dz=(\mathcal{F}w)(ya^{1/2}).\end{aligned}$$

Hence,

$$(\mathcal{F}g)(y)=h_{\varepsilon,\rho}(y)(\mathcal{F}f)(y),\qquad h_{\varepsilon,\rho}(y)=\int\limits_{\varepsilon I_m}^{\rho I_m}\frac{(\mathcal{F}w)(ya^{1/2})}{|a|^{\alpha/2}}\,d_*a.$$

By taking into account that $(\mathcal{F}w)(y) = u_0(y'y)$ where $u_0(r)$ is symmetric, we obtain

$$\begin{aligned} h_{\varepsilon,\rho}(y) &= \int\limits_{\varepsilon I_m}^{\rho I_m} \frac{u_0(a^{1/2}y'ya^{1/2})}{|a|^{\alpha/2}}\, d_*a \\ &= \frac{2^m|y'y|^{\alpha/2}}{\sigma_{n,m}} \int\limits_{\{z\in\mathfrak{M}_{n,m}:\, \varepsilon(y'y)<z'z<\rho(y'y)\}} \frac{(\mathcal{F}w)(z)}{|z|_m^{n+\alpha}}\, dz \\ &= |y|_m^\alpha \psi^\alpha_{\varepsilon,\rho}(y) \end{aligned} \tag{4.9}$$

(see the argument in the proof of Theorem **??**). This gives

$$(\mathcal{F}g)(y) = |y|_m^\alpha \psi^\alpha_{\varepsilon,\rho}(y)(\mathcal{F}f)(y). \tag{4.10}$$

Since $w \in \mathcal{S}$ and u_0 is supported away from the boundary $\partial\mathcal{P}_m$, it follows that $(\mathcal{F}g)(y) \equiv h_{\varepsilon,\rho}(y)(\mathcal{F}w)(y) \in \mathcal{S}$, and therefore, $g \in \mathcal{S}$. Hence, by (**??**), (**??**), and (**??**), for any compactly supported C^∞ function ϕ, we have

$$\begin{aligned} (T^\alpha_{\varepsilon,\rho} I^\alpha f,\ \phi) &= (I^\alpha g,\ \phi) \\ &= (2\pi)^{-nm}(\mathcal{F}I^\alpha g,\ \mathcal{F}\phi) \\ &= (2\pi)^{-nm}(|y|_m^{-\alpha}(\mathcal{F}g)(y),\ (\mathcal{F}\phi)(y)) \\ &= (2\pi)^{-nm}(\psi^\alpha_{\varepsilon,\rho}(y)(\mathcal{F}f)(y),\ (\mathcal{F}\phi)(y)). \end{aligned}$$

Thus, by the Parseval equality,

$$(T^\alpha_{\varepsilon,\rho} I^\alpha f,\ \phi) = (\mathcal{F}^{-1}[\psi^\alpha_{\varepsilon,\rho}\mathcal{F}f],\ \phi) \tag{4.11}$$

(note that $\psi^\alpha_{\varepsilon,\rho}(y)(\mathcal{F}f)(y) \in L^2$ in view of the boundedness of $\psi^\alpha_{\varepsilon,\rho}(y)$). Since $\mathcal{F}^{-1}[\psi^\alpha_{\varepsilon,\rho}\mathcal{F}f] \in L^2$ and $T^\alpha_{\varepsilon,\rho} I^\alpha f = I^\alpha g$ is a locally integrable function (see (**??**) and (**??**)), then (**??**) implies the pointwise equality (**??**) for any $f \in \mathcal{S}$.

To complete the proof, it remains to extend (**??**) to all $f \in L^2 \cap L^p$. Following Theorem **??**, we introduce the weighted space

$$X = \{\varphi :\ \|\varphi\|_X = \int\limits_{\mathfrak{M}_{n,m}} |\varphi(x)|\,\omega(x)\,dx < \infty\}$$

where $\omega(x) = \exp(-\mathrm{tr}(x'x))$ if $Re\,\alpha > m-1$, and $\omega(x) = |I_m + x'x|^{-\lambda/2}$ if $\alpha = k,\ k = 1, 2, \dots, n-m$; see (**??**). It may happened that these domains of α overlap, but this is not important. By Hölder's inequality, $||\varphi||_X \le c\,||\varphi||_2$. Since by Theorem **??**,

$$\|T^\alpha_{\varepsilon,\rho} I^\alpha f\|_X = \|I^\alpha g\|_X \le c\,\|g\|_p = c\,\|T^\alpha_{\varepsilon,\rho} f\|_p \le c_{\varepsilon,\rho}\|f\|_p,$$

and
$$\|\mathcal{F}^{-1}[\psi^{\alpha}_{\varepsilon,\rho}\mathcal{F}f]\|_X \le c\|\mathcal{F}^{-1}[\psi^{\alpha}_{\varepsilon,\rho}\mathcal{F}f]\|_2 \le c'_{\varepsilon,\rho}\|f\|_2,$$
operators
$$T^{\alpha}_{\varepsilon,\rho}I^{\alpha} : L^p \to X, \qquad \mathcal{F}^{-1}\psi^{\alpha}_{\varepsilon,\rho}\mathcal{F} : L^2 \to X$$
are bounded. This remark allows us to extend (**??**) to all $f \in L^2 \cap L^p$ by taking into account that there is a sequence $\{f_j\} \subset \mathcal{S}$ such that the quantities $\|f - f_j\|_p$ and $\|f - f_j\|_2$ tend to 0 as $j \to \infty$ simultaneously. Such a sequence can be explicitly constructed using the standard "averaging-truncating" procedure. □

5. Continuous ridgelet transforms and inversion of the Radon transform

5.1. **Intertwining operators.** Given a sufficiently good function w on $\mathfrak{M}_{n-k,m}$, consider the intertwining operator

$$(Wf)(\xi,t) \;=\; \int\limits_{\mathfrak{M}_{n,m}} f(x)\, w(t-\xi' x)\, dx \tag{5.1}$$

which transforms a function f on $\mathfrak{M}_{n,m}$ into a function Wf on the "cylinder" $V_{n,n-k}\times\mathfrak{M}_{n-k,m}$. The corresponding dual operator is defined by

$$(W^*\varphi)(x) \;=\; \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\, w(\xi' x - t)\, dt, \tag{5.2}$$

so that

$$\int\limits_{\mathfrak{M}_{n,m}} f(x)(W^*\varphi)(x)dx = \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)(Wf)(\xi,t)dt \tag{5.3}$$

(at least, formally). We shall see that $\mathcal{P}_m$-scaled versions of W and W^* can be regarded as matrix modifications of continuous k-plane ridgelet transforms (see [**?**], [**?**], and references therein), and used for explicit and approximate inversion of the Radon transform (**??**). We start with some preparations.

Lemma 5.1. *Given a function $\varphi(\xi,t)$ on $V_{n,n-k}\times\mathfrak{M}_{n-k,m}$, let*

$$(W_0\,\varphi)(\xi,t) = \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,z)\, w(t-z)\, dz \tag{5.4}$$

be a convolution in the t-variable. Then

$$(Wf)(\xi,t) = (W_0\hat{f})(\xi,t), \tag{5.5}$$

$$(W^*\varphi)(x) = (W_0\varphi)^\vee(x) \tag{5.6}$$

provided that either side of the corresponding equality is finite for f, φ, and w replaced by $|f|$, $|\varphi|$, and $|w|$, respectively.

Proof. The equality (**??**) follows immediately from (**??**). To prove (**??**), we choose a rotation $g_\xi \in SO(n)$ satisfying

$$g_\xi \xi_0 = \xi, \qquad \xi_0 = \begin{bmatrix} 0 \\ I_{n-k} \end{bmatrix} \in V_{n,n-k}.$$

The change of variable $x = g_\xi y$ in (**??**) gives

$$(Wf)(\xi, t) = \int\limits_{\mathfrak{M}_{n,m}} f(g_\xi y)\, w(t - \xi_0' y)\, dy.$$

By setting

$$y = \begin{bmatrix} \omega \\ z \end{bmatrix}, \qquad \omega \in \mathfrak{M}_{k,m}, \qquad z \in \mathfrak{M}_{n-k,m},$$

so that $\xi_0' y = z$, owing to (**??**), we obtain

$$\begin{aligned} (Wf)(\xi, t) &= \int\limits_{\mathfrak{M}_{n-k,m}} w(t - z)\; dz \int\limits_{\mathfrak{M}_{k,m}} f\left(g_\xi \begin{bmatrix} \omega \\ z \end{bmatrix}\right) d\omega \\ &= \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, z)\, w(t - z)\; dz = (W_0\hat{f})(\xi, t). \end{aligned}$$

$\square$

Lemma 5.2. *Let $f(x)$ and $w(z)$ be integrable functions on $\mathfrak{M}_{n,m}$ and $\mathfrak{M}_{n-k,m}$, respectively. Then $(W^*\hat{f})(x)$ is a locally integrable function on $\mathfrak{M}_{n,m}$ which belongs to $\mathcal{S}'(\mathfrak{M}_{n,m})$ and satisfies*

$$\int\limits_{\mathfrak{M}_{n,m}} (W^*\hat{f})(x)\, \phi(x)\, dx = \int\limits_{\mathfrak{M}_{n,m}} f(x)\, (W^*\hat{\phi})(x)\, dx, \quad \phi \in \mathcal{S}(\mathfrak{M}_{n,m}). \tag{5.7}$$

Proof. We have

$$
\begin{aligned}
\text{l.h.s} &\overset{(??)}{=} \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi,t)(W\phi)(\xi,t)dt \\
&\overset{(??)}{=} \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi,t)(W_0\hat{\phi})(\xi,t)dt \qquad (5.8)\\
&\overset{(??)}{=} \int\limits_{\mathfrak{M}_{n,m}} f(x)\,(W_0\hat{\phi})^{\vee}(x)\,dx \\
&\overset{(??)}{=} \text{r.h.s.}
\end{aligned}
$$

These calculations are well justified and all statements of the lemma become clear, owing to the following estimate of the expression (**??**):

$$
\begin{aligned}
&\int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} |\hat{f}(\xi,z)|\,dz \int\limits_{\mathfrak{M}_{n-k,m}} |\hat{\phi}(\xi,t)\,w(t-z)|\,dt \qquad (5.9)\\
\leq\ & \|\hat{\phi}\|_\infty \|w\|_1 \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} |\hat{f}(\xi,z)|\,dz \\
\overset{(??)}{=}\ & \|\hat{\phi}\|_\infty \|w\|_1 \|f\|_1.
\end{aligned}
$$

□

In the sequel, it is convenient to use different notations for the Fourier transform on $\mathfrak{M}_{n,m}$ and $\mathfrak{M}_{n-k,m}$. For the first one we write $\mathcal{F}$ as before, and the second will be denoted by $\tilde{\mathcal{F}}$.

Lemma 5.3. *If* $w \in L^1(\mathfrak{M}_{n-k,m})$ *and* $\phi \in \mathcal{S}(\mathfrak{M}_{n,m})$*, then*

$$
\begin{aligned}
(W^*\hat{\phi})(x) &= (2\pi)^{(k-n)m} \int\limits_{V_{n,n-k}} d_*\xi \qquad (5.10)\\
&\times \int\limits_{\mathfrak{M}_{n-k,m}} \exp(-\mathrm{tr}(ix'\xi z))(\tilde{\mathcal{F}}w)(z)(\mathcal{F}\phi)(\xi z)dz.
\end{aligned}
$$

Proof. Since $w \in L^1(\mathfrak{M}_{n-k,m})$ and $\phi \in \mathcal{S}(\mathfrak{M}_{n,m})$, the convolution

$$
(W_0\,\hat{\phi})(\xi,t) = \int\limits_{\mathfrak{M}_{n-k,m}} \hat{\phi}(\xi,t-z)\,w(z)\,dz
$$

has the Fourier transform (in the t-variable) belonging to $L^1(\mathfrak{M}_{n-k,m})$. Hence (see, e.g., [**?**, p. 11]) one can write

$$\begin{aligned}(W_0\,\hat{\phi})(\xi,t) &= (2\pi)^{(k-n)m} \int\limits_{\mathfrak{M}_{n-k,m}} \exp(-\mathrm{tr}(it'z))[\tilde{\mathcal{F}}(W_0\,\hat{\phi})(\xi,\cdot)](z)dz \\ &= (2\pi)^{(k-n)m} \int\limits_{\mathfrak{M}_{n-k,m}} \exp(-\mathrm{tr}(it'z))(\tilde{\mathcal{F}}w)(z)[\tilde{\mathcal{F}}\hat{\phi}(\xi,\cdot)](z)dz.\end{aligned}$$

By the projection-slice theorem (see (**??**)),

$$(W_0\,\hat{\phi})(\xi,t) = (2\pi)^{(k-n)m} \int\limits_{\mathfrak{M}_{n-k,m}} \exp(-\mathrm{tr}(it'z))(\tilde{\mathcal{F}}w)(z)(\mathcal{F}\phi)(\xi z)dz.$$

This proves the statement. □

5.2. **Continuous ridgelet transforms.** Let $w(z)$ be a sufficiently good function on $\mathfrak{M}_{n-k,m}$, $1 \le k \le n-m$. We consider the $\mathcal{P}_m$-scaled version of w defined by $w_a(z) = |a|^{(k-n)/2} w(za^{-1/2})$, $a \in \mathcal{P}_m$, and introduce the following dual pair of intertwining operators

$$(W_a f)(\xi,t) = \int\limits_{\mathfrak{M}_{n,m}} f(x)\, w_a(t-\xi'x)\, dx, \tag{5.11}$$

$$(W_a^*\varphi)(x) = \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\, w_a(\xi'x-t)\, dt. \tag{5.12}$$

If $w(z)$ oscillates in a certain sense we call (**??**) the *continuous ridgelet transform* of f, and (**??**) the *dual continuous ridgelet transform* of φ.

Operators (**??**) and (**??**) generalize usual k-plane ridgelet transforms [**?**], [**?**] to the higher rank case $m > 1$. The function $x \to w(\xi'x - t)$ is constant on each matrix plane $\tau = \{x \in \mathfrak{M}_{n,m} : \xi'x = t\}$ and represents a "plane wave".

The following definition will be useful in the sequel.

Definition 5.4. A function $w(z)$ on $\mathfrak{M}_{n-k,m}$ is called an *admissible wavelet function* if it obeys the following conditions:
(i) $w(z)$ is radial, i.e., $w(z) \equiv w_0(z'z)$, and belongs to $L^1(\mathfrak{M}_{n-k,m})$.
(ii) The Fourier transform of w has the form $(\tilde{\mathcal{F}}w)(y) = u_0(y'y)$, where u_0 satisfies the symmetry condition (**??**).

(iii) The integral

$$c_w = \frac{2^{m(k+1)}\pi^{km}}{\sigma_{n,m}} \int\limits_{\mathfrak{M}_{n-k,m}} \frac{(\tilde{\mathcal{F}}w)(\zeta)}{|\zeta|_m^n}\, d\zeta = \lim_{\substack{A\to 0\\ B\to\infty}} \frac{2^{m(k+1)}\pi^{km}}{\sigma_{n,m}} \int\limits_{\{\zeta\in\mathfrak{M}_{n-k,m}:\, A<\zeta'\zeta<B\}} \frac{(\tilde{\mathcal{F}}w)(\zeta)}{|\zeta|_m^n}\, d\zeta \tag{5.13}$$

$(A, B \in \mathcal{P}_m)$ is finite.

5.3. **Inversion of the Radon transform.**

5.3.1. *Discussion of the problem.* There exist different approaches to inversion of the Radon transform (**??**); see [**?**]. The consideration below sheds new light on this problem and provides essential progress. To explain our strategy, we use intertwining fractional integrals $P^\alpha f$ and $\overset{*}{P}{}^\alpha\varphi$ of the Semyanistyi type, which link together the Radon transform $f(x) \to \hat{f}(\xi,t)$, the dual Radon transform $\varphi(\xi,t) \to \check{\varphi}(x)$, and Riesz potentials. Namely, we define

$$P^\alpha f = \tilde{I}^\alpha \hat{f}, \qquad \overset{*}{P}{}^\alpha\varphi = (\tilde{I}^\alpha\varphi)^\vee, \tag{5.14}$$

$$\alpha\in\mathbb{C}, \qquad \alpha\neq n-k-m+1,\, n-k-m+2, \ldots .$$

Here, $1\le k\le n-m$ and $\tilde{I}^\alpha$ denotes the Riesz potential on $\mathfrak{M}_{n-k,m}$ acting in the t-variable. Operators (**??**) were introduced in [**?**]. If $Re\,\alpha > m-1$, they are represented as absolutely convergent integrals

$$(P^\alpha f)(\xi,t) = \frac{1}{\gamma_{n-k,m}(\alpha)} \int\limits_{\mathfrak{M}_{n,m}} f(x)\, |\xi' x - t|_m^{\alpha+k-n}\, dx, \tag{5.15}$$

$$(\overset{*}{P}{}^\alpha\varphi)(x) = \frac{1}{\gamma_{n-k,m}(\alpha)} \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\, |\xi' x - t|_m^{\alpha+k-n}\, dt, \tag{5.16}$$

where $\gamma_{n-k,m}(\alpha)$ is the normalizing constant in the definition of the Riesz potential on $\mathfrak{M}_{n-k,m}$; cf. (**??**), (**??**).

The following statement determines our way of thinking.

Theorem 5.5 ([**?**], Section 5.3)**.** *Let* $1\le k\le n-m$, $\alpha\in\mathbf{W}_{n,m}$. *Suppose that*

$$f\in L^p(\mathfrak{M}_{n,m}), \qquad 1\le p < \frac{n}{Re\,\alpha + k + m - 1}\,.$$

Then

$$(\overset{*}{P}{}^\alpha \hat{f})(x) = c_{n,k,m}(I^{\alpha+k} f)(x), \tag{5.17}$$

$$c_{n,k,m} = 2^{km}\pi^{km/2}\Gamma_m\left(\frac{n}{2}\right)/\Gamma_m\left(\frac{n-k}{2}\right). \tag{5.18}$$

In particular, for $\alpha = 0$,

$$(\hat{f})^\vee(x) = c_{n,k,m}(I^k f)(x) \tag{5.19}$$

(the generalized Fuglede formula).

Formula (**??**) paves two ways to the inversion of the Radon transform. Following the first one, we use (**??**) as it is and invert the Riesz potential $I^{\alpha+k}f$ by choosing α as we wish. For instance, one can set $\alpha = 0$ and apply (**??**). This program can be realized using results of the previous section. The second way is to set (formally) $\alpha = -k$ in (**??**). This gives

$$c_{n,k,m}f(x) = (\overset{*}{P}{}^{-k}\hat{f})(x) = (\tilde{I}^{-k}\hat{f})^\vee(x), \tag{5.20}$$

and we have to find "good" representation for the inverse of the Riesz potential $\tilde{I}^k$ applied to $\hat{f}(\xi, t)$ in the t-variable. In the first case, we just apply the left inverse operator to $I^{\alpha+k}$. In the second one, we do not know in advance whether $\hat{f}(\xi, \cdot)$ lies in the range of the Riesz potential $\tilde{I}^k$. To circumvent this difficulty, we make use of continuous ridgelet transforms.

Below we consider both approaches.

5.3.2. *The first method.* We utilize the generalized Fuglede formula (**??**) and invert the Riesz potential according to Theorem **??**. This gives the following result for the Radon transform.

Theorem 5.6. *Let $1 \le k \le n - m$,*

$$f \in L^2 \cap L^p, \qquad 1 \le p < \frac{n}{k+m-1}. \tag{5.21}$$

*Suppose that $\mathcal{W}_a$ is the continuous wavelet transform (**??**) generated by the wavelet w satisfying conditions of Theorem **??**. Then the Radon transform $f(x) \to \hat{f}(\xi, t)$ can be inverted by the formula*

$$d_w(k)\, f = c_{n,k,m} \int\limits_{\mathcal{P}_m} \frac{\mathcal{W}_a \hat{f}}{|a|^{k/2}}\, d_*a = c_{n,k,m} \underset{\substack{\varepsilon\to 0\\ \rho\to\infty}}{\overset{(L^2)}{\lim}} \int\limits_{\varepsilon I_m}^{\rho I_m} \frac{W_a \hat{f}}{|a|^{k/2}}\, d_*a \tag{5.22}$$

where

$$c_{n,k,m} = \frac{2^{km}\pi^{km/2}\Gamma_m(n/2)}{\Gamma_m((n-k)/2)}, \qquad d_w(k) = \frac{2^m}{\sigma_{n,m}} \int\limits_{\mathfrak{M}_{n,m}} \frac{(\mathcal{F}w)(y)}{|y|_m^{n+k}}\, dy.$$

5.3.3. *The second method.* By (**??**) and (**??**), it is natural to expect, that the Radon transform can be inverted as

$$f(x) = \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathcal{P}_m} \frac{(\hat{f}(\xi,\cdot) * w_a)(\xi' x)}{|a|^{k/2}}\, d_*a = \int\limits_{\mathcal{P}_m} \frac{(W_a^* \hat{f})(x)}{|a|^{k/2}}\, d_*a$$

(up to a constant multiple) where W_a^* is the dual ridgelet transform (**??**). Below we justify this formula.

Theorem 5.7. *Let $f \in L^1(\mathfrak{M}_{n,m}) \cap L^2(\mathfrak{M}_{n,m})$, $1 \le k \le n-m$. If w is an admissible wavelet function (see Definition* **??***), then the Radon transform $f(x) \to \hat{f}(\xi,t)$ can be inverted by the formula*

$$c_w\, f = \int\limits_{\mathcal{P}_m} \frac{W_a^* \hat{f}}{|a|^{k/2}}\, d_*a = \overset{(L^2)}{\lim_{\substack{\varepsilon \to 0 \\ \rho \to \infty}}} \int\limits_{\varepsilon I_m}^{\rho I_m} \frac{W_a^* \hat{f}}{|a|^{k/2}}\, d_*a, \tag{5.23}$$

*where c_w is defined by (**??**), and*

$$(W_a^* \hat{f})(x) = \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi,t)\, w_a(\xi' x - t)\, dt. \tag{5.24}$$

Proof. Consider the truncated integral

$$I_{\varepsilon,\rho} f = \int\limits_{\varepsilon I_m}^{\rho I_m} \frac{W_a^* \hat{f}}{|a|^{k/2}}\, d_*a, \qquad 0 < \varepsilon < \rho < \infty. \tag{5.25}$$

Owing to (**??**), for any test function $\phi \in \mathcal{S}$, the expression $(I_{\varepsilon,\rho} f,\ \phi)$ is finite when f, w, and ϕ are replaced by $|f|$, $|w|$ and $|\phi|$, respectively. Hence, we can change the order of integration, and (**??**) yields

$$(I_{\varepsilon,\rho} f,\ \phi) = (f,\ \dot{I}_{\varepsilon,\rho} \phi). \tag{5.26}$$

where $\dot{I}_{\varepsilon,\rho}$ has the same meaning as in (**??**) but with w replaced by its complex conjugate $\bar{w}$. Let us show that

$$(\dot{I}_{\varepsilon,\rho} \phi)(x) = \mathcal{F}^{-1}[\, \overline{m_{\varepsilon,\rho}}\, \mathcal{F}\phi\,](x), \tag{5.27}$$

where $m_{\varepsilon,\rho}(y) = \tilde{m}_{\varepsilon,\rho}(y' y)$,

$$\tilde{m}_{\varepsilon,\rho}(r) = \frac{2^{m(k+1)} \pi^{km}}{\sigma_{n,m}} \int\limits_{\{\zeta \in \mathfrak{M}_{n-k,m} :\, \varepsilon r < \zeta' \zeta < \rho r\}} \frac{(\tilde{\mathcal{F}} w)(\zeta)}{|\zeta|_m^n}\, d\zeta, \tag{5.28}$$

$r \in \mathcal{P}_m$. Suppose that the Fourier transform of w has the form $(\tilde{\mathcal{F}} w)(\zeta) = u_0(\zeta' \zeta)$ where u_0 obeys the symmetry condition (**??**). By

(**??**) (with w replaced by $\overline{w_a}$), we have

$$(\dot{I}_{\varepsilon,\rho}\phi)(x)=(2\pi)^{(k-n)m}\int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \exp(-\mathrm{tr}(ix'\xi z))k^\alpha_{\varepsilon,\rho}(z)(\mathcal{F}\phi)(\xi z)dz,$$

where

$$k^\alpha_{\varepsilon,\rho}(z) \;=\; \int\limits_{\varepsilon I_m}^{\rho I_m} \frac{(\tilde{\mathcal{F}}\bar{w})(za^{1/2})}{|a|^{k/2}}\, d_*a = \int\limits_{\varepsilon I_m}^{\rho I_m} \frac{\overline{u_0(a^{1/2}ra^{1/2})}}{|a|^{k/2}}\, d_*a, \qquad r=z'z.$$

Owing to the symmetry (**??**), we have $u_0(a^{1/2}ra^{1/2})=u_0(r^{1/2}ar^{1/2})$ (without loss of generality, one can assume $\mathrm{rank}(z)=m$). Then by changing variable $r^{1/2}ar^{1/2}=s$ and making use of Lemma **??**, we obtain

$$\begin{aligned} k^\alpha_{\varepsilon,\rho}(z) \;&=\; \int\limits_{\varepsilon I_m}^{\rho I_m} \frac{\overline{u_0(r^{1/2}ar^{1/2})}}{|a|^{k/2}}\, d_*a = |r|^{k/2}\int\limits_{\varepsilon r}^{\rho r} \frac{\overline{u_0(s)}}{|s|^{k/2}}\, d_*s \\ &=\; \frac{2^m}{\sigma_{n-k,m}}\,|r|^{k/2} \int\limits_{\{\zeta\in\mathfrak{M}_{n-k,m}:\,\varepsilon r<\zeta'\zeta<\rho r\}} \frac{(\tilde{\mathcal{F}}\bar{w})(\zeta)}{|\zeta|^n_m}\, d\zeta. \end{aligned}$$

Replacing ζ by $-\zeta$ and using the equality $(\tilde{\mathcal{F}}\bar{w})(-\zeta)=\overline{(\tilde{\mathcal{F}}w)(\zeta)}$, we have

$$k^\alpha_{\varepsilon,\rho}(z)=\frac{\sigma_{n,m}\,(2\pi)^{-km}}{\sigma_{n-k,m}}\,|r|^{k/2}\,\overline{\tilde{m}_{\varepsilon,\rho}(r)}.$$

Hence

$$(\dot{I}_{\varepsilon,\rho}\phi)(x)=c\int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \exp(-\mathrm{tr}(ix'\xi z))\,|z|^k_m\,\overline{\tilde{m}_{\varepsilon,\rho}(z'z)}(\mathcal{F}\phi)(\xi z)\,dz,$$

where $c=(2\pi)^{-nm}\sigma_{n,m}/\sigma_{n-k,m}$. By Lemma **??**,

$$(\dot{I}_{\varepsilon,\rho}\phi)(x)=(2\pi)^{-nm}\int\limits_{\mathfrak{M}_{n,m}} \exp(-\mathrm{tr}(ix'y))\,\overline{\tilde{m}_{\varepsilon,\rho}(y'y)}\,(\mathcal{F}\phi)(y)\,dy,$$

and (**??**) follows.

The rest of the proof is standard. Since $f\in L^2$ and $m_{\varepsilon,\rho}$ is uniformly bounded, then by the Parseval equality,

$$(I_{\varepsilon,\rho}f,\ \phi)=(\mathcal{F}^{-1}m_{\varepsilon,\rho}\mathcal{F}f,\ \phi), \qquad \forall\phi\in\mathcal{S}(\mathfrak{M}_{n,m}). \tag{5.29}$$

By Lemma **??**, $I_{\varepsilon,\rho}f$ is a locally integrable function. Since $\mathcal{F}^{-1}m_{\varepsilon,\rho}\mathcal{F}f$ is locally integrable too, then (**??**) implies a pointwise equality $I_{\varepsilon,\rho}f=$

$\mathcal{F}^{-1}m_{\varepsilon,\rho}\mathcal{F}f$, and we have

$$\begin{aligned}\|I_{\varepsilon,\rho}f - c_w f\|_2 &= \|\mathcal{F}^{-1}m_{\varepsilon,\rho}\mathcal{F}f - c_w f\|_2 \\ &= (2\pi)^{-nm/2}\|(m_{\varepsilon,\rho} - c_w)\mathcal{F}f\|_2 \to 0\end{aligned}$$

as $\varepsilon \to 0$, $\rho \to \infty$. □

5.4. **Reproducing formula for the ridgelet transform.** Given two functions $u(z)$ and $v(z)$ on $\mathfrak{M}_{n-k,m}$, we set

$$u_a(z) = |a|^{(k-n)/2}u(za^{-1/2}), \qquad v_a(z) = |a|^{(k-n)/2}v(za^{-1/2}), \quad a \in \mathcal{P}_m,$$

and consider the corresponding ridgelet transforms

$$(5.30) \qquad (U_a f)(\xi,t) = \int\limits_{\mathfrak{M}_{n,m}} f(x)\, u_a(t - \xi' x)\, dx,$$

$$(5.31) \qquad (V_a^*\varphi)(x) = \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \varphi(\xi,t)\, v_a(\xi' x - z)\, dz,$$

cf. (**??**), (**??**).

Theorem 5.8. *Let u and v be integrable radial functions on $\mathfrak{M}_{n-k,m}$, $1 \le k \le n-m$, such that their convolution $u * v$ is admissible (see Definition **??**). Let*

$$\begin{aligned}(5.32)\quad c_{u,v} &= \frac{2^{m(k+1)}\pi^{km}}{\sigma_{n,m}} \int\limits_{\mathfrak{M}_{n-k,m}} \frac{(\tilde{\mathcal{F}}u)(\zeta)(\tilde{\mathcal{F}}v)(\zeta)}{|\zeta|_m^n}\, d\zeta \\ &= \lim_{\substack{A\to 0\\ B\to\infty}} \frac{2^{m(k+1)}\pi^{km}}{\sigma_{n,m}} \int\limits_{\{\zeta\in\mathfrak{M}_{n-k,m}:\, A<z'z<B\}} \frac{(\tilde{\mathcal{F}}u)(\zeta)(\tilde{\mathcal{F}}v)(\zeta)}{|\zeta|_m^n}\, dz,\end{aligned}$$

$(A, B \in \mathcal{P}_m)$. Then for $f \in L^1(\mathfrak{M}_{n,m}) \cap L^2(\mathfrak{M}_{n,m})$,

$$(5.33) \qquad c_{u,v} f = \int\limits_{\mathcal{P}_m} \frac{V_a^* U_a f}{|a|^{k/2}}\, d_*a = \overset{(L^2)}{\lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}}} \int\limits_{\varepsilon I_m}^{\rho I_m} \frac{V_a^* U_a f}{|a|^{k/2}}\, d_*a.$$

Proof. Let us show that $V_a^* U_a f$ coincides with the dual ridgelet transform $W_a^* \hat{f}$ (see (**??**)) generated by the function $w = u * v$. We have

$$\begin{aligned}
(W_a^* \hat{f})(x) &= \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, z)\, dz \int\limits_{\mathfrak{M}_{n-k,m}} u_a(t)\, v_a(\xi' x - z - t) dt \\
&= \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} \hat{f}(\xi, z)\, dz \int\limits_{\mathfrak{M}_{n-k,m}} u_a(\zeta - z)\, v_a(\xi' x - \zeta) d\zeta \\
&= \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} v_a(\xi' x - \zeta) d\zeta \int\limits_{\mathfrak{M}_{n-k,m}} u_a(\zeta - z) \hat{f}(\xi, z)\, dz \\
&\overset{(??)}{=} \int\limits_{V_{n,n-k}} d_*\xi \int\limits_{\mathfrak{M}_{n-k,m}} v_a(\xi' x - \zeta) d\zeta \int\limits_{\mathfrak{M}_{n,m}} f(x)\, u_a(\zeta - \xi' x)\, dx \\
&= (V_a^* U_a f)(x).
\end{aligned}$$

Now the result follows by Theorem **??**. □

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