PRODUCTS OF SPECIAL SETS OF REAL NUMBERS

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Abstract. We describe a simple machinery which translates results on algebraic sums of sets of reals into the corresponding results on their cartesian product. Some consequences are:

(1) The product of a meager/null-additive set and a strong measure zero/strongly meager set in the Cantor space has strong measure zero/is strongly meager, respectively. This extends a result of Scheepers from [12].
(2) Using Scheepers' notation for selection principles:
    (a) Borel's Conjecture for $\mathsf{S}_1(\Omega,\Omega)$ implies Borel's Conjecture.
    (b) $\mathsf{S}_{fin}(\Omega,\Omega^{gp})\cap\mathsf{S}_1(\mathcal{O},\mathcal{O})=\mathsf{S}_1(\Omega,\Omega^{gp})$.

1. Products in the Cantor space

The Cantor space $\mathcal{C}=\{0,1\}^{\mathbb{N}}$ is equipped with the product topology. For distinct $x,y\in\mathcal{C}$, write $N(x,y)=\min\{n:x(n)\neq y(n)\}$. Then the topology of $\mathcal{C}$ is generated by the following metric:

$$d(x,y)=\begin{cases}0 & x=y\\ \frac{1}{N(x,y)+1} & \text{otherwise}\end{cases}$$

(so that $d(x,y)\leq 1$ for all $x,y\in\mathcal{C}$). A canonical measure μ is defined on $\mathcal{C}$ by taking the product of the uniform probability measure on $\{0,1\}$. Fix a natural number k, and consider the product space $\mathcal{C}^k$. Define the *product metric* d_k on $\mathcal{C}^k$ by

$$d_k((x_0,\dots,x_{k-1}),(y_0,\dots,y_{k-1}))=\max\{d(x_0,y_0),\dots,d(x_{k-1},y_{k-1})\}.$$

Then d_k generates the topology of $\mathcal{C}^k$. The measure on $\mathcal{C}^k$ is the product measure $\mu\times\ldots\times\mu$ (k times).

$\mathcal{C}$, with the operation $\oplus$ defined by $(x\oplus y)(n)=x(n)+y(n)\bmod 2$ is a topological group, and therefore so is $\mathcal{C}^k$ for all k.

Lemma 1.1. *The function $\Psi_k:\mathcal{C}^k\to\mathcal{C}$ defined by*

$$\Psi_k(x_0,\dots,x_{k-1})(mk+i)=x_i(m)$$

for each m and each $i<k$, is a bi-Lipschitz measure preserving group isomorphism.

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Proof. Clearly Ψ_k is bijective. Assume that $\vec{x} = (x_0, \dots, x_{k-1})$ and $\vec{y} = (y_0, \dots, y_{k-1})$ are members of $\mathcal{C}^k$. Then for each m and each $i < k$,

$$\begin{aligned} &\Psi_k(\vec{x} \oplus \vec{y})(mk+i) = \\ &= \Psi_k(x_0 \oplus y_0, \dots, x_{k-1} \oplus y_{k-1})(mk+i) = (x_i \oplus y_i)(m) = \\ &= x_i(m) + y_i(m) \bmod 2 = \Psi_k(\vec{x})(mk+i) + \Psi_k(\vec{y})(mk+i) \bmod 2. \end{aligned}$$

Thus, $\Psi_k(\vec{x} \oplus \vec{y}) = \Psi_k(\vec{x}) \oplus \Psi_k(\vec{y})$, and Ψ_k is a group isomorphism.

Now, assume that $\vec{x} = (x_0, \dots, x_{k-1})$ and $\vec{y} = (y_0, \dots, y_{k-1})$ are distinct members of $\mathcal{C}^k$, and let i be such that $d(x_i, y_i)$ is maximal, that is, $N = N(x_i, y_i)$ is minimal. Then $N(\Psi_k(\vec{x}), \Psi_k(\vec{y})) \geq kN$, and therefore

$$d(\Psi_k(\vec{x}), \Psi_k(\vec{y})) \leq \frac{1}{kN+1} < d(x, y).$$

Similarly, for distinct $x, y \in \mathcal{C}$, if $N(x, y) = mk+i$ where $i < k$, then $N(\Psi_k^{-1}(x), \Psi_k^{-1}(y)) \geq m$ and

$$d(\Psi_k^{-1}(x), \Psi_k^{-1}(y)) \leq \frac{1}{m+1} \leq \frac{k}{mk+i+1} = k \cdot d(x, y).$$

To see that Ψ_k is measure preserving, observe that the measure of a basic open set U in $\mathcal{C}$ is 2^{-m}, where m is the number of coordinates of U which are not equal to $\{0, 1\}$. Consequently, the same assertion is true for $\mathcal{C}^k$, where m is the sum of numbers of such coordinates within each of the k coordinates of $\mathcal{C}^k$. It remains to observe that this number m is invariant under Ψ_k. Thus Ψ_k preserves measures of basic open sets, and therefore of all measurable sets. □

This allows us, in many cases, to restrict attention to subsets of $\mathcal{C}$ rather than talking about subsets of $\mathcal{C}^k$ for arbitrary k.

Notation 1.2. Throughout the paper, we will use the notation $\tilde{P}(A)$ for the family $\bigcup_{k \in \mathbb{N}} P(A^k)$ (so that $X \in \tilde{P}(A)$ if, and only if, $X \subseteq A^k$ for some k).

Theorem 1.3. *Assume that $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ are subsets of $\tilde{P}(\mathcal{C})$, and that $\mathcal{K}$ is closed under taking Lipschitz images. If*

for each $X \in \mathcal{I} \cap P(\mathcal{C}^k)$ and $Y \in \mathcal{J} \cap P(\mathcal{C}^k)$, $X \times Y \in \mathcal{K}$,

then

for each $X \in \mathcal{I} \cap P(\mathcal{C}^k)$ and $Y \in \mathcal{J} \cap P(\mathcal{C}^k)$, $X \oplus Y \in \mathcal{K}$.

Proof. The mapping $(x, y) \mapsto x \oplus y$ is a Lipschitz mapping from $X \times Y$ onto $X \oplus Y$. □

The converse of Theorem 1.3 also holds, and in a much stronger form. For simplicity, we introduce the following notions.

Definition 1.4. A collection $\mathcal{P} \subseteq \tilde{P}(\mathcal{C})$ is *semiproductive* if:

(1) For each k, l, and $X \in \mathcal{P} \cap \mathcal{C}^k$, if 0 is the zero element of $\mathcal{C}^l$, then $X \times \{0\} \in \mathcal{P}$; and
(2) For each k, l, $X \in \mathcal{P} \cap \mathcal{C}^k$, and a bi-Lipschitz measure preserving group isomorphism $\Phi : \mathcal{C}^k \to \mathcal{C}^l$, $\Phi[X] \in \mathcal{P}$.

We will say that $\mathcal{P}$ is 0-*productive* if we only require that (1) is satisfied, and *iso-productive* if we only require that (2) is satisfied.

As changing the order of coordinates is a bi-Lipschitz measure preserving group isomorphism, we have the following.

Lemma 1.5. *Assume that $\mathcal{P} \subseteq \tilde{P}(\mathcal{C})$ is semiproductive. Then for each k, l, and $X \in \mathcal{P} \cap \mathcal{C}^k$, if 0 is the zero element of $\mathcal{C}^l$, then $\{0\} \times X \in \mathcal{P}$.*

Theorem 1.6. *Assume that $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ are subsets of $\tilde{P}(\mathcal{C})$ such that $\mathcal{I}$ and $\mathcal{J}$ are semiproductive, and $\mathcal{K}$ is iso-productive. If*

for each $X \in \mathcal{I} \cap P(\mathcal{C})$ and $Y \in \mathcal{J} \cap P(\mathcal{C})$, $X \oplus Y \in \mathcal{K}$,

then

for each $X \in \mathcal{I}$ and $Y \in \mathcal{J}$, $X \times Y \in \mathcal{K}$.

Proof. Assume that $X \in \mathcal{I} \cap P(\mathcal{C}^k)$ and $Y \in \mathcal{J} \cap P(\mathcal{C}^l)$. Then $\tilde{X} = \Psi_k[X] \in \mathcal{I} \cap P(\mathcal{C})$, and $\tilde{Y} = \Psi_l[Y] \in \mathcal{J} \cap P(\mathcal{C})$.

$$\begin{aligned}\Psi_2[\tilde{X} \times \tilde{Y}] &= \Psi_2[(\tilde{X} \times \{0\}) \oplus (\{0\} \times \tilde{Y})] = \\ &= \Psi_2[\tilde{X} \times \{0\}] \oplus \Psi_2[\{0\} \times \tilde{Y}] = X' \oplus Y',\end{aligned}$$

Thus, $X' \oplus Y' \in \mathcal{K}$. As Ψ_2 is bijective, $\tilde{X} \times \tilde{Y} = \Psi_2^{-1}[X' \oplus Y'] \in \mathcal{K}$. As $\Psi_k \times \Psi_l$ is a bi-Lipschitz measure preserving group isomorphism and $\Psi_k^{-1} \times \Psi_l^{-1} : \tilde{X} \times \tilde{Y} \to X \times Y$ is surjective, $X \times Y \in \mathcal{K}$. $\square$

Almost all properties of special sets of reals which were considered in the literature are closed under taking bi-Lipschitz images and products with singletons, e.g., Hausdorff dimension, strong measure zero, and all properties in the special-sets version of the Cichoń Diagram [8] as well as in the Scheepers Diagram (see Section 4) and its extensions [13, 14]; see, e.g., [5, 11] for many more examples. The property of having measure zero is not closed under taking Lipschitz images: $\mathcal{C} \times \{0\}$ has measure zero in $\mathcal{C}^2$, but its Lipschitz image $\mathcal{C}$ does not. However this property is by definition closed under taking measure preserving images.

Assume that $\mathcal{I} \subseteq \tilde{P}(\mathcal{C})$. A subset X of $\mathcal{C}^k$ is $\mathcal{I}$-*additive* if for each $I \in \mathcal{I} \cap P(\mathcal{C}^k)$, $X \oplus I \in \mathcal{I}$. Clearly if $X, Y \subseteq \mathcal{C}$ are $\mathcal{I}$-additive, then $X \oplus Y$ is $\mathcal{I}$-additive. More generally, we will say that a subset X of $\mathcal{C}^k$ is $(\mathcal{I}, \mathcal{J})$-*additive* if for each $I \in \mathcal{I} \cap \mathcal{C}^k$, $X \oplus I \in \mathcal{J}$. Let $\mathcal{I}^\star$ and $(\mathcal{I}, \mathcal{J})^\star$ denote the classes of all $\mathcal{I}$-additive sets and $(\mathcal{I}, \mathcal{J})$-additive sets, respectively.

Theorem 1.7. *Assume that $\mathcal{I}, \mathcal{J}, \mathcal{K} \subseteq \tilde{P}(\mathcal{C})$ are iso-productive, and $(\mathcal{J}, \mathcal{K})^\star, (\mathcal{I}, \mathcal{J})^\star$ are 0-productive. Then for each $X \in (\mathcal{J}, \mathcal{K})^\star$ and $Y \in (\mathcal{I}, \mathcal{J})^\star$, $X \times Y \in (\mathcal{I}, \mathcal{K})^\star$. In particular, if $\mathcal{I}$ is iso-productive and $\mathcal{I}^\star$ is 0-productive, then $\mathcal{I}^\star$ is closed under taking finite products.*

Proof. Assume that $X \in (\mathcal{J}, \mathcal{K})^\star \cap P(\mathcal{C})$, $Y \in (\mathcal{I}, \mathcal{J})^\star \cap P(\mathcal{C})$, and $I \in \mathcal{I}$. Then $Y \oplus I \in \mathcal{J}$ and therefore $X \oplus (Y \oplus I) \in \mathcal{K}$. Thus $X \oplus Y \in (\mathcal{I}, \mathcal{K})^\star$.

Lemma 1.8. *Assume that $\mathcal{I}, \mathcal{J} \subseteq \tilde{P}(\mathcal{C})$ are iso-productive. Then $(\mathcal{I}, \mathcal{J})^\star$ is iso-productive.*

Proof. Assume that $X \in (\mathcal{I}, \mathcal{J})^\star \cap P(\mathcal{C}^k)$ and $\Phi : \mathcal{C}^k \to \mathcal{C}^l$ is a bi-Lipschitz measure preserving group isomorphism. Then for each $I \in \mathcal{I} \cap P(\mathcal{C}^l)$,

$$\Phi[X] \oplus I = \Phi[X \oplus \Phi^{-1}[I]].$$

As $\Phi^{-1}[I] \in \mathcal{I}$, $X \oplus \Phi^{-1}[I] \in \mathcal{J}$ and therefore $\Phi[X] \oplus I \in \mathcal{J}$, that is, $\Phi[X] \in (\mathcal{I}, \mathcal{J})^\star$. $\square$

Thus $(\mathcal{J}, \mathcal{K})^\star$ and $(\mathcal{I}, \mathcal{J})^\star$ are semiproductive, and the theorem follows from Theorem 1.6. $\square$

Following is a useful criterion for the 0-productivity required in Theorem 1.7.

Lemma 1.9. *Assume that $\mathcal{I}, \mathcal{J} \subseteq \tilde{P}(\mathcal{C})$ are iso-productive, and for the element $0 \in \mathcal{C}$ and each $X \in (\mathcal{I}, \mathcal{J})^\star \cap P(\mathcal{C})$, $\Psi_2[X \times \{0\}] \in (\mathcal{I}, \mathcal{J})^\star$. Then $(\mathcal{I}, \mathcal{J})^\star$ is 0-productive. In particular, if $\mathcal{I}$ is iso-productive and and for each $X \in \mathcal{I}^\star \cap P(\mathcal{C})$, $\Psi_2[X \times \{0\}] \in \mathcal{I}^\star$, then $\mathcal{I}^\star$ is 0-productive.*

Proof. Assume that $X \in (\mathcal{I}, \mathcal{J})^\star \cap P(\mathcal{C}^k)$ and fix l. By Lemma 1.8, $\tilde{X} = \Psi_k[X] \in (\mathcal{I}, \mathcal{J})^\star \cap P(\mathcal{C})$. Thus $\Psi_2 \circ (\Psi_k \times \Psi_l)[X \times \{0\}] = \Psi_2[\tilde{X} \times \{0\}] \in (\mathcal{I}, \mathcal{J})^\star$. Applying Lemma 1.8 again, we get that $X \times \{0\} \in (\mathcal{I}, \mathcal{J})^\star$. $\square$

We now give some applications. Let X be a metric space. Following Borel, we say that X has *strong measure zero* if for each sequence of positive reals $\{\epsilon_n\}_{n\in\mathbb{N}}$, there exists a cover $\{I_n\}_{n\in\mathbb{N}}$ of X such that $\operatorname{diam}(I_n) < \epsilon_n$ for all n. X has *Hausdorff dimension zero* if for each positive reals ϵ, δ there exists a cover $\{I_n\}_{n\in\mathbb{N}}$ of X with $\sum_n \operatorname{diam}(I_n)^\delta < \epsilon$. Clearly strong measure zero implies Hausdorff dimension zero. X has the *Hurewicz property* if for each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X there exist finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $X \subseteq \bigcup_n \bigcap_{m>n} \cup\mathcal{F}_n$. Let SMZ (respectively, $\mathcal{H}$) denote the collections of metric spaces having strong measure zero (respectively, the Hurewicz property).

In [12], Scheepers proves the following Theorem 1.10. In [7], the corresponding theorem is proved with $X \oplus Y$ instead of $X \times Y$ (for a direct, combinatorial proof of this result, see Theorem A.3 in the appendix). It is easy to see that the collections $\mathcal{H} \cap \tilde{P}(\mathcal{C})$ and $\mathrm{SMZ} \cap \tilde{P}(\mathcal{C})$ (and, therefore, $\mathcal{H} \cap \mathrm{SMZ} \cap \tilde{P}(\mathcal{C})$) are semiproductive. By Theorem 1.6, the first result actually follows from the second. (For yet another proof of Theorem 1.10, see Theorem A.2 in the appendix.)

Theorem 1.10 (Scheepers [12])**.** *Assume that $X \in \mathcal{H} \cap \mathrm{SMZ} \cap \tilde{P}(\mathcal{C})$ and $Y \in \mathrm{SMZ} \cap \tilde{P}(\mathcal{C})$. Then $X \times Y \in \mathrm{SMZ}$.*

We now treat the classes of meager-additive and null-additive sets. Let $\mathcal{M}$ and $\mathcal{N}$ denote the meager (i.e., first category) and null (i.e., measure zero) sets, respectively.

Theorem 1.11. *The classes $\mathcal{M}^\star \cap \tilde{P}(\mathcal{C})$ and $\mathcal{N}^\star \cap \tilde{P}(\mathcal{C})$ are semiproductive, and are closed under taking finite products.*

Proof. Clearly, the classes $\mathcal{M} \cap \tilde{P}(\mathcal{C})$ and $\mathcal{N} \cap \tilde{P}(\mathcal{C})$ are semiproductive. By Theorem 1.7, it is enough to show that the classes $\mathcal{M}^\star \cap \tilde{P}(\mathcal{C})$ and $\mathcal{N}^\star \cap \tilde{P}(\mathcal{C})$ are 0-productive. We first treat $\mathcal{M}^\star$.

Lemma 1.12 (Bartoszynski-Judah [1, Theorem 2.7.17]). *A subset X of $\mathcal{C}$ is meager-additive if, and only if, for each increasing sequence $\{m_n\}_{n\in\mathbb{N}}$ there exist a sequence $\{l_n\}_{n\in\mathbb{N}}$ and $y \in \mathcal{C}$ such that for each $x \in X$ and all but finitely many n,*

$$l_n \le m_k < m_{k+1} \le l_{n+1} \text{ and } x{\restriction}[m_k, m_{k+1}) = y{\restriction}[m_k, m_{k+1})$$

for some k. (In this case we say that $\{l_n\}_{n\in\mathbb{N}}$ and y are appropriate for $\{m_n\}_{n\in\mathbb{N}}$ and X.)

We will prove the sufficient criterion of Lemma 1.9. Assume that $X \in \mathcal{M}^\star \cap P(\mathcal{C})$, and let $\tilde{X} = \Psi_2[X \times \{0\}]$. We must show that $\tilde{X} \in \mathcal{M}^\star$. Let an increasing sequence $\{m_n\}_{n\in\mathbb{N}}$ be given. Choose an increasing sequence $\{m'_n\}_{n\in\mathbb{N}}$ of even numbers such that for all but finitely many n, there exists k such that $m'_n \le m_k < m_{k+1} \le m'_{n+1}$.

Apply Lemma 1.12 to obtain $\{l_n\}_{n\in\mathbb{N}}$ and y which are appropriate for $\{m'_n/2\}_{n\in\mathbb{N}}$ and X. By the definition of Ψ_2, we get that $\{2l_n\}_{n\in\mathbb{N}}$ and $\Psi_2(y, 0)$ are appropriate for $\{m'_n\}_{n\in\mathbb{N}}$ and $\Psi_2[X \times \{0\}]$. In particular, they are appropriate for $\{m_n\}_{n\in\mathbb{N}}$ and $\Psi_2[X \times \{0\}]$. This shows that $\mathcal{M}^\star$ is 0-productive.

Using similar arguments, the fact that $\mathcal{N}^\star \cap \tilde{P}(\mathcal{C})$ is 0-productive follows from the following.

Lemma 1.13 (Shelah [1, Theorem 2.7.18]). *A subset X of $\mathcal{C}$ is null-additive if, and only if, for every increasing sequence $\{m_n\}_{n\in\mathbb{N}}$ there exists a sequence $\{S_n\}_{n\in\mathbb{N}}$ such that each S_n is a set of at most n functions from $[m_n, m_{n+1})$ to $\{0, 1\}$, and for each $x \in X$ and all but finitely many n, $x{\restriction}[m_n, m_{n+1}) \in S_n$.*

This finishes the proof of Theorem 1.11. □

The literature contains numerous results on sums which can be translated this way into theorems about products.

Proposition 1.14 (folklore). *For all k, a set $X \subseteq \mathcal{C}^k$ has strong measure zero if, and only if, for each meager $M \subseteq \mathcal{C}^k$, $X \oplus M \neq \mathcal{C}^k$.*

Proof. Assume that $X \subseteq \mathcal{C}^k$ has strong measure, and $M \subseteq \mathcal{C}^k$ is meager. Then by the Galvin-Mycielski-Solovay Theorem, $\Psi_k[X \oplus M] = \Psi_k[X] \oplus \Psi_k[M] \neq \mathcal{C}$, and therefore $X \oplus M \neq \mathcal{C}^k$.

Conversely, assume that $X \subseteq \mathcal{C}^k$ and for each meager $M \subseteq \mathcal{C}^k$, $X \oplus M \neq \mathcal{C}^k$. Then for each meager $M \subseteq \mathcal{C}$, $X \oplus \Psi_k^{-1}[M] \neq \mathcal{C}^k$, therefore $\Psi_k[X \oplus \Psi_k^{-1}[M]] = \Psi_k[X] \oplus M \neq \mathcal{C}$, thus $\Psi_k[X]$ has strong measure zero, and therefore X has strong measure zero. □

In [7] it is proved that every set of reals with the Hurewicz proeprty as well as strong measure zero is meager-additive (for a direct proof see Theorem A.3 in the appendix). Consequently, the following theorem extends Theorem 1.10.

Theorem 1.15. *Assume that $X \in \mathcal{M}^\star \cap \tilde{P}(\mathcal{C})$ and $Y \in \mathrm{SMZ} \cap \tilde{P}(\mathcal{C})$. Then $X \times Y \in \mathrm{SMZ}$.*

Proof. By Theorem 1.11, $\mathcal{M}^\star \cap \tilde{P}(\mathcal{C})$ is semiproductive. Recall that $\mathrm{SMZ} \cap \tilde{P}(\mathcal{C})$ is semiproductive too. By the Galvin-Mycielski-Solovay Theorem (Proposition 1.14 for $k = 1$), the conditions of Theorem 1.6 hold, and its consequence tells what we are looking for. □

To prove the dual result, we need the following lemma. For a set J denote $J_x = \{y : (x,y) \in J\}$ and $J_y = \{x : (x,y) \in J\}$. Say that a family $\mathcal{J}$ which does not contain any $\mathcal{C}^k$ as element is a *Fubini family* if, whenever $J \in \mathcal{J} \cap \mathcal{C}^{k+l}$, we have that

$$\{x \in \mathcal{C}^k : J_x \notin \mathcal{J}\}, \{y \in \mathcal{C}^l : J^y \notin \mathcal{J}\} \in \mathcal{J}. \tag{1}$$

The most important examples for Fubiny families are $\mathcal{M}$ (Kuratowski-Ulam Theorem) and $\mathcal{N}$ (Fubini Theorem). To understand what we really prove, we will say that $\mathcal{J}$ is a *weakly Fubini family* if "$\in \mathcal{J}$" is replaced by "$\neq \mathcal{C}^k$" in (1). Clearly, if $\mathcal{J}$ is a Fubini family then it is a weakly Fubini family.

A set $X \subseteq \mathcal{C}^k$ is *not $\mathcal{J}$-covering* if for each $J \in \mathcal{J} \cap P(\mathcal{C}^k)$, $X \oplus J \neq \mathcal{C}^k$.

Lemma 1.16. *Assume that $\mathcal{J}$ is a weakly Fubini family. Then the family of not $\mathcal{J}$-covering sets is* 0*-productive.*

Proof. Assume that $X \subseteq \mathcal{C}^k$ is not $\mathcal{J}$-covering, $0 \in \mathcal{C}^l$, and $J \in \mathcal{J} \cap P(\mathcal{C}^{k+l})$. As $\mathcal{J}$ is a weakly Fubini family, there exists $y \in \mathcal{C}^l$ such that $J^y \in \mathcal{J}$. Thus, $((X \times \{0\}) \oplus J)^y = X \oplus J^y \neq \mathcal{C}^k$, therefore $(X \times \{0\}) \oplus J \neq \mathcal{C}^{k+l}$. □

A set $X \subseteq \mathcal{C}^k$ is *strongly meager* if it is not $\mathcal{N}$-covering. Using the same proof as in Theorem 1.15, we get the following.

Theorem 1.17. *The product of a null-additive set in $\mathcal{C}^k$ and a strongly meager set in $\mathcal{C}^l$ is strongly meager.*

It is folklore that a product of strong measure zero sets need not have strong measure zero (e.g., [5]). Also, the product of strongly meager sets need not be strongly meager [9]. The following slightly extends the result of [9] (Observe that the diagonal is null in $\mathcal{C} \times \mathcal{C}$).

Theorem 1.18. *Assume that $\mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N}) = \mathfrak{b}$. Then there exists a strongly meager set of reals $S \subseteq \mathcal{C}$ such that $(S \times S) \oplus \{(x,x) : x \in \mathcal{C}\} = \mathcal{C} \times \mathcal{C}$.*

Proof. Construct, as in [13], a $\mathfrak{b}$-Sierpinski set S such that $S \oplus S = \mathcal{C}$. Then $(S \times S) \oplus \{(x,x) : x \in \mathcal{C}\} = \mathcal{C} \times \mathcal{C}$: Given $y, z \in \mathcal{C}$, choose $s_1, s_2 \in S$ such that $s_1 \oplus s_2 = y \oplus z$, and take $x = s_1 \oplus y$. Then $s_1 \oplus x = y$, and

$$s_2 \oplus x = s_2 \oplus (s_1 \oplus s_1) \oplus x = (s_1 \oplus s_2) \oplus (s_1 \oplus x) = (y \oplus z) \oplus y = z.$$

By [13], every Borel image of S is bounded. Moreover, for each null set N, $|S \cap N| < \mathsf{cov}(\mathcal{N})$. By a result of Pawlikowski (see [1] – Definition 8.5.7, the observation after it, and Theorem 8.5.12), these two properties imply that S is strongly meager. □

2. Products in the Euclidean space

We first observe that as the mapping from $\mathbb{R}^k \times \mathbb{R}^k$ to $\mathbb{R}^k$ defined by $(x,y) \mapsto x+y$ is Lipschitz, Theorem 1.3 remains true in the *Euclidean space* $\mathbb{R}^k$.

Theorem 2.1. *Assume that $\mathcal{I}$, $\mathcal{J}$, and $\mathcal{K}$ are subsets of $\tilde{P}(\mathbb{R})$, and that $\mathcal{K}$ is closed under taking Lipschitz images. For each k, if*

for each $X \in \mathcal{I} \cap P(\mathbb{R}^k)$ and $Y \in \mathcal{J} \cap P(\mathbb{R}^k)$, $X \times Y \in \mathcal{K}$, *then*

for each $X \in \mathcal{I} \cap P(\mathbb{R}^k)$ *and* $Y \in \mathcal{J} \cap P(\mathbb{R}^k)$, $X + Y \in \mathcal{K}$.

However, we are unable to prove Theorem 1.6 (in its current form) for the Euclidean space, despite the fact that for each k, the group $\langle \mathbb{R}^k, + \rangle$ is isomorphic to $\langle \mathbb{R}, + \rangle$, both being vector spaces over the rationals $\mathbb{Q}$ with dimension continuum. The problem is that $\langle \mathbb{R}^k, + \rangle$ and $\langle \mathbb{R}, + \rangle$ are not isomorphic as *topological groups*. In fact, they are not even homeomorphic: $\mathbb{R}^k$ remains connected after removing a point. We can, though, obtain similar results.

Definition 2.2. A collection $\mathcal{P} \subseteq \tilde{P}(\mathbb{R})$ is *bi-0-productive* if for each k, l, and $X \in \mathcal{P} \cap \mathcal{C}^k$, if 0 is the zero element of $\mathbb{R}^l$, then $X \times \{0\}, \{0\} \times X \in \mathcal{P}$.

Theorem 2.3. *Assume that* $\mathcal{I}$ *and* $\mathcal{J}$ *are bi-0-productive subsets of* $\tilde{P}(\mathbb{R})$ *and* $\mathcal{K} \subseteq \tilde{P}(\mathbb{R})$. *If*

for each k *and all* $X \in \mathcal{I} \cap P(\mathbb{R}^k)$ *and* $Y \in \mathcal{J} \cap P(\mathbb{R}^k)$, $X + Y \in \mathcal{K}$,

then

for each $X \in \mathcal{I}$ *and* $Y \in \mathcal{J}$, $X \times Y, Y \times X \in \mathcal{K}$.

Proof. Assume that $X \in \mathcal{I} \cap P(\mathbb{R}^k)$ and $Y \in \mathcal{J} \cap P(\mathbb{R}^l)$. Then

$$X \times Y = (X \times \{0\}) + (\{0\} \times Y).$$

As $\mathcal{I}$ and $\mathcal{J}$ are bi-0-productive, $X \times \{0\} \in \mathcal{I} \cap P(\mathbb{R}^{k+l})$ and $\{0\} \times Y \in \mathcal{J} \cap P(\mathbb{R}^{k+l})$, therefore $X \times Y \in \mathcal{K}$. Similarly, $Y \times X \in \mathcal{K}$. $\square$

We now prove the analogue of Theorem 1.7.

Theorem 2.4. *Assume that* $\mathcal{I}, \mathcal{J}, \mathcal{K} \subseteq \tilde{P}(\mathbb{R})$, *and that* $(\mathcal{J}, \mathcal{K})^\star$ *and* $(\mathcal{I}, \mathcal{J})^\star$ *are bi-0-productive. Then for each* $X \in (\mathcal{J}, \mathcal{K})^\star$ *and* $Y \in (\mathcal{I}, \mathcal{J})^\star$, $X \times Y, Y \times X \in (\mathcal{I}, \mathcal{K})^\star$. *In particular, if* $\mathcal{I}^\star$ *is bi-0-productive, then* $\mathcal{I}^\star$ *is closed under taking finite products.*

Proof. Assume that $X \in (\mathcal{J}, \mathcal{K})^\star \cap P(\mathbb{R}^k)$, $Y \in (\mathcal{I}, \mathcal{J})^\star \cap P(\mathbb{R}^k)$, and $I \in \mathcal{I} \cap P(\mathbb{R}^k)$. Then $Y + I \in \mathcal{J}$ and therefore $X + (Y + I) \in \mathcal{K}$. Thus $X + Y \in (\mathcal{I}, \mathcal{K})^\star$. As $(\mathcal{J}, \mathcal{K})^\star$ and $(\mathcal{I}, \mathcal{J})^\star$ are bi-0-productive, our theorem follows from Theorem 2.3. $\square$

It was noticed by Marcin Kysiak that one can use our arguments to show that the analogue of theorem 1.10 in the Euclidean space also holds.

Theorem 2.5 (Scheepers [12])**.** *Assume that* $X \subseteq \mathbb{R}^k$ *has the Hurewicz property and has strong measure zero. Then for each strong measure zero set* $Y \subseteq \mathbb{R}^l$, *the set* $X \times Y$ *has strong measure zero.*

Proof. In [7] it is proved that for $k = 1$, the assumptions of the theorem imply that $X + Y$ has strong measure zero. But the proof there actually holds for $\mathbb{R}^k$ for any k. Apply Theorem 2.3. $\square$

A complete proof of Theorem 2.5 is given in Theorem 3.6.

In particular, if X is a γ-set of reals and Y has strong measure zero, then $X \times Y$ has strong measure zero. This settles several problems mentioned just before Theorem 14 of [17].

We do not know whether the analogue of Theorem 1.11 in the Euclidean space holds.

Problem 2.6. *Are the classes $\mathcal{M}^\star \cap \tilde{P}(\mathbb{R})$ and $\mathcal{N}^\star \cap \tilde{P}(\mathbb{R})$ 0-productive?*

If $\mathcal{M}^\star \cap \tilde{P}(\mathbb{R})$ is 0-productive, then it is closed under taking finite products. (The same assertion holds for $\mathcal{N}^\star \cap \tilde{P}(\mathbb{R})$.)

3. The Euclidean space through the looking glass

The results in the Section 2 are not easy to use, as one should verify first that the additive results given in the literature for $\mathbb{R}$ actually hold in $\mathbb{R}^k$ for all k. We suggest here another approach, which covers some of the cases of interest.

Definition 3.1. The function $T : \{0,1\}^{\mathbb{N}} \to [0,1]$ is defined by

$$x \mapsto \sum_{i\in\mathbb{N}} \frac{x(i)}{2^{i+1}} = 0.x(0)x(1)x(2)\ldots,$$

where the last term is in base 2. Let C denote the collection of all eventually constant elements of $\{0,1\}^{\mathbb{N}}$, and $\mathbb{Q}_2 = T[C]$ denote the 2-adic rational numbers in $[0,1]$.

Lemma 3.2 (folklore)**.**
1. *T is a uniformly continuous surjection.*
2. *C is countable.*
3. *$T : \{0,1\}^{\mathbb{N}} \setminus C \to [0,1] \setminus \mathbb{Q}_2$ is a homeomorphism which preserves measure in both directions.*

Proof. (1) T is continuous on its compact domain, and clearly it is onto.

(2) is obvious, and the only nontrivial part of (3) is that T^{-1} is continuous on $[0,1]\setminus\mathbb{Q}_2$ (it is not *uniformly* continuous: Take $x_n = 0.10^n\overline{01}$ and $y_n = 0.01^n\overline{01}$, then $x_n - y_n \to 0$, but $d(T^{-1}(x_n), T^{-1}(y_n)) = 1$ for all n.). Write $\tilde{y}$ for $T^{-1}(y)$. If $x_n \to x$ are elements of $[0,1]\setminus\mathbb{Q}_2$ then from some n onwards, the $\tilde{x}_n(0) = \tilde{x}(0)$: Assume that this is not the case. Then by moving to a subsequence we may assume that for all n, $\tilde{x}(0) \neq \tilde{x}_n(0)$. Assume that $\tilde{x}(0) = 0$ and $\tilde{x}_n(0) = 1$ for all n (the other case is similar). Let $k = \min\{m > 0 : x(m) = 0\}$ (recall that $\tilde{x}$ is not eventually constant). Then $x = 0.01^{k-1}0\ldots$, thus

$$x_n - x \geq 0.1 - 0.01^{k-1}0\overline{1} = 0.1 - 0.01^k = 0.0^k1 = \frac{1}{2^{k+1}}$$

for all n, a contradiction.

An inductive argument shows that for each k, $\tilde{x}_n(k) = \tilde{x}(k)$ for all large enough n. □

In principle, we can use the function T to translate questions about products in $\mathbb{R}$ into questions about products in $\mathcal{C}$, apply the results of Section 1, and translate back to $\mathbb{R}$. The problem is that in the first test-case that comes to mind, namely, Scheepers' Theorem 2.5, we must deal with strong measure zero sets. By (1) of Lemma 3.2, if Y has strong measure zero then so does $T[Y]$. The other direction does not follow from

Lemma 3.2, since a homeomorphic image of a strong measure zero set need not have strong measure zero [10].

Proposition 3.3. *For each k, $T^k : \mathcal{C}^k \to [0,1]^k$ preserves strong measure zero sets in both directions.*

Proof. Since T is uniformly continuous, T^k is uniformly continuous. Thus, if $X \subseteq \mathcal{C}^k$ has strong measure zero, then so does $T^k[X]$.

We now prove the other direction. The following is an easy exercise.

Lemma 3.4 (folklore)**.** *Assume that there exists $f : \mathbb{N} \to \mathbb{N}$ such that the metric space (X, d) has the following property: For each sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive reals, there exist a cover $\{I^n_m : n \in \mathbb{N}, m \leq f(n)\}$ of X satisfying* $\mathrm{diam}(I^n_m) < \epsilon_n$ *for each n and m. Then (X, d) has strong measure zero.*

To use Lemma 3.4, we make the following observation.

Lemma 3.5. *Assume that $I \subseteq [0,1]$ and* $\mathrm{diam}(I) < \epsilon$*. Then there exist $A_0, A_1 \subseteq \mathcal{C}$, both with diameter $< \epsilon$, such that $T^{-1}[I] \subseteq A_0 \cup A_1$.*

Proof. Let n be the maximal such that $2^{n-1}\epsilon \leq 1$. Then there exists $k < 2^n - 1$ such that

$$I \subseteq \left[\frac{k}{2^n}, \frac{k+2}{2^n}\right] = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right] \cup \left[\frac{k+1}{2^n}, \frac{k+2}{2^n}\right].$$

There exist sequences $s_0, s_1 \in \{0,1\}^n$ such that for $i = 0, 1$, if $A_i = \{x \in \mathcal{C} : s_i \subseteq x\}$, then $T[A_i] = [(k+i)/2^n, (k+i+1)/2^n]$. $\square$

Assume that $Y \subseteq [0,1]^k$ has strong measure zero, and $\{\epsilon_n\}_{n\in\mathbb{N}}$ is a sequence of positive reals. Let $I_n \subseteq [0,1]^k$ be such that $\mathrm{diam}(I_n) < \epsilon_n$ and $Y \subseteq \bigcup_n I_n$. Fix n. Then I_n is contained in the product of k intervals, $I^n_1, \ldots I^n_k \subseteq [0,1]$, each with diameter $< \epsilon_n$. For each $i = 1, \ldots, k$, use Lemma 3.5 to obtain sets $A^{n,i}_0, A^{n,i}_1$ as in the lemma. Then

$$I_n \subseteq \bigcup_{s\in\{0,1\}^k} \Pi^k_{i=1} T[A^{n,i}_{s(i)}] = T^k\left[\bigcup_{s\in\{0,1\}^k} \Pi^k_{i=1} A^{n,i}_{s(i)}\right],$$

so that $T^{-k}[I_n]$ is covered by 2^k sets of diameter $< \epsilon_n$. Since the sets I_n cover Y, we have by Lemma 3.4 that $T^{-k}[Y]$ has strong measure zero. $\square$

We are now ready to give an alternative proof of Theorem 2.5, which is more self-contained then the one given in Section 2. Observe that Scheeper's [12] proof of this theorem is more direct and simple. The proof given here is only meant to demonstrate the applicability of our translation machinery, which is perhaps applicable to other results which appear in the literature for algebraic sums of sets of reals.

Theorem 3.6. *Assume that $X \in \mathcal{H} \cap \mathrm{SMZ} \cap \tilde{P}(\mathbb{R})$ and $Y \in \mathrm{SMZ} \cap \tilde{P}(\mathbb{R})$. Then $X \times Y \in \mathrm{SMZ}$.*

Proof. We first use the following.

Lemma 3.7. *Assume that $\mathcal{I}$ is preserved under taking closed subsets, uniformly continuous images, and countable unions, and that $\mathbb{R} \notin \mathcal{I}$. Then for each $X \in \mathcal{I} \cap P(\mathbb{R}^k)$ and a countable set $Q \subseteq \mathbb{R}$ there exists $x \in \mathbb{R}^k$ such that $(x+X) \cap \bigcup_{m<k} \mathbb{R}^m \times Q \times \mathbb{R}^{k-m-1} = \emptyset$.*

Proof. The assumptions imply that for each $i<k$, the projection $X_i = \pi_i[X]$ on the ith coordinate is a member of $\mathcal{I}$. As Q is countable, $Q - X_i \neq \mathbb{R}$. Choose $x_i \notin Q - X_i$. Then $(x_i + X_i) \cap Q = \emptyset$. Take $x = (x_0, \dots, x_{k-1})$. Then $(x+X) \cap \bigcup_{m<k} \mathbb{R}^m \times Q \times \mathbb{R}^{k-m-1} = \emptyset$. □

Assume that $X \in \mathcal{H} \cap \mathrm{SMZ} \cap P(\mathbb{R}^k)$ and $Y \in \mathrm{SMZ} \cap P(\mathbb{R}^l)$. It is well known that $\mathcal{H}$ and SMZ satisfy the assumptions of Lemma 3.7. Take

$$Q = \bigcup_{m,n \in \mathbb{Z}} (m \cdot \mathbb{Q}_2 + n).$$

Then by Lemma 3.7, we may assume that X is disjoint from $\bigcup_{m<k} \mathbb{R}^m \times Q \times \mathbb{R}^{k-m-1}$, and Y is disjoint from $\bigcup_{m<l} \mathbb{R}^m \times Q \times \mathbb{R}^{l-m-1}$.

For each n set $X_n = X \cap [-n,n]^k$ and $Y_n = Y \cap [-n,n]^l$. Then X_n is a closed subset of X and therefore has the Hurewicz property. Moreover, X_n and Y_n have strong measure zero, and $X \times Y = \bigcup_n X_n \times Y_n$. Since SMZ is preserved under countable unions, it is enough to show that $X_n \times Y_n$ has strong measure zero for each n. Transforming (each coordinate of) X_n, Y_n with the bi-Lipschitz homeomorphism $x \mapsto (n+x)/2n$, we may assume (by our choice of Q!) that $X_n, Y_n \subseteq ([0,1] \setminus \mathbb{Q}_2)^k$.

So assume that $X \in \mathcal{H} \cap \mathrm{SMZ} \cap P(([0,1] \setminus \mathbb{Q}_2)^k)$, and $Y \in \mathrm{SMZ} \cap P(([0,1] \setminus \mathbb{Q}_2)^l)$ Since $T^k : \mathcal{C}^k \to ([0,1] \setminus \mathbb{Q}_2)^k$ is a homeomorphism, $T^{-k}[X]$ has the Hurewicz property. By Proposition 3.3, $T^{-k}[X], T^{-l}[Y] \subseteq \mathcal{C}$ have strong measure zero. By Theorem 1.10, $T^{-k}[X] \times T^{-l}[Y]$ has strong measure zero. As $T^k \times T^l$ is uniformly continuous, $X \times Y$ has strong measure zero. □

4. Borel's Conjecture and conjunction of properties

To put things in a wider context, we briefly describe the general framework. Let X be a topological space. An open cover $\mathcal{U}$ of X is an *ω-cover* of X if X is not in $\mathcal{U}$ and for each finite subset F of X, there is a set $U \in \mathcal{U}$ such that $F \subseteq U$. $\mathcal{U}$ is a *γ-cover* of X if it is infinite and for each x in X, $x \in U$ for all but finitely many $U \in \mathcal{U}$. Let $\mathcal{O}$, Ω, and Γ denote the collections of all countable open covers, ω-covers, and γ-covers of X, respectively. Let $\mathfrak{U}$ and $\mathfrak{V}$ be collections of covers of a space X. Following are selection hypotheses which X might satisfy or not satisfy.

- $\mathsf{S}_1(\mathfrak{U}, \mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$, there is a sequence $\{U_n\}_{n\in\mathbb{N}}$ such that for each n $U_n \in \mathcal{U}_n$, and $\{U_n\}_{n\in\mathbb{N}} \in \mathfrak{V}$.
- $\mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$, there is a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ such that each $\mathcal{F}_n$ is a finite (possibly empty) subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \mathfrak{V}$.
- $\mathsf{U}_{fin}(\mathfrak{U}, \mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$ which do not contain a finite subcover, there exists a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ such that for each n $\mathcal{F}_n$ is a finite (possibly empty) subset of $\mathcal{U}_n$, and $\{\cup\mathcal{F}_n\}_{n\in\mathbb{N}} \in \mathfrak{V}$.

Using this notation, $\mathsf{U}_{fin}(\mathcal{O},\Gamma)$ is the Hurewicz property, $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ is the Menger property, $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ is Rothberger's property C'', and $\mathsf{S}_1(\Omega,\Gamma)$ is the γ-property.

Many equivalences hold among these properties, and the surviving ones appear in Figure 1 (where an arrow denotes implication), to which no arrow can be added except perhaps from $\mathsf{U}_{fin}(\mathcal{O},\Gamma)$ or $\mathsf{U}_{fin}(\mathcal{O},\Omega)$ to $\mathsf{S}_{fin}(\Gamma,\Omega)$ [3].

$$\begin{tikzcd}
 & & \mathsf{U}_{fin}(\mathcal{O},\Gamma) \arrow[r] & \mathsf{U}_{fin}(\mathcal{O},\Omega) \arrow[r] & \mathsf{S}_{fin}(\mathcal{O},\mathcal{O}) \\
 & & \mathsf{S}_{fin}(\Gamma,\Omega) \arrow[ur] & & \\
\mathsf{S}_1(\Gamma,\Gamma) \arrow[r] \arrow[uurr] & \mathsf{S}_1(\Gamma,\Omega) \arrow[ur] \arrow[rr] & & \mathsf{S}_1(\Gamma,\mathcal{O}) \arrow[uur] & \\
 & & \mathsf{S}_{fin}(\Omega,\Omega) \arrow[uu] & & \\
\mathsf{S}_1(\Omega,\Gamma) \arrow[r] \arrow[uu] & \mathsf{S}_1(\Omega,\Omega) \arrow[uu] \arrow[ur] \arrow[rr] & & \mathsf{S}_1(\mathcal{O},\mathcal{O}) \arrow[uu] &
\end{tikzcd}$$

FIGURE 1. The Scheepers Diagram

Let us write $BC(P)$ for the Borel Conjecture for *metric spaces* with property P, that is, the hypothesis that every metric space with property P is countable. Laver proved that $BC(\text{SMZ})$ is consistent. Since $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ implies strong measure zero, it follows that $BC(\mathsf{S}_1(\Omega,\Gamma))$, $BC(\mathsf{S}_1(\Omega,\Omega))$, and $BC(\mathsf{S}_1(\mathcal{O},\mathcal{O}))$ are all consistent. On the other hand, all other classes in the Scheepers Diagram provably contain uncountable sets of reals [3, 2], and therefore cannot satisfy BC.

In [6] Miller proves that $BC(\mathsf{S}_1(\mathcal{O},\mathcal{O}))$ implies (and is therefore equivalent to) $BC(\text{SMZ})$, but $BC(\mathsf{S}_1(\Omega,\Gamma))$ does not imply $BC(\text{SMZ})$. We will use our results to extend this result. With regards to the Scheepers Diagram 1, the best we can get is that $BC(\mathsf{S}_1(\Omega,\Omega))$ implies $BC(\text{SMZ})$. We will prove a stronger result.

Definition 4.1. For a fixed topological space X, Ω^{gp} denotes the collection of open ω-covers $\mathcal{U}$ of X such that: There exists a partition $\mathcal{P}$ of $\mathcal{U}$ into finite sets such that for each finite $F \subseteq X$ and all but finitely many $\mathcal{F} \in \mathcal{P}$, there exists $U \in \mathcal{F}$ such that $F \subseteq U$ [4].

$\mathsf{S}_1(\Omega,\Omega^{gp})$ is strictly stronger than $\mathsf{S}_1(\Omega,\Omega)$ [4].

Theorem 4.2. *$BC(\mathsf{S}_1(\Omega,\Omega^{gp}))$ implies (and is therefore equivalent to) $BC(\text{SMZ})$.*

Proof. If $\aleph_1 = \mathfrak{b}$ then by [2] there exists an uncountable set of reals X satisfying $\mathsf{S}_1(\Omega,\Omega^{gp})$.

Assume that $\aleph_1 < \mathfrak{b}$, and $BC(\text{SMZ})$ fails. Take any strong measure zero set X with $|X| = \aleph_1$. Then $|X| < \mathfrak{c}$ and by a result of Carlson [1, Lemma 8.1.9], we may assume that $X \subseteq \mathbb{R}$. As $|X| < \mathfrak{b}$, all finite powers of X have the Hurewicz property. By Theorem 2.5, $X^2 = X \times X$ has strong measure zero, therefore $X^3 = X \times X^2$ has strong measure zero, etc.

By [4], $\mathsf{S}_1(\Omega,\Omega^{gp})$ is equivalent to satisfying the Hurewicz property as well as having strong measure zero in all finite powers. $\square$

Thus, the arguments in the last proof actually establish the following.

Theorem 4.3. *For a set of reals X, the following are equivalent:*

1. *X satisfies $\mathsf{S}_{fin}(\Omega, \Omega^{gp})$ and has strong measure-zero,*
2. *X satisfies $\mathsf{S}_{fin}(\Omega, \Omega^{gp})$ and $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$,*
3. *X satisfies $\mathsf{S}_{fin}(\Omega, \Omega^{gp})$ and is meager-additive,*
4. *X satisfies $\mathsf{S}_{fin}(\Omega, \Omega^{gp})$ and $\mathsf{S}_1(\Omega, \Omega)$,*
5. *X satisfies $\mathsf{S}_1(\Omega, \Omega^{gp})$.*

Proof. Clearly, $5 \Rightarrow 4 \Rightarrow 2 \Rightarrow 1$, and $3 \Rightarrow 1$.

($1 \Rightarrow 5$) Assume that (1) holds. In [4] it is proved that $\mathsf{S}_{fin}(\Omega, \Omega^{gp})$ is equivalent to satisfying the Hurewicz property $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$ in all finite powers. By Theorem 2.5, all finite powers of X satisfy $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$ and have strong measure zero. By [4], X satisfies $\mathsf{S}_1(\Omega, \Omega^{gp})$

($1 \Rightarrow 3$) In [7] it is proved that every strong measure zero set of reals with the Hurewicz property is meager additive. $\square$

The theorem also holds when $X \subseteq \mathcal{C}$. In this case, the quoted assertion in the last proof can be proved directly – see Theorem A.3 in the appendix.

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Appendix A. Direct proofs of quoted theorems

Following is Bartoszynski's (unpublished) combinatorial proof of theorem 1.10. The proof uses the following characterization of strong measure zero.

Lemma A.1 ([1, Lemma 8.1.13]). *For $X \subseteq \mathcal{C}$, the following are equivalent:*

(1) *X has strong measure zero,*
(2) *For each $f \in \mathbb{N}^\mathbb{N}$ there exists a function g such that $g(n) \in \{0,1\}^{f(n)}$ for all n, and for each $x \in X$ there exist infinitely many n such that $x \restriction f(n) = g(n)$,*
(3) *For each increasing sequence $\{m_n\}_{n\in\mathbb{N}}$ there exists $z \in \mathcal{C}$ such that for each $x \in X$ there exist infinitely many n such that $x\restriction[m_n, m_{n+1}) = z\restriction[m_n, m_{n+1})$.*

Let $\mathbb{N}^{\nearrow\mathbb{N}}$ denote the subspace of the Baire space $\mathbb{N}^\mathbb{N}$ consisting of the increasing functions in $\mathbb{N}^\mathbb{N}$. In [16] it is proved that if X has the Hurewicz property and $\Psi : X \to \mathbb{N}^{\nearrow\mathbb{N}}$ is continuous, then $\Psi[X]$ admits some slalom $h \in \mathbb{N}^{\nearrow\mathbb{N}}$, that is, such that for each $x \in X$ and all but finitely many n, there exists k such that $h(n) \le \Psi(x)(k) < h(n+1)$. This fact will be used in the proof.

Theorem A.2. *Assume that $X \subseteq \{0,1\}^\mathbb{N}$ has the Hurewicz property and strong measure zero, and $Y \subseteq \mathcal{C}$ has strong measure zero. Then $X \times Y$ has strong measure zero.*

Proof. Fix $f \in \mathbb{N}^{\nearrow\mathbb{N}}$ and let g be as in Lemma A.1 for X and f. Define a function $\Psi : X \to \mathbb{N}^{\nearrow\mathbb{N}}$ so that for each $x \in X$, $\Psi(x)$ is the increasing enumeration of the set $\{n : x \restriction f(n) = g(n)\}$. Then Ψ is continuous, thus there exists $h \in \mathbb{N}^{\nearrow\mathbb{N}}$ such that for each $x \in X$ and all but finitely many n, there exists k such that $h(n) \le \Psi(x)(k) < h(n+1)$.

Consider a mapping Φ defined on Y by

$$\Phi(y)(n) = \langle (g(k), y \restriction f(k)) : h(n) \le k < h(n+1)\rangle.$$

Then Φ is uniformly continuous. Thus (essentially, by Lemma A.1) there exists a function r such that for all $y \in Y$ there exist infinitely many n such that $\Phi(y)(n) =$

$r(n)$. From r we decode a function s such that $s(n) \in \{0,1\}^{f(n)} \times \{0,1\}^{f(n)}$ by $s(k) = r(n)(k)$ where n is such that $h(n) \leq k < h(n+1)$.

Then for all $x \in X$ and $y \in Y$ there exist infinitely many n such that $(x \restriction f(n), y \restriction f(n)) = s(n)$, which shows that $X \times Y$ has strong measure zero. $\square$

Using the bi-Lipschitz transformations Ψ_k of Section 1, we obtain the general Theorem 1.10 from Theorem A.2.

Bartoszynski's proof of Theorem A.2 can be modified to obtain a direct, combinatorial proof of one of the main theorems in [7] when restricted to the Cantor space.

Theorem A.3 ([7]). *Assume that $X \in \tilde{P}(\mathcal{C})$, X has the Hurewicz property $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$, and strong measure zero. Then X is meager-additive.*

Proof. We prove the result for $X \subseteq \mathcal{C}$ and use the machinery of Section 1 to deduce the general theorem.

Assume that $\{m_n\}_{n\in\mathbb{N}}$ is an arbitrary increasing sequence. By Lemma 1.12, it suffices to find a sequence $\{l_n\}_{n\in\mathbb{N}}$ and $z \in \mathcal{C}$ such that for each $x \in X$ and all but finitely many n, $l_n \leq m_k < m_{k+1} \leq l_{n+1}$ and $x\restriction[m_k, m_{k+1}) = z\restriction[m_k, m_{k+1})$ for some k.

By Lemma A.1, there exists $z \in \mathcal{C}$ such that for each $x \in X$ there exist infinitely many n such that $x\restriction[m_n, m_{n+1}) = z\restriction[m_n, m_{n+1})$. Again, for each $x \in X$ let $\Psi(x)$ be the increasing enumeration of these ns, and use the fact that X has the Hurewicz property to find a slalom $h \in \mathbb{N}^{\nearrow\mathbb{N}}$ for $\Psi[X]$.

Take $l_n = m_{h(n)}$ for each n. Fix $x \in X$. Since h is a slalom for $\Psi[X]$, for all but finitely many n there exists k such that for $i = \Psi(x)(k)$, $h(n) \leq i < h(n+1)$. Then

$$l_n = m_{h(n)} \leq m_i < m_{i+1} \leq m_{h(n+1)} = l_{n+1},$$

and by the definition of $\Psi(x)$, $x\restriction[m_i, m_{i+1}) = z\restriction[m_i, m_{i+1})$. $\square$

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