THE HUREWICZ COVERING PROPERTY AND SLALOMS IN THE BAIRE SPACE

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ABSTRACT. According to a result of Kočinac and Scheepers, the Hurewicz covering property is equivalent to a somewhat simpler selection property: For each sequence of large open covers of the space one can choose finitely many elements from each cover to obtain a groupable cover of the space. We simplify the characterization further by omitting the need to consider sequences of covers: A set of reals X satisfies the Hurewicz property if, and only if, each large open cover of X contains a groupable subcover. This solves in the affirmative a problem of Scheepers.

The proof uses a rigorously justified abuse of notation and a "structure" counterpart of a combinatorial characterization, in terms of slaloms, of the minimal cardinality $\mathfrak{b}$ of an unbounded family of functions in the Baire space. In particular, we obtain a new characterization of $\mathfrak{b}$.

1. INTRODUCTION

A separable zero-dimensional metrizable space X satisfies the *Hurewicz property* [3] if:

> For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of open covers of X there exist finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $X \subseteq \bigcup_n \bigcap_{m>n} \cup\mathcal{F}_n$.

This property is a generalization of σ-compactness.

Much effort was put in the past in order to find a simpler characterization of this property. In particular, it was desired to avoid the need to glue the elements of each $\mathcal{F}_n$ together (that is, by taking their union) in the definition of the Hurewicz property.

The first step toward simplification was the observation that one may restrict attention to sequences of *large* (rather than arbitrary) open covers of X in the above definition [10] ($\mathcal{U}$ is a *large cover* of X if each member of X is contained in infinitely many members of $\mathcal{U}$.)

The main ingredient in the next major step toward this goal was implicitly studied in [11, 8, 5] while considering spaces having analogous properties in all finite powers and a close relative of the Reznichenko (or: weak Fréchet-Urysohn) property. Finally, this ingredient was isolated and analyzed in [6]: A large cover $\mathcal{U}$ of X is *groupable* if there exists a partition $\mathcal{P}$ of $\mathcal{U}$ into finite sets such that for each $x \in X$ and all but finitely many $\mathcal{F} \in \mathcal{P}$, $x \in \cup\mathcal{F}$. Observe that ignoring all but countably many

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elements of the partition, we get that each groupable cover contains a countable groupable cover. Moreover, in [13] it is proved that for the types of spaces considered here, each large open cover contains a countable large cover.

One of the main results in Kočinac-Scheepers' [6] is that the Hurewicz property is equivalent to the following one:

> For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of large open covers of X there exist finite subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_n \mathcal{F}_n$ is a groupable cover of X.

This characterization is misleading in its pretending to be a mere result of unstitching the unions $\cup\mathcal{F}_n$: The sets $\mathcal{F}_n$ need not be disjoint in the original definition, and overcoming this difficulty requires a deep analysis involving infinite game-theoretic methods – see [6].

In this paper we take the task of simplification one step further by removing the need to consider *sequences* of covers. We prove that the Hurewicz property is equivalent to:

> $(\star)$ Each large cover of X contains a groupable cover of X.

This solves in the affirmative a problem of Scheepers [11, Problem 1], which asks whether, for strong measure zero sets, $(\star)$ is equivalent to the Hurewicz property.

Another way to view this simplification is as follows: The Kočinac-Scheepers characterization is equivalent to requiring that the resulting cover $\bigcup_n \mathcal{F}_n$ is large *together* with the property that each large open cover of X contains a groupable cover of X. The first requirement has also appeared in the literature, and is equivalent to a property introduced by Menger in [7] (see [10]). Our result says that it is enough to require only that the second property holds, or in other words, that the second property actually implies the first.

2. Two possible interpretations

Our proof relies on a delicate interplay between two possible interpretations of the term "large open cover of X" when X is a subspace of another space Y:

(1) A large open cover of X by subsets of X which are relatively open in X; and
(2) A large open cover of X by open subsets of Y.

The notions do not coincide, because a large cover of the second type, when restricted to X, need not be large – it can even be finite.

For shortness, we will use the following notation. For a space X, denote the property that each large cover of X by open subsets of X contains a groupable cover of X by $\binom{\Lambda_X}{\Lambda^{gp}_X}$. We write $\binom{\Lambda}{\Lambda^{gp}}$ instead of $\binom{\Lambda_X}{\Lambda^{gp}_X}$ when the space X is clear from the context. It is easy to see that $\binom{\Lambda_X}{\Lambda^{gp}_X}$ implies that each *countable* large open cover of X is groupable (divide the countably many remaining elements between the sets in the partition so they remain finite).

We will need the following simple fact.

Lemma 1. *The property $\binom{\Lambda}{\Lambda^{gp}}$ is preserved under taking closed subsets and continuous images, that is:*

(1) *If X satisfies $\binom{\Lambda_X}{\Lambda_X^{gp}}$ and C is a closed subset of X, then C satisfies $\binom{\Lambda_C}{\Lambda_C^{gp}}$.*
(2) *If X satisfies $\binom{\Lambda_X}{\Lambda_X^{gp}}$ and Y is a continuous image of X, then Y satisfies $\binom{\Lambda_Y}{\Lambda_Y^{gp}}$.*

Proof. (1) Assume that $\mathcal{U}$ is a large open cover of X. Then $\tilde{\mathcal{U}} = \{U \cup (X \setminus C) : U \in \mathcal{U}\}$ is a large open cover of X. Applying the groupability of $\tilde{\mathcal{U}}$ for X and forgetting the $X \setminus C$ part of the open sets shows the groupability of $\mathcal{U}$ for C.

(2) Assume that $f : X \to Y$ is a continuous surjection and that $\mathcal{U} = \{U_n\}_{n\in\mathbb{N}}$ is a large open cover of Y (by open subsets of Y). Then $\{f^{-1}[U_n]\}_{n\in\mathbb{N}}$ is a large open cover of X, whose groupability implies the groupability of $\mathcal{U}$ for X. □

The following theorem tells us that *for our purposes*, it does not matter which notion of large covers we use (so that we can switch between the two notions at our convenience).

Theorem 2. *Assume that X is a subspace of Y and X satisfies $\binom{\Lambda_X}{\Lambda_X^{gp}}$. Then each countable collection $\mathcal{U}$ of open sets in Y which is a large cover of X is groupable for X.*

Proof. We will repeatedly use the following lemma.

Lemma 3. *Assume that X is a subspace of Y, and $\mathcal{U} = \{U_n\}_{n\in\mathbb{N}}$ is a large open cover of X by open subsets of Y. Define an equivalence relation $\sim$ on $\mathbb{N}$ by*

$$n \sim m \quad \text{if} \qquad X \cap U_n = X \cap U_m.$$

Let $A = \{n : [n] \text{ is infinite}\}$, and $V = \bigcup_{n\in A} U_n$. Then $\{U_n : n \in A\}$ is a groupable cover of $X \cap V$, and $\{U_n : n \notin A\}$ is a large cover of $X \setminus V$ (by open subsets of Y).

Proof. Define a partition of A as follows: Let $[n_0], [n_1], \dots$ enumerate the elements of $A/\sim$. Let F_0 contain the first element of $[n_0]$, F_1 contain the second element of $[n_0]$ and the first element of $[n_1]$, F_2 contain the third element of $[n_0]$, the second element of $[n_1]$ and the first element of $[n_2]$, etc. Fix $x \in X \cap V$, and let i be such that $x \in U_{n_i}$. Then for all but finitely many n, there exists $k \in F_n \cap [n_i]$ and therefore $x \in U_k$. This proves the first assertion.

Assume that $X \not\subseteq V$. As $\mathcal{U}$ is a large cover of $X \setminus V$ and for $x \in X \setminus V$ and $n \in A$, $x \notin U_n$, there must exist infinitely many $n \notin A$ such that $x \in U_n$. □

We now prove Theorem 2. Enumerate $\mathcal{U}$ bijectively as $\{U_n\}_{n\in\mathbb{N}}$. We make the following definition by transfinite induction on $\alpha < \aleph_1$ (and make sure that indeed it terminates at some $\alpha < \aleph_1$). Carry out the following construction as long as A_α is not empty.

(1) *First step:* Set $X_0 = X$, $B_0 = \mathbb{N}$, and $V_0 = \emptyset$.
(2) *Successor step:* Assume that X_α, B_α, and V_α are defined, and $\{U_n : n \in B_\alpha\}$ is a large cover of $X_\alpha \setminus V_\alpha$. Set $X_{\alpha+1} = X_\alpha \setminus V_\alpha$, and define an equivalence relation $\sim_{\alpha+1}$ on B_α by $n \sim_{\alpha+1} m$ if $X_{\alpha+1} \cap U_n = X_{\alpha+1} \cap U_m$. Let $A_{\alpha+1} = \{n \in B_\alpha : [n]_{\sim_{\alpha+1}} \text{ is infinite}\}$, $B_{\alpha+1} = B_\alpha \setminus A_{\alpha+1}$, and $V_{\alpha+1} = \bigcup_{n\in A_{\alpha+1}} U_n$. Use Lemma 3 to obtain a partition $\{F_n^{\alpha+1}\}_{n\in\mathbb{N}}$ of $A_{\alpha+1}$ into finite sets witnessing

that $\{U_n : n \in A_{\alpha+1}\}$ is a groupable cover of $X_{\alpha+1} \cap V_{\alpha+1}$ (and $\{U_n : n \in B_{\alpha+1}\}$ is a large cover of $X_{\alpha+1} \setminus V_{\alpha+1}$).

(3) *Limit step:* Assume that α is a limit and the construction was carried upto step α. Set $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$, $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$, $B_\alpha = \bigcap_{\beta<\alpha} B_\beta = \mathbb{N} \setminus A_\alpha$, and $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$. For each $x \in X_\alpha$ and each $\beta < \alpha$, $x \notin V_\beta$, that is, $\{n : x \in U_n\}$ is disjoint from A_β. Thus, $\{n : x \in U_n\}$ is infinite, and is a subset of B_α. In other words, $\{U_n : n \in B_\alpha\}$ is a large cover of X_α. Observe that in this case, X_α is disjoint from V_α.

As long as the construction continues, A_α is not empty and therefore $B_{\alpha+1}$ is a proper subset of B_α. Thus, as $B_0 \subseteq \mathbb{N}$, the construction cannot continue uncountably many steps. Let $\alpha < \aleph_1$ be the step where the construction terminated (this can only happen when α is a successor). Then A_α is empty, therefore V_α is empty, thus $\{U_n : n \in B_\alpha\}$ is a large cover of X_α. The definition of B_α implies that in this case, $\{U_n \cap X_\alpha : n \in B_\alpha\}$ is a large cover of X_α by open subsets of X_α. By the construction, X_α is a closed subset of X (an intersection of closed sets). By Lemma 1, $\{U_n \cap X_\alpha : n \in B_\alpha\}$ is groupable for X_α – let $\{F_n^{\alpha+1}\}_{n\in\mathbb{N}}$ be a partition of B_α into finite sets that witnesses that.

The partitions $\{F_n^{\beta+1}\}_{n\in\mathbb{N}}$ where $\beta \leq \alpha$ form a countable family of partitions of disjoint subsets of $\mathbb{N}$. Re-enumerate all these partitions as $\{\{G_n^m\}_{n\in\mathbb{N}} : m \in \mathbb{N}\}$, and define a partition $\{H_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ into finite sets by

$$H_n = \bigcup_{\max\{i,j\}=n} G_j^i.$$

Observe that $X \subseteq X_\alpha \cup \bigcup_{\beta<\alpha} V_{\beta+1} \setminus V_\beta$, where each $X \cap (V_{\beta+1} \setminus V_\beta)$ is taken care by $\{F_n^{\beta+1}\}_{n\in\mathbb{N}}$, and X_α is taken care by $\{F_n^{\alpha+1}\}_{n\in\mathbb{N}}$. Consequently, for each $x \in X$ there exists m such that $x \in \cup\{U_k : k \in G_n^m\} \subseteq \cup\{U_k : k \in H_n\}$ for all but finitely many n. This shows that $\mathcal{U}$ is a groupable cover of X. □

3. The main theorem

Theorem 4. *For a separable and zero-dimensional metrizable space X, the following are equivalent:*

1. *X has the Hurewicz property; and*
2. *Every large open cover of X contains a groupable cover of X.*

Proof. $(1 \Rightarrow 2)$ By Proposition 1.1 of [13], every large open cover of X contains a *countable* large open cover of X. In [11, Lemma 3] and [6, Lemma 8] it is proved that the Hurewicz property implies that each countable large open cover of X is groupable.

$(2 \Rightarrow 1)$ We will prove this direction by a sequence of small steps, using the results of the previous section.

The *Baire space* ${}^{\mathbb{N}}\mathbb{N}$ of infinite sequences of natural numbers is equipped with the product topology (where the topology of $\mathbb{N}$ is discrete). A quasiordering $\leq^*$ is defined on the Baire space ${}^{\mathbb{N}}\mathbb{N}$ by eventual dominance:

$$f \leq^* g \quad \text{if} \qquad f(n) \leq g(n) \text{ for all but finitely many } n.$$

We say that a subset Y of ${}^{\mathbb{N}}\mathbb{N}$ is *bounded* if there exists g in ${}^{\mathbb{N}}\mathbb{N}$ such that for each $f \in Y$, $f \leq^* g$. Otherwise, we say that Y is *unbounded*. According to a theorem of Hurewicz [3] (see also Recław [9]), X has the Hurewicz property if, and only if, each continuous image of X in ${}^{\mathbb{N}}\mathbb{N}$ is bounded. Let ${}^{\mathbb{N}\nearrow}\mathbb{N}$ denote the subspace of ${}^{\mathbb{N}}\mathbb{N}$ consisting of the strictly increasing elements of ${}^{\mathbb{N}}\mathbb{N}$. The mapping from ${}^{\mathbb{N}}\mathbb{N}$ to ${}^{\mathbb{N}\nearrow}\mathbb{N}$ defined by

$$f(n) \mapsto g(n) = f(0) + \cdots + f(n) + n$$

is a homeomorphism which preserves boundedness in both directions. Consequently, Hurewicz' theorem can be stated using ${}^{\mathbb{N}\nearrow}\mathbb{N}$ instead of ${}^{\mathbb{N}}\mathbb{N}$.

For $f, g \in {}^{\mathbb{N}\nearrow}\mathbb{N}$, we say that f *goes through the slalom defined by* g if for all but finitely many n, there exists m such that $f(m) \in [g(n), g(n+1))$. A subset Y of ${}^{\mathbb{N}\nearrow}\mathbb{N}$ *admits a slalom* if there exists $g \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ such that each $f \in Y$ goes through the slalom g.

Lemma 5 (folklore)**.** *Assume that $Y \subseteq {}^{\mathbb{N}\nearrow}\mathbb{N}$. The following are equivalent:*

1. *Y is bounded,*
2. *Y admits a slalom; and*
3. *There exists a partition $\{F_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ such that for each $f \in Y$ and all but finitely many n, there exists m such that $f(m) \in F_n$.*

For completeness, we give a short proof.

Proof. $(1 \Rightarrow 2)$ Assume that $g \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ bounds Y. Define inductively $h \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ by

$$\begin{aligned} h(0) &= g(0) \\ h(n+1) &= g(h(n)) + 1 \end{aligned}$$

Then for each $f \in Y$ and all but finitely many n, $h(n) \leq f(h(n)) \leq g(h(n)) < h(n+1)$, that is, $f(h(n)) \in [h(n), h(n+1))$.

$(2 \Rightarrow 1)$ Assume that Y admits a slalom g. Let h be a function which eventually dominates all functions of the form $f(n) = g(n_0 + n)$, $n_0 \in \mathbb{N}$. Let f be any element of Y and choose n_0 such that for each $n \geq n_0$, there exists m such that $f(m) \in [g(n), g(n+1))$. Choose m_0 such that $f(m_0) \in [g(n_0), g(n_0+1))$. By induction on n, we have that $(f(n) \leq) f(m_0 + n) \leq g(n_0 + 1 + n)$ for all n. For large enough n, we have that $g(n_0 + 1 + n) \leq h(n)$, thus $f \leq^* h$.

Clearly $(2 \Rightarrow 3)$. We will show that $(3 \Rightarrow 2)$. Let $\{F_n\}_{n\in\mathbb{N}}$ be as in (2). Define $g \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ as follows: Set $g(0) = 0$. Having defined $g(0), \dots, g(n-1)$, let m be the minimal such that $F_m \cap [0, g(n-1)) = \emptyset$, and set $g(n) = \max F_m + 1$. Then for each n there exists F_m such that $F_m \subseteq [g(n), g(n+1))$. Consequently, Y admits the slalom defined by g. $\square$

The *Cantor space* $\{0,1\}^{\mathbb{N}}$ is also equipped with the product topology. Identify $P(\mathbb{N})$ with $\{0,1\}^{\mathbb{N}}$ by characteristic functions. The *Rothberger space* $P_\infty(\mathbb{N})$ is the subspace of $P(\mathbb{N})$ consisting of all infinite sets of natural numbers. The space ${}^{\mathbb{N}\nearrow}\mathbb{N}$ is homeomorphic to $P_\infty(\mathbb{N})$ by identifying each $f \in {}^{\mathbb{N}\nearrow}\mathbb{N}$ with its image $f[\mathbb{N}]$ (so that f is the increasing enumeration of $f[\mathbb{N}]$).

Translating Lemma 5 into the language of $P_\infty(\mathbb{N})$ and using Hurewicz' theorem, we obtain the following characterization of the Hurewicz property in terms of continuous images in the Rothberger space.

Lemma 6. *For a separable and zero-dimensional metrizable space X, the following are equivalent:*

1. *X has the Hurewicz property,*
2. *For each continuous image Y of X in $P_\infty(\mathbb{N})$ there exists $g \in {}^{\mathbb{N}}\!\nearrow\!\mathbb{N}$ such that for each $y \in Y$, $y \cap [g(n), g(n+1)) \neq \emptyset$ for all but finitely many n; and*
3. *For each continuous image Y of X in $P_\infty(\mathbb{N})$ there exists a partition $\{F_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ into finite sets such that for each $y \in Y$, $y \cap F_n \neq \emptyset$ for all but finitely many n.*

Assume that every large open cover of X is groupable. We will show that (3) of Lemma 6 holds. Let Y be a continuous image of X in $P_\infty(\mathbb{N})$. Then by Lemma 1, Y satisfies $\binom{\Lambda_Y}{\Lambda_Y^{gp}}$. Thus, by Theorem 2, every large open cover of $P_\infty(\mathbb{N})$ is groupable as a cover of Y.

Let $\mathcal{U} = \{O_n\}_{n\in\mathbb{N}}$ where for each n,

$$O_n = \{a \in P_\infty(\mathbb{N}) : n \in a\}.$$

Then $\mathcal{U}$ is a large open cover of $P_\infty(\mathbb{N})$. Thus, there exists a partition $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ of $\mathcal{U}$ into finite sets, such that for each $y \in Y$, $y \in \cup\mathcal{F}_n$ for all but finitely many n. For each n set $F_n = \{m : O_m \in \mathcal{F}_n\}$. Then $\{F_n\}_{n\in\mathbb{N}}$ is a partition of $\mathbb{N}$ into finite sets. For each $y \in Y$ and for all but finitely many n, there exists k such that $y \in O_k \in \mathcal{F}_n$, that is, $k \in y \cap F_n$, therefore $y \cap F_n \neq \emptyset$. $\square$

Remark 7. A strengthening of the Hurewicz property for X, considering *countable Borel* covers instead of open covers, was given the following simple characterization in [12]:

> For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of countable (large) Borel covers of X, there exist elements $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $X \subseteq \bigcup_n \bigcap_{m>n} U_n$.

(Notice that the analogue equivalence for the open case does not hold [4].) Using the same proof as in Theorem 4, we get that this property is also equivalent to requiring that every countable large Borel cover of X is groupable.

Forgetting about the topology and considering only countable covers, we get the following characterization of the minimal cardinality $\mathfrak{b}$ of an unbounded family in the Baire space ${}^{\mathbb{N}}\mathbb{N}$. For a cardinal κ, denote by Λ_κ (respectively, Λ_κ^{gp}) the collection of countable large (respectively, groupable) covers of κ.

Corollary 8. *For an infinite cardinal κ, the following are equivalent:*

1. *$\kappa < \mathfrak{b}$,*
2. *Each subset of ${}^{\mathbb{N}}\!\nearrow\!\mathbb{N}$ of cardinality κ admits a slalom,*
3. *For each family $Y \subseteq P_\infty(\mathbb{N})$ of cardinality κ, there exists a partition $\{F_n\}_{n\in\mathbb{N}}$ of $\mathbb{N}$ into finite sets such that for each $y \in Y$, $y \cap F_n \neq \emptyset$ for all but finitely many n; and*

(4) *κ satisfies $\binom{\Lambda_\kappa}{\Lambda_\kappa^{gp}}$ (that is, every countable large cover of κ is a groupable cover of κ).*

Remark 9. The underlying combinatorics in this paper is similar to that appearing in Bartoszynski's characterization of $\mathsf{add}(\mathcal{N})$ (the minimal cardinality of a family of measure zero sets whose union is not a measure zero set) [1]. The equivalence $(1 \Leftrightarrow 2)$ in Corollary 8 is folklore. The equivalence of these with (4) seems to be new.

The only other covering property we know of which enjoys the possibility of considering subcovers of a given cover instead of selecting from a given sequence of covers is the Gerlits-Nagy γ-property. The proof for this fact is much easier – see [2].

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