GENERALIZATIONS OF THE BUSEMANN-PETTY PROBLEM FOR SECTIONS OF CONVEX BODIES

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0. Introduction

Let M be a compact convex set of dimension i in $\mathbb{R}^n$ that contains the origin in its relative interior. When $i = n$, M is called a convex body. For $1 \le k \le i$, when a subspace $\eta \subset \mathbb{R}^n$ has dimension $n - i + k$, the intersection $M \cap \eta$ is a k-dimensional compact convex set in general. The k-th *dual volume* $\tilde{V}_k(M)$ is defined as the average of the k-dimensional volume $\mathrm{vol}_k(M \cap \eta)$ about η,

$$\tilde{V}_k(M) = \int_{G_{n,n-i+k}} \mathrm{vol}_k(M \cap \eta) d\eta, \tag{0.1}$$

where $\mathrm{vol}_k(\cdot)$ denotes the k-dimensional volume functional, and $d\eta$ is the invariant probability measure on the Grassmannian $G_{n,n-i+k}$. Note that $\tilde{V}_i(M) = \mathrm{vol}_i(M)$. Dual volumes are important geometric invariants of sections of convex sets. One of the subjects for sections of convex sets is to study how dual volumes behave. In this paper, we investigate inequalities of dual volumes of origin-symmetric convex bodies. We consider the following problem:

Problem A. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, $1 \le k \le i$, and $k < l \le n$. If*

$$\tilde{V}_k(K \cap \xi) \le \tilde{V}_k(L \cap \xi)$$

for any $\xi \in G_{n,i}$, does it follow that

$$\tilde{V}_l(K) \le \tilde{V}_l(L)?$$

The special case, $k = i = n - 1$, $l = n$, is the well-known Busemann-Petty problem [BP], which has been resolved by a series of works, see [LR], [Lu], [B], [Bo], [Gi], [P], [G1], [G2], [K1], [Z2], [GKS]. Simplified proofs were presented in [BFM], [K3], [R3].

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The Busemann-Petty problem deals with hyperplane sections of a convex body. It has a positive solution in $\mathbb{R}^3$ and in $\mathbb{R}^4$, and has a negative solution in higher dimensions. The case $k=i$, $l=n$, of Problem A, called the generalized Busemann-Petty problem in [Z1], deals with intermediate sections. It was shown in [BZh] that the generalized Busemann-Petty problem has a negative solution when $i \geq 4$. A different proof of this result was given in [K2]. For $i=2$, or 3, the answer is still open. For the special case that K is a body of revolution, the answer is positive [GrZ], [Z1].

Problem A can be reformulated in an equivalent form which is more convenient to handle and has generalizations. We denote by S^{n-1} the unit sphere in $\mathbb{R}^n$ endowed with the rotation-invariant probability measure du. Let K be a convex body in $\mathbb{R}^n$ that contains the origin in its interior. The *radial function of* K is defined by

$$\rho_K(u) = \sup\{\lambda \geq 0 : \lambda u \in K\}, \qquad u \in S^{n-1}.$$

One can show (see Lemma 2.1) that

$$\tilde{V}_k(K \cap \xi) = \kappa_k \int_{S^{n-1}\cap\xi} \rho_K(u)^k d_\xi u$$

where κ_k is the volume of the k-dimensional unit ball, and $d_\xi u$ denotes the relevant induced measure. In view of this, we introduce the following functionals:

$$I_k(K,\xi) = \int_{S^{n-1}\cap\xi} \rho_K(u)^k d_\xi u, \qquad J_l(K) = \int_{S^{n-1}} \rho_K(u)^l du. \tag{0.2}$$

Problem B. *Let K and L be origin-symmetric convex bodies in $\mathbb{R}^n$, $0 < k < l < \infty$. If*

$$I_k(K,\xi) \leq I_k(L,\xi)$$

for any $\xi \in G_{n,i}$, does it follow that

$$J_l(K) \leq J_l(L)?$$

In Problem B, k and l are any positive numbers. If they are integers satisfying $1 \leq k \leq i$ and $k < l \leq n$, then Problem B is a reformulation of Problem A. Particular cases of Problem B were considered by Hadwiger [H] ($n=3$, $l-k \leq 1$) for bodies of revolution, and Koldobsky [K2, Theorem 8] ($k=n-1, i=3$) for any origin-symmetric convex bodies. For these cases they gave affirmative answers. Problem B has trivially an affirmative answer when $i=1$. So it is always assumed that $2 \leq i \leq n-1$.

Main results.

(a) Problem B (and therefore Problem A) has negative answers when (i) $i \geq 4$; (ii) $l-k > n-i$ for $i=2$ or 3. (Theorem 6.1)

(b) Problem B has a positive answer when $l=k+1$ for $i=2$ or 3. (Theorem 5.2)

(c) Other cases of $i=2$ or 3 remain open in general, but have positive answers when K is a body of revolution. (Corollary 5.4)

These results are generalizations of the full solution to the Busemann-Petty problem. The main tools we use are totally geodesic Radon transforms and harmonic analysis on the sphere. We also apply some techniques of fractional calculus [R1], [SKM], in particular, the Erdelyi-Kober fractional integrals which arise in our problem in a natural way. The curvature of a convex body plays a role in the proof of negative results. The paper is almost self-contained.

The paper is organized as follows. In Section 1 we give explicit formulas for the spherical Radon transforms and their dual transforms of $SO(n-1)$ invariant functions (zonal functions). In Section 2 we define a class of star bodies called (i,p)-intersection bodies and show the connections between these bodies and Problems A and B. Section 3 includes a few elementary geometric lemmas. In Section 4, using dual Radon transforms, we derive important formulas that represent the radial function of a convex body in terms of volumes of parallel sections. These are generalizations of the corresponding formulas in [G1], [Z2] and [GKS]. In Section 5 we prove positive answers for Problems A and B. Negative answers are obtained in Section 6. All results for Problem A in the case of bodies of revolution are summarized in Table 1 at the end of Section 6.

As we have already mentioned, a negative answer to the generalized Busemann-Petty problem for $i \geq 4$ was first given in [BZh]. However, the proof of one of the lemmas in [BZh] has certain gap. We will give another proof of that lemma in Appendix.

Notations.

Denote by $\sigma_{p-1} = 2\pi^{p/2}/\Gamma(p/2)$, $p > 0$, and $\kappa_p = \sigma_{p-1}/p$. Then σ_{n-1} is the surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$, and κ_n is the volume of the n-dimensional unit ball B^n. Let $G_{n,i}$, $1 \leq i \leq n-1$, be the Grassmann manifold of i-dimensional subspaces of $\mathbb{R}^n$. For $u \in S^{n-1}$ and $\xi \in G_{n,i}$, we denote by du and $d\xi$ the corresponding $SO(n)$-invariant measures with total mass 1.

A convex body in $\mathbb{R}^n$ is a compact convex set with nonempty interior. The class of all convex bodies in $\mathbb{R}^n$ containing the origin o in its interior will be denoted by $\mathcal{K}^n$. Denote by $\mathcal{K}^n_e$ the subclass of origin-symmetric convex bodies. A star body L in $\mathbb{R}^n$ is a star shaped set that has continuous radial function $\rho_L(u) = \sup\{\lambda \geq 0 : \lambda u \in L\}, \quad u \in S^{n-1}$. The class of star bodies is denoted by $\mathcal{S}^n$. The subclass of origin-symmetric star bodies is denoted by $\mathcal{S}^n_e$. $\mathbb{N}$ is the set of positive integers.

1. Spherical Radon transforms

Various problems about sections of centrally symmetric bodies are intimately connected with totally geodesic Radon transforms on the unit sphere S^{n-1}. In this section we present some auxiliary statements about these transforms. More information can be found in [He], [R2], [R3], and other sources.

For continuous functions $f(u)$ on S^{n-1} and $\varphi(\xi)$ on $G_{n,i}$, the totally geodesic Radon transform $\mathrm{R}_i f$ and its dual transform $\mathrm{R}^*_i \varphi$ are defined by

$$(\mathrm{R}_i f)(\xi) = \int_{S^{n-1}\cap\xi} f(u) d_\xi u, \qquad (\mathrm{R}^*_i \varphi)(u) = \int_{u\in\xi} \varphi(\xi) d_u \xi, \tag{1.1}$$

where $d_\xi u$ and $d_u \xi$ denote the induced normalized measures on the corresponding manifolds $S^{n-1}\cap\xi$ and $\{\xi \in G_{n,i} : u \in \xi\}$. These manifolds can be identified with S^{i-1} and

$G_{n-1,i-1}$, respectively. The duality between R_i and R_i^* is expressed by

$$\int_{G_{n,i}} (\mathrm{R}_i f)(\xi)\varphi(\xi)d\xi = \int_{S^{n-1}} f(u)(\mathrm{R}_i^*\varphi)(u)du. \tag{1.2}$$

This allows to define $\mathrm{R}_i\mu$ and $\mathrm{R}_i^*\nu$ for arbitrary finite Borel measures μ on S^{n-1} and ν on $G_{n,i}$.

Let $e_1, e_2, \dots, e_n$ be coordinate unit vectors, and $x \cdot y$ the usual inner product in $\mathbb{R}^n$. Given $u \in S^{n-1}$ and $\xi \in G_{n,i}$, we write $d(u,\xi)$ for the geodesic distance between u and $S^{n-1} \cap \xi$, and denote by $|\mathrm{P}_\xi u|$ the length of the orthogonal projection of u onto ξ. It is easily seen that

$$\sin d(u,\xi) = |\mathrm{P}_{\xi^\perp} u|, \qquad \cos d(u,\xi) = |\mathrm{P}_\xi u|. \tag{1.3}$$

A function on the unit sphere S^{n-1} or on the Grassmannian $G_{n,i}$ is called a *zonal function* if it is invariant under the group $SO(n-1)$ of rotations preserving the north pole e_n.

Lemma 1.1.
(i) *A function $f(u)$ on S^{n-1} is zonal if and only if there exist $f_0(t)$, $t \in [-1,1]$, such that $f(u) = f_0(\cos\omega)$, $\omega = d(e_n, u)$. If $f(u)$ is zonal then $f(u) = f(e_1 \sin\omega + e_n \cos\omega)$.*
(ii) *A function $\varphi(\xi)$ on $G_{n,i}$ is zonal if and only if there exist $\varphi_0(s)$, $s \in [0,1]$, such that $\varphi(\xi) = \varphi_0(\sin\theta)$, $\theta = d(e_n,\xi)$. If $\varphi(\xi)$ is zonal then $\varphi(\xi) = \varphi(\xi_\theta)$ where $\xi_\theta = \mathrm{span}(e_1, \dots, e_{i-1}, e_i \sin\theta + e_n \cos\theta)$.*

The statement (i) and the "if" part in (ii) are obvious. The "only if" part in (ii) can be understood by geometric reasoning and proved analytically following more general Lemma 2.7 from [GrR].

The next lemma provides Abel type representations for Radon transforms and there dual transforms of zonal functions.

Lemma 1.2. *For $u \in S^{n-1}$ and $\xi \in G_{n,i}$, we denote*

$$\omega = d(e_n, u), \quad t = \cos\omega, \qquad \theta = d(e_n,\xi), \quad s = \sin\theta.$$

If $f(u) = f_0(t)$ and $\varphi(\xi) = \varphi_0(s)$ then

$$(\mathrm{R}_i f)(\xi) = \frac{c_1}{\cos^{i-2}\theta} \int_{-\cos\theta}^{\cos\theta} (\cos^2\theta - t^2)^{(i-3)/2} f_0(t)\, dt, \tag{1.4}$$

$$(\mathrm{R}_i^*\varphi)(u) = \frac{c_2}{\sin^{n-3}\omega} \int_0^{\sin\omega} (\sin^2\omega - s^2)^{(i-3)/2} s^{n-i-1} \varphi_0(s)\, ds, \tag{1.5}$$

$$c_1 = \frac{\sigma_{i-2}}{\sigma_{i-1}}, \qquad c_2 = \frac{\sigma_{i-2}\,\sigma_{n-i-1}}{\sigma_{n-2}},$$

provided that the corresponding integrals exist in the Lebesgue sense. If f is an even function then (1.4) becomes

$$(\mathrm{R}_i f)(\xi) = \frac{2c_1}{\cos^{i-2}\theta} \int_0^{\cos\theta} (\cos^2\theta - t^2)^{(i-3)/2} f_0(t)\, dt. \tag{1.6}$$

Proof. The formula (1.4) can be found in many sources in different forms. Both formulas were presented in [R2], Lemma 2.4. For the sake of completeness we give another proof which is more geometric. Let $\mathrm{P}_\xi e_n = (\cos\theta) u_0$, ψ be the angle between u_0 and u. Using spherical trigonometry on the triangle (e_n, u, u_0), we get $u \cdot e_n = \cos\omega = \cos\psi\cos\theta$, and

$$\begin{aligned}(\mathrm{R}_i f)(\xi) &= \int_{S^{n-1}\cap\xi} f_0(u\cdot e_n)\, du \\ &= \frac{\sigma_{i-2}}{\sigma_{i-1}} \int_0^\pi f_0(\cos\psi\cos\theta)\sin^{i-2}\psi\, d\psi.\end{aligned}$$

This coincides with (1.4). In order to prove (1.5) we fix $\xi_0 \in G_{n,i}$ that contains e_n and u, and denote by G_u the group of rotations in $u^\perp$. For $\alpha \in G_u$, let $\alpha e_n = v\sin\omega + u\cos\omega$, $\quad v \in S^{n-1}\cap u^\perp$. Then $|\mathrm{P}_{\xi_0^\perp}(\alpha e_n)| = |\mathrm{P}_{\xi_0^\perp} v|\sin\omega$, and by (1.3),

$$\begin{aligned}(\mathrm{R}_i^* a_2)(u) &= \int_{u\in\xi} a(|\mathrm{P}_{\xi^\perp} e_n|) d\xi \\ &= \int_{G_u} a(|\mathrm{P}_{\alpha^{-1}\xi_0^\perp} e_n|) d\alpha \\ &= \int_{G_u} a(|\mathrm{P}_{\xi_0^\perp}(\alpha e_n)|) d\alpha \\ &= \int_{S^{n-1}\cap u^\perp} a(|\mathrm{P}_{\xi_0^\perp} v|\sin\omega) dv.\end{aligned}$$

Using the bi-spherical coordinates [VK, pp. 12, 22]

$$\begin{aligned} &v = x\cos\psi + y\sin\psi, \qquad dv = c_2 \sin^{n-i-1}\psi\cos^{i-2}\psi\, d\psi dx dy, \\ &x \in S^{i-2} \subset \xi_0 \cap u^\perp, \qquad y \in S^{n-i-1} \subset \xi_0^\perp, \qquad 0 < \psi < \pi/2,\end{aligned}$$

we have $|\mathrm{P}_{\xi_0^\perp} v| = \sin\psi$, and

$$(\mathrm{R}_i^* a_2)(u) = c_2 \int_0^{\pi/2} a(\sin\omega\sin\psi)\sin^{n-i-1}\psi\cos^{i-2}\psi d\psi.$$

This gives (1.5). $\square$

Let $\mathcal{C}_i$ be the class of C^∞ zonal functions on the Grassmannian $G_{n,i}$. Note that $\mathcal{C}_1$ is the class of even C^∞-functions on S^{n-1}.

Lemma 1.3. *The Radon transform* R_i *and the dual transform* R_i^* *are bijective mapings from* $\mathcal{C}_1$ *onto* $\mathcal{C}_i$ *and from* $\mathcal{C}_i$ *onto* $\mathcal{C}_1$*, respectively.*

Proof. It is known (see [He], Proposition 2.4, p. 60) that R_i and R_i^* act from $C^\infty(S^{n-1})$ into $C^\infty(G_{n,i})$ and from $C^\infty(G_{n,i})$ into $C^\infty(S^{n-1})$, respectively. Since both transforms commute with rotations, then $\mathrm{R}_i(\mathcal{C}_1) \subset \mathcal{C}_i$ and $\mathrm{R}_i^*(\mathcal{C}_i) \subset \mathcal{C}_1$. Injectivity of R_i on $\mathcal{C}_1$ and

R_i^* on $\mathcal{C}_i$ follows from Lemma 1.1 and uniqueness of inversion of Abel type integrals (1.6), (1.5) [SKM]. It was also proved in [GZ] (Lemma 2.2) using harmonic analysis on Grassmannians. Note that R_i is injective on $C^\infty_{even}(S^{n-1})$ [He, p. 99], [R2], whereas R_i^* is not injective on $C^\infty(G_{n,i})$.

In order to prove surjectivity of $\mathrm{R}_i : \mathcal{C}_1 \to \mathcal{C}_i$ and $\mathrm{R}_i^* : \mathcal{C}_i \to \mathcal{C}_1$ we introduce the spherical Riesz potential (or the generalized sine transform) [R2]

(1.7)
$$\begin{aligned}(Q^\alpha f)(u) &= \frac{\sigma_{n-1}\Gamma((n-1-\alpha)/2)}{2\pi^{(n-1)/2}\Gamma(\alpha/2)} \int_{S^{n-1}} (1-|u\cdot v|^2)^{(\alpha-n+1)/2} f(v)dv \\ &= \frac{\sigma_{n-1}\Gamma((n-1-\alpha)/2)}{2\pi^{(n-1)/2}\Gamma(\alpha/2)} \int_{S^{n-1}} (\sin d(u,v))^{\alpha-n+1} f(v)dv,\end{aligned}$$
$$\alpha > 0, \qquad \alpha - n + 1 \neq 0, 2, 4, \dots .$$

Here $d(u,v)$ is the geodesic distance between the points $u, v \in S^{n-1}$. The normalizing coefficient in (1.7) is chosen so that if Y_j is a spherical harmonic of degree j, then by the Funk-Hecke formula and [PBM, 2.21.2(3)], $Q^\alpha(Y_j) = q_\alpha(j)Y_j$ where

$$q_\alpha(j) = \frac{\Gamma((j+n-1-\alpha)/2)\,\Gamma((j+1)/2)}{\Gamma((j+\alpha+1)/2)\,\Gamma((j+n-1)/2)} \qquad (\sim (j/2)^{-\alpha} \quad \text{as} \quad j \to \infty) \tag{1.8}$$

for j even, and $q_\alpha(j) = 0$ for j odd. Since $q_\alpha(j)$ is finite for $\alpha - n + 1 \neq 0, 2, 4, \dots$, different from zero for j even, and has a power behavior as $j \to \infty$, then Q^α maps $C^\infty_{even}(S^{n-1})$ into $C^\infty_{even}(S^{n-1})$ and is injective. By the same reason, for any $g \in C^\infty_{even}(S^{n-1})$, if $g = \sum Y_j$, then $f = \sum_{j\ even} q_\alpha(j)^{-1} Y_j$ belongs to $C^\infty_{even}(S^{n-1})$ and $Q^\alpha f = g$. Thus Q^α is an automorphism of $C^\infty_{even}(S^{n-1})$. It is known [He, p. 94], [R2, Theorem 1.1], that

$$\mathrm{R}_i^*\mathrm{R}_i f = c\, Q^{i-1} f, \qquad c = \frac{\Gamma((n-1)/2)\Gamma(i/2)}{\Gamma((n-i)/2)\Gamma(1/2)}. \tag{1.9}$$

Let us show that each function $\varphi \in \mathcal{C}_i$ is represented by the Radon transform $\varphi = \mathrm{R}_i f$ for some $f \in \mathcal{C}_1$. We have $\mathrm{R}_i^*\varphi \in \mathcal{C}_1$. Hence there exists $f \in \mathcal{C}_1$ so that $\mathrm{R}_i^*\varphi = cQ^{i-1}f$. By (1.9), $\mathrm{R}_i^*\varphi = \mathrm{R}_i^*\mathrm{R}_i f$. Since φ and $\mathrm{R}_i f$ are zonal and R_i^* has a trivial kernel in $\mathcal{C}_i$, this implies $\varphi = \mathrm{R}_i f$. Now let us show that each function $f \in \mathcal{C}_1$ is represented by the dual Radon transform $f = \mathrm{R}_i^*\varphi$ for some $\varphi \in \mathcal{C}_i$. By (1.9),

$$f = Q^{i-1}(Q^{i-1})^{-1} f = \mathrm{R}_i^*\varphi, \quad \varphi = c^{-1}\mathrm{R}_i(Q^{i-1})^{-1} f.$$

Since R_i and $(Q^{i-1})^{-1}$ preserve smoothness and zonality, we are done. $\square$

2. Dual intrinsic volumes and intersection bodies

The following lemma contains various representations of the dual volume functional $\tilde{V}_k$. For $\xi \in G_{n,i}$, we denote by $G_{i,k}(\xi)$ the Grassmann manifold of all k-dimensional subspaces η in ξ.

Lemma 2.1. *Let $K \in \mathcal{S}^n$, $\xi \in G_{n,i}$, $1 \le k \le i \le n$. Then the dual volume*

$$\tilde{V}_k(K \cap \xi) = \int_{G_{n,n-i+k}} \mathrm{vol}_k(K \cap \xi \cap \eta) d\eta \tag{2.1}$$

has the following representations:

$$\tilde{V}_k(K \cap \xi) = \int_{G_{i,k}(\xi)} \mathrm{vol}_k(K \cap \zeta) d\zeta \tag{2.2}$$

$$= \kappa_k \int_{S^{n-1}\cap\xi} \rho_K(u)^k d_\xi u \tag{2.3}$$

$$= \kappa_k (\mathrm{R}_i \rho_K^k)(\xi) \tag{2.4}$$

$$= \frac{\sigma_{k-1}}{\sigma_{i-1}} \int_{K\cap\xi} |x|^{k-i} dx. \tag{2.5}$$

Proof. Let us prove (2.2). For almost all $\eta \in G_{n,n-i+k}$, the intersection $\zeta = \xi \cap \eta$ is an element of $G_{i,k}(\xi)$. Given $\zeta \in G_{i,k}(\xi)$, denote by $d_\zeta \eta$ the invariant probability measure on the homogeneous manifold $\{\eta \in G_{n,n-i+k} : \eta \supset \zeta\}$. By the formulas (14.40) and (14.42) in [Sa], for any $f \in L^1(G_{n,n-i+k})$ we have

$$\int_{G_{n,n-i+k}} f(\eta) d\eta = \int_{G_{i,k}(\xi)} d\zeta \int_{\{\eta\in G_{n,n-i+k} : \eta\supset\zeta\}} f(\eta)\, \Delta(\eta) d_\zeta \eta \tag{2.6}$$

where the factor $\Delta(\eta)$ satisfies

$$\int_{\{\eta\in G_{n,n-i+k} : \eta\supset\zeta\}} \Delta(\eta)\, d_\zeta \eta = c \equiv \mathrm{const}.$$

The value of c depends on normalization of measures involved. If we set $f(\eta) = \mathrm{vol}_k(K\cap \xi \cap \eta)$ then for $\eta \supset \zeta$ we get $f(\eta) = f(\zeta)$, and (2.6) yields

$$\int_{G_{n,n-i+k}} \mathrm{vol}_k(K \cap \xi \cap \eta) d\eta = c \int_{G_{i,k}(\xi)} \mathrm{vol}_k(K \cap \zeta) d\zeta. \tag{2.7}$$

If K is a unit ball then (2.7) gives $c = 1$ and (2.2) follows.

Let us prove (2.3) and (2.4), which are equivalent. By passing to polar coordinates, we have $\mathrm{vol}_k(K \cap \zeta) = \kappa_k(\mathrm{R}_k \rho_K^k)(\zeta)$, and therefore

$$\int_{G_{i,k}(\xi)} \mathrm{vol}_k(K \cap \zeta) d\zeta = \kappa_k \int_{G_{i,k}(\xi)} (\mathrm{R}_k \rho_K^k)(\zeta) d\zeta = \kappa_k \int_{S^{n-1}\cap\xi} \rho_K(u)^k d_\xi u.$$

The last step is clear in view of duality (1.2) (set $\varphi \equiv 1$). The equivalence of (2.5) and (2.3) follows if we write (2.5) in polar coordinates. $\square$

Remark 2.2.

(i) If M is a compact convex set of dimension i in $\mathbb{R}^n$ that contains the origin in its relative interior, and $M \subset \xi \in G_{n,i}$, then $\tilde{V}_k(M \cap \xi) = \tilde{V}_k(M)$, cf. (0.1). By (2.3), the dual volume $\tilde{V}_k(M)$ is independent of the dimension n of the ambient space, and can be called *the kth dual intrinsic volume of* M. This notion is dual in a sense to the intrinsic volume $V_k(M)$ introduced by McMullen in [Mc1]; see also [Mc2], [S, p. 210]. Whereas $V_k(M)$ corresponds to projections of convex bodies, $\tilde{V}_k(M)$ is appropriate to studying central sections. The representation (2.5) shows that $\tilde{V}_k(M)$, $k = 1, 2, \dots,$ are k-homogeneous rotation invariant valuations on $\mathcal{K}^n$; see [A], [Kl].

(ii) The integrals in (2.3) and (2.4) are meaningful for all positive k and star bodies. We will keep the same notation $\tilde{V}_k(K \cap \xi)$ and $\tilde{V}_k(K)$ for these cases. If $i = n$, then (2.3) becomes

$$\tilde{V}_k(K) = \kappa_k \int_{S^{n-1}} \rho_K(u)^k du. \tag{2.8}$$

For $k \le n$, $k \in \mathbb{N}$, (2.8) is the re-normalized version of the dual Quermassintegral

$$\tilde{V}_k(K) = \frac{\kappa_k}{\kappa_n} \tilde{W}_{n-k}(K). \tag{2.9}$$

Dual volumes of star bodies were introduced by Lutwak; see [Lu] and [BZ, p. 158]. The definition here differs by a normalization constant factor, which was resulted from a discusson with R.J. Gardner.

(iii) In the case $i = n$, $k \le n$, $k \in \mathbb{N}$, (2.2) yields

$$\tilde{V}_k(K) = \int_{G_{n,k}(\xi)} \mathrm{vol}_k(K \cap \zeta) d\zeta. \tag{2.10}$$

This quantity measures the mean volume of k-dimensional central sections of K.

(iv) From (0.2) and (2.3), we have

$$\tilde{V}_k(K \cap \xi) = \kappa_k I_k(K, \xi), \qquad \tilde{V}_k(K) = \kappa_k J_k(K). \tag{2.11}$$

These equalities give the connection between Problems A and B.

Definition 2.3. For $1 \le i \le n-1$, $p \in \mathbb{R}$, an origin-symmetric star body $K \in \mathcal{S}_e^n$ is called an (i,p)-intersection body if there exists a non-negative measure μ on the Grassmannian $G_{n,i}$ such that

$$\rho_K^p = \mathrm{R}_i^* \mu. \tag{2.12}$$

We denote by $\mathcal{I}_{i,p}$ the class of all (i,p)-intersection bodies in $\mathbb{R}^n$.

An $(n-1,1)$-intersection body is simply called intersection body. This notion was introduced by Lutwak [Lu]. The $(i, n-i)$-intersection bodies were studied in [GrZ] and [Z1]. The connection of Problems A and B with (i,p)-intersection bodies is given by the following two lemmas. The basic idea comes from Lutwak [Lu].

Lemma 2.4. *Let $0 < k < l$, and $K, L \in \mathcal{S}^n$. If $K \in \mathcal{I}_{i,\,l-k}$, and*

$$\tilde{V}_k(K \cap \xi) \le \tilde{V}_k(L \cap \xi)$$

for any $\xi \in G_{n,i}$, then

$$\tilde{V}_l(K) \le \tilde{V}_l(L).$$

Proof. Since $K \in \mathcal{I}_{i,\,l-k}$, there exists a non-negative measure μ on $G_{n,i}$ such that $\rho_K^{l-k} = \mathrm{R}_i^* \mu$. By (2.8), (1.2), and the Hölder inequality, we have

$$\begin{aligned}
\tilde{V}_l(K) &= \kappa_l \int_{S^{n-1}} \rho_K(u)^l du \\
&= \kappa_l \int_{S^{n-1}} (\mathrm{R}_i^* \mu)(u) \rho_K(u)^k du \\
&= \kappa_l \int_{G_{n,i}} (\mathrm{R}_i \rho_K^k)(\xi) d\mu(\xi) \\
&= \frac{\kappa_l}{\kappa_k} \int_{G_{n,i}} \tilde{V}_k(K \cap \xi) d\mu(\xi) \\
&\le \frac{\kappa_l}{\kappa_k} \int_{G_{n,i}} \tilde{V}_k(L \cap \xi) d\mu(\xi) \\
&= \kappa_l \int_{S^{n-1}} \rho_K(u)^{l-k} \rho_L(u)^k du \\
&\le \tilde{V}_l(K)^{1-k/l} \tilde{V}_l(L)^{k/l}.
\end{aligned}$$

This gives the inequality $\tilde{V}_l(K) \le \tilde{V}_l(L)$. □

Lemma 2.5. *Let $0 < k < l$, and let L be a C^∞ origin-symmetric convex body of revolution that has positive curvature. If $L \notin \mathcal{I}_{i,l-k}$ then there exists another C^∞ origin-symmetric convex body of revolution K having positive curvature so that*

$$\tilde{V}_k(K \cap \xi) < \tilde{V}_k(L \cap \xi)$$

for any $\xi \in G_{n,i}$, but

$$\tilde{V}_l(K) > \tilde{V}_l(L).$$

Proof. By Lemma 1.3, there is a unique C^∞ zonal function f on $G_{n,i}$ so that

$$\rho_L^{l-k} = \mathrm{R}_i^* f.$$

Since $L \notin \mathcal{I}_{i,l-k}$, f is negative somewhere. Therefore, there is a C^∞ zonal function $g > 0$ on $G_{n,i}$ such that

$$\int_{G_{n,i}} fg < 0.$$

Define an origin-symmetric body of revolution K by

$$\rho_K^k = \rho_L^k - \varepsilon g_1$$

where $\varepsilon > 0$ is sufficiently small and $g = \mathrm{R}_i g_1$. Using the argument from [G2, p. 439], one can show that K is convex and its boundary has positive curvature. It follows that

$$\mathrm{R}_i \rho_K^k = \mathrm{R}_i \rho_L^k - \varepsilon g.$$

By (2.4), this gives

$$\tilde{V}_k(K \cap \xi) < \tilde{V}_k(L \cap \xi)$$

for any $\xi \in G_{n,i}$. But

$$\int_{S^{n-1}} \rho_L^{l-k}(\rho_L^k - \rho_K^k) = \varepsilon \int_{S^{n-1}} (\mathrm{R}_i^* f) g_1 = \varepsilon \int_{S^{n-1}} fg < 0.$$

Thus, the Hölder inequality gives

$$\int_{S^{n-1}} \rho_L^l < \int_{S^{n-1}} \rho_L^{l-k} \rho_K^k \leq \left(\int_{S^{n-1}} \rho_L^l \right)^{1-k/l} \left(\int_{S^{n-1}} \rho_K^l \right)^{k/l}.$$

This yields $\tilde{V}_l(K) > \tilde{V}_l(L)$. $\square$

3. Some geometric lemmas

Given a point $z \in int(K)$ (the interior of K), define the extended radial function of K with respect to z,

$$\rho(z, v) = \sup\{\lambda > 0 : z + \lambda v \in K\}, \quad (z, v) \in \Omega = int(K) \times S^{n-1}.$$

Lemma 3.1. *If a convex body $K \in \mathcal{K}^n$ has C^m boundary ∂K, $1 \leq m \leq \infty$, then the extended radial function $\rho(z, v)$ is C^m in Ω.*

Proof. Consider the function

$$v = g(z, x) = \frac{x - z}{|x - z|}, \quad z \in int(K),\ x \in \partial K.$$

Since ∂K is C^m, $g(z, x)$ is a C^m function in $int(K) \times \partial K$. When z is fixed, $g(z, \cdot)$ is a C^m diffeomorphism from ∂K to S^{n-1}. By the implicit function theorem, $x = f(z, v)$ is a C^m function in Ω. Thus, $\rho(z, v) = |x - z| = |f(z, v) - z|$ is a C^m function in Ω. $\square$

Lemma 3.2. *For $K \in \mathcal{K}_e^n$, there exist origin-symmetric convex bodies $K_j \subset K$ of positive curvature such that $\rho_{K_j}(u) \in C^\infty(S^{n-1})$ and $\rho_{K_j}(u) \to \rho_K(u)$ uniformly on S^{n-1} as $j \to \infty$.*

Proof. Without loss of generality, assume that $\rho_K(u) \geq 1$. It is well known that every origin-symmetric convex body can be approximated in the Hausdorff metric by origin-symmetric convex bodies having C^∞-boundary of positive curvature; see [S], pp. 158-160. Therefore, there exist C^∞ origin-symmetric convex bodies of positive curvature K'_j such that

$$|\rho_{K'_j}(u) - \rho_K(u)| < \frac{1}{j+1} \quad \forall u \in S^{n-1}.$$

Let $K_j = \frac{j}{j+1}K'_j$. Then, obviously, $\rho_{K_j}(u) \to \rho_K(u)$ uniformly on S^{n-1} as $j \to \infty$, and

$$\rho_{K_j} = \frac{j}{j+1}\rho_{K'_j} < \frac{j}{j+1}(\rho_K + \frac{1}{j+1}) \leq \rho_K.$$

Thus, $K_j \subset K$. $\square$

Lemma 3.3. *If K is a C^∞ convex body of revolution with the radial function ρ_K, then the star body K_ε, $\varepsilon > 0$, defined by*

$$\rho_{K_\varepsilon}^p = \rho_K^p - \varepsilon p \rho_K^{p+1}, \quad p > 0,$$

is a C^∞ convex body that has positive curvature when $\varepsilon > 0$ is small enough.

Proof. We only need to prove that the boundary of K_ε has positive curvature. Note that

$$\rho_{K_\varepsilon} = \rho_K - \varepsilon\rho_K^2 + O(\varepsilon^2).$$

Let $\rho = \rho_K - \varepsilon\rho_K^2$. Abusing notation, one can write $\rho(u) \equiv \rho(\theta)$, θ being the angle between u and the hyperplane $x_n = 0$. Since $\rho_K^2 + 2{\rho'_K}^2 - \rho_K\rho''_K \geq 0$, an elementary calculation gives

$$\rho^2 + 2{\rho'}^2 - \rho\rho'' = (1 - 3\varepsilon\rho_K)(\rho_K^2 + 2{\rho'_K}^2 - \rho_K\rho''_K) + \varepsilon\rho_K^3 + O(\varepsilon^2) > 0$$

when ε is small enough. This gives the desired result. $\square$

Lemma 3.4. *Let C be a C^2 closed convex curve in the plane that encloses the origin. If (ρ, θ) are the polar coordinates of a point on the curve and $\kappa(\theta)$ is the curvature, then*

$$\int_0^{2\pi} \kappa(\theta)(\rho(\theta)^2 + {\rho'(\theta)}^2)^{\frac{1}{2}} d\theta = 2\pi.$$

Proof. Let s be the parameter of arc length of C, and let φ be the angle between the tangent line and the x-axis. Then

$$\frac{ds}{d\theta} = (\rho(\theta)^2 + {\rho'(\theta)}^2)^{\frac{1}{2}}, \qquad \frac{d\varphi}{ds} = \kappa.$$

Thus

$$2\pi = \int_0^{2\pi} d\varphi = \int_0^{\ell} \kappa ds = \int_0^{2\pi} \kappa(\theta)(\rho(\theta)^2 + {\rho'(\theta)}^2)^{\frac{1}{2}} d\theta,$$

where ℓ is the perimeter of the curve. $\square$

4. Dual Radon transform formulas for radial functions of convex bodies

It was discovered in [G1] and [Z2] that positive solutions to the Busemann-Petty problem in $\mathbb{R}^3$ and in $\mathbb{R}^4$ are intimately connected with the volume of parallel hyperplane sections of convex bodies. For $i = n-1$, this connection evolves through appropriate representation of the inverse spherical Radon transform of the radial function $\rho_K(u)$. The corresponding formulas were obtained in [G1] (n=3), [Z2] (n=4), and [GKS] (all $n \geq 3$). A different proof of the formulas was given in [BFM]. In this section, we generalize these formulas to i-dimensional sections for all $2 \leq i \leq n-1$.

For each convex body $K \in \mathcal{K}^n$, define the following functions:

$$A_i(t,\xi) = \int_{S^{n-1}\cap\xi^\perp} \mathrm{vol}_i(K \cap \{tu+\xi\})du, \qquad \xi \in G_{n,i}, \quad t \in \mathbb{R}, \tag{4.1}$$

$$a(t,v) = \int_{S^{n-1}\cap v^\perp} \mathrm{vol}_1(K \cap \{tu+\mathbb{R}v\})du, \qquad v \in S^{n-1}, \quad t \in \mathbb{R}. \tag{4.2}$$

The function (4.1) averages volumes of all i-dimensional sections of K parallel to ξ at distance $|t|$ from the origin. The function (4.2) is the mean length of chords parallel to v at distance $|t|$ from the origin. Note that

$$a(0,v) = \rho_K(v). \tag{4.3}$$

Lemma 4.1. *Let $K \in \mathcal{K}^n$, $2 \leq i < n$. Then*

$$\mathrm{R}_i^* A_i(t,\cdot)(v) = \sigma_{i-2} \int_t^\infty a(r,v)(r^2-t^2)^{\frac{i-3}{2}} r dr, \quad t \geq 0. \tag{4.4}$$

Proof. This equality was established in [BFM] for $i = n-1$. In the general case the proof is as follows. We denote

$$g_i(t,v) = \mathrm{R}_i^* A_i(t,\cdot)(v), \quad v \in S^{n-1},$$
$$b(x,v) = \mathrm{vol}_1(K \cap \{x+\mathbb{R}v\}), \quad x \in \mathbb{R}^n.$$

Fix a unit vector in $v_1 \in v^\perp$ and let ξ_0 be a subspace of dimension i that contains v and v_1. Let $SO(n-1)$ be the group of rotations about v. Then

$$\begin{aligned}
g_i(t,v) &= \int_{SO(n-1)} A_i(t,\alpha\xi_0)d\alpha \\
&= \int_{SO(n-1)} d\alpha \int_{S^{n-1}\cap\xi_0^\perp} du_0 \int_{x\in K\cap\{\alpha\xi_0+t\alpha u_0\}} dx \\
&= \int_{SO(n-1)} d\alpha \int_{S^{n-1}\cap\xi_0^\perp} du_0 \int_{y\in\xi_0\cap v^\perp} b(\alpha(y+tu_0),v)dy \\
&= \int_{SO(n-1)} d\alpha \int_{S^{n-1}\cap\xi_0^\perp} du_0 \int_{y\in\xi_0\cap v^\perp} b(\sqrt{|y|^2+t^2}\alpha v_1,v)dy \\
&= \sigma_{i-2} \int_{SO(n-1)} d\alpha \int_0^\infty b(\sqrt{s^2+t^2}\alpha v_1,v)s^{i-2}ds \\
&= \sigma_{i-2} \int_{S^{n-1}\cap v^\perp} du \int_t^\infty b(ru,v)(r^2-t^2)^{\frac{i-3}{2}} rdr,
\end{aligned}$$

which gives (4.4). $\square$

Let

$$r_K = \sup\{t > 0 :\ tB \subset K\} \tag{4.5}$$

be the radius of inscribed ball in K.

Lemma 4.2. *If convex body $K \in \mathcal{K}^n$ is C^m, $1 \le m \le \infty$, then the derivatives*

$$A_i^{(j)}(t,\xi) = \left(\frac{d}{dt}\right)^j A_i(t,\xi), \qquad 1 \le j \le m,$$

are continuous in $(-r_K, r_K) \times G_{n,i}$.

Proof. Let $z = tu$, $|t| < r_K$, $u \in S^{n-1} \cap \xi^\perp$. Then

$$\mathrm{vol}_i(K \cap \{\xi + tu\}) = \frac{\sigma_{i-1}}{i} \int_{S^{n-1}\cap\xi} \rho(tu,v)^i dv, \tag{4.6}$$

and by (4.1) we have

$$A_i(t,\xi) = \frac{\sigma_{i-1}}{i} \int_{S^{n-1}\cap\xi^\perp} du \int_{S^{n-1}\cap\xi} \rho(tu,v)^i dv.$$

The lemma follows from Lemma 3.1. $\square$

Theorem 4.3. *If convex body $K \in \mathcal{K}_e^n$ is C^2, then*

$$\rho_K = \frac{1}{2\pi}\mathrm{R}_2^* \int_0^\infty \frac{A_2(0,\cdot) - A_2(t,\cdot)}{t^2}\, dt, \tag{4.7}$$

$$\rho_K = -\frac{1}{4\pi}\mathrm{R}_3^* A_3''(0,\cdot). \tag{4.8}$$

Proof. When u, v are fixed, $b(ru,v) = \mathrm{vol}_1(K \cap \{ru + \mathbb{R}v\})$ as a function of r has compact support and is continuously differentiable except on the boundary of the support. By (4.4) and the Fubini theorem,

$$\begin{aligned} g_2(t,v) &= \sigma_{i-2} \int_t^\infty (r^2 - t^2)^{-\frac12} r\,dr \int_{S^{n-1}\cap v^\perp} b(ru,v)du \\ &= -\sigma_{i-2} \int_{S^{n-1}\cap v^\perp} du \int_t^\infty (r^2 - t^2)^{\frac12} b'(ru,v)dr. \end{aligned}$$

By changing variable, we get

$$\begin{aligned} &\frac{1}{t^2}(g_2(t,v) - g_2(0,v)) \\ &\quad= \sigma_{i-2} \int_{S^{n-1}\cap v^\perp} du\Big(-\int_1^\infty (s^2-1)^{\frac12} b'(tsu,v)ds + \int_0^\infty s b'(tsu,v)ds \Big) \\ &\quad= \sigma_{i-2} \int_{S^{n-1}\cap v^\perp} du\Big(-\int_1^\infty (s - (s^2-1)^{\frac12}) b'(tsu,v)ds + \int_0^1 s b'(tsu,v)ds \Big). \end{aligned}$$

Thus, by Fubini's theorem,

$$\begin{aligned}
\mathrm{R}_2^* \int_0^\infty \frac{A_2(t,\cdot)-A_2(0,\cdot)}{t^2}dt &= \int_0^\infty \frac{g_2(t,v)-g_2(0,v)}{t^2}dt \\
&= \sigma_{i-2}\int_{S^{n-1}\cap v^\perp} du\Big(-b(0,v)\int_1^\infty \frac{s-(s^2-1)^{\frac{1}{2}}}{s}ds - b(0,v)\Big) \\
&= -2\pi\rho_K(v).
\end{aligned}$$

This yields (4.7).

To show (4.8), we differentiate (4.4) and obtain

$$\frac{d}{dt}\mathrm{R}_3^* A_3(t,\cdot)(v) = -2\pi t a(t,v).$$

Hence

$$\frac{d^2}{dt^2}\mathrm{R}_3^* A_3(t,\cdot)(v)\Big|_{t=0} = -2\pi a(0,v) = -4\pi\rho_K(v).$$

By Lemma 4.2, one can differentiate under the sign of R_3^* that gives (4.8). □

The special case $n=3$ of formula (4.7) was obtained in [G1], and the special case $n=4$ of formula (4.8) was proved in [Z2]. The proof of formula (4.7) above requires that the convex body is C^1. In fact, formula (4.7) holds for any origin-symmetric convex bodies.

Theorem 4.4. *If $K \in \mathcal{K}_e^n$ then the function*

$$\varphi_2(\xi) = \frac{1}{2\pi}\int_0^\infty \frac{A_2(0,\xi)-A_2(t,\xi)}{t^2}\,dt, \tag{4.9}$$

is well defined for almost all $\xi \in G_{n,2}$, non-negative, and integrable on $G_{n,2}$. Furthermore, for all $v \in S^{n-1}$,

$$\rho_K(v) = (\mathrm{R}_2^*\varphi_2)(v). \tag{4.10}$$

Proof. We fix $v \in S^{n-1}$, and set $g(t) = \mathrm{R}_2^* A_2(t,\cdot)(v)$, $a(r) = a(r,v)$. By (4.4),

$$\begin{aligned}
g(0)-g(t) &= 2t\Big[\int_0^\infty a(ts)ds - \int_1^\infty a(ts)(s^2-1)^{-1/2}sds\Big] \\
&= 2t\int_0^\infty a(ts)k(s)ds
\end{aligned}$$

where $k(s)=1$ if $s<1$ and $k(s) = 1 - s(s^2-1)^{-1/2}$ if $s>1$. Then for $\varepsilon > 0$,

$$\begin{aligned}
\frac{1}{2}\int_\varepsilon^\infty \frac{g(0)-g(t)}{t^2}\,dt &= \int_\varepsilon^\infty \frac{dt}{t}\int_0^\infty a(ts)k(s)ds \\
&= \int_0^\infty a(y)dy\int_\varepsilon^\infty k\Big(\frac{y}{t}\Big)\frac{dt}{t^2} \\
&= \int_0^\infty a(y)\frac{dy}{y}\int_0^{y/\varepsilon} k(s)ds.
\end{aligned}$$

This gives

$$\frac{1}{2}\int_{\varepsilon}^{\infty}\frac{g(0)-g(t)}{t^2}\,dt=\int_0^{\infty}a(\varepsilon y)\lambda(y)\,dy, \tag{4.11}$$

$$\lambda(y)=\frac{1}{y}\int_0^y k(s)ds=\frac{y-(y^2-1)_+^{-1/2}}{y}\in L^1(0,\infty),$$

where $(y^2-1)_+^{-1/2}=(y^2-1)^{-1/2}$ if $y>1$ and 0 otherwise. Since $a(r)$ is bounded and continuous at $r=0$, the Lebesgue theorem on dominated convergence yields

$$\frac{1}{2}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty}\frac{g(0)-g(t)}{t^2}\,dt=a(0)\int_0^{\infty}\lambda(y)dy=\frac{\pi}{2}\,a(0)=\pi\rho_K. \tag{4.12}$$

To finish the proof we denote

$$\varphi_\varepsilon(\xi)=\frac{1}{2\pi}\int_{\varepsilon}^{\infty}\frac{A_2(0,\xi)-A_2(t,\xi)}{t^2}\,dt.$$

It is clear that φ_ε is bounded on $G_{n,2}$ for each $\varepsilon>0$. Moreover, since K is convex and origin-symmetric, then $A_2(0,\xi)-A_2(t,\xi)\geq 0\quad \forall t,\xi$, and therefore $\varphi_\varepsilon(\xi)$ represents a sequence of non-decreasing (in ε) non-negative functions. The integrals

$$\mathrm{R}_2^*\varphi_\varepsilon=\frac{1}{2\pi}\int_{\varepsilon}^{\infty}\frac{g(0)-g(t)}{t^2}\,dt$$

are uniformly bounded in ε because by (4.11), $\mathrm{R}_2^*\varphi_\varepsilon\leq\pi^{-1}||a||_\infty\int_0^\infty\lambda(y)dy$. Applying the Beppo Levi theorem [KF, p. 58] and using (4.12), we conclude that the limit

$$\varphi_2(\xi)=\lim_{\varepsilon\to 0}\varphi_\varepsilon(\xi)=\frac{1}{2\pi}\int_0^{\infty}\frac{A_2(0,\xi)-A_2(t,\xi)}{t^2}\,dt$$

exists a.e. on $G_{n,2}$, φ_2 is integrable on $G_{n,2}$, and $\mathrm{R}_2^*\varphi_2=\lim\limits_{\varepsilon\to 0}\mathrm{R}_2^*\varphi_\varepsilon=\rho_K$. $\square$

The next theorem extends (4.7) and (4.8) to the case of sections of arbitrary dimensions $1\leq i<n$. It generalizes known formulas for $i=n-1$ which were obtained in [GKS] using Fourier transform techniques. Our proof is based on another idea, and might be useful in different occurrences.

Theorem 4.5. *If $K\in\mathcal{K}_e^n$ is C^∞, then the radial function ρ_K can be represented by the dual Radon transform $\rho_K=\mathrm{R}_i^*\varphi_i$ where*

(4.13)

$$\varphi_i(\xi)=\frac{\pi^{\frac{1-i}{2}}}{\Gamma\left(\frac{1-i}{2}\right)}\int_0^{\infty}t^{1-i}\Big[A_i(t,\xi)-\sum_{j=0}^{\frac{i-2}{2}}\frac{t^{2j}}{(2j)!}A_i^{(2j)}(0,\xi)\Big]dt,\quad i \textit{ even},$$

(4.14)

$$\varphi_i(\xi)=\frac{(-1)^{\frac{i-1}{2}}\,\pi^{1-\frac{i}{2}}}{2^i\,\Gamma\left(\frac{i}{2}\right)}A_i^{(i-1)}(0,\xi),\quad i \textit{ odd}.$$

Proof. Consider the following analytic family of functions associated to the body K:

$$g_\lambda(\xi) = \frac{1}{\Gamma(\lambda/2)} \int_K |P_{\xi^\perp} x|^{\lambda+i-n} dx, \quad \xi \in G_{n,i}, \quad 0 < Re\lambda < n-i, \tag{4.15}$$

where $|P_{\xi^\perp} x|$ denotes the length of the orthogonal projection of x on $\xi^\perp$. Integration in (4.15) over slices parallel to ξ gives

$$\begin{aligned} g_\lambda(\xi) &= \frac{1}{\Gamma(\lambda/2)} \int_{\xi^\perp} |y|^{\lambda+i-n} \mathrm{vol}_i(K \cap \{\xi + y\}) dy \\ &= \frac{\sigma_{n-i-1}}{\Gamma(\lambda/2)} \int_0^\infty t^{\lambda-1} A_i(t,\xi) dt. \end{aligned} \tag{4.16}$$

On the other hand, by passing to polar coordinates we have

$$g_\lambda(\xi) = \frac{\sigma_{n-i-1}}{(\lambda+i)\,\Gamma(\lambda/2)} \int_{S^{n-1}} |P_{\xi^\perp} u|^{\lambda+i-n} \rho_K^{\lambda+i}(u) du. \tag{4.17}$$

For $f \in C^\infty_{even}(S^{n-1})$, consider the generalized cosine transform [R2]

$$(\mathrm{R}_i^\lambda f)(\xi) = \frac{\sigma_{n-1}\,\Gamma((n-i-\lambda)/2)}{2\pi^{(n-1)/2}\Gamma(\lambda/2)} \int_{S^{n-1}} f(u) |P_{\xi^\perp} u|^{\lambda+i-n} du. \tag{4.18}$$

By Theorem 1.1 from [R2],

$$\frac{\Gamma((n-i)/2)}{\Gamma((n-1)/2)} \mathrm{R}_i^* \mathrm{R}_i^\lambda f = Q^{\lambda+i-1} f, \tag{4.19}$$

where $Q^{\lambda+i-1} f$ is the spherical Riesz potential defined by (1.7). For $f = \rho_K^{\lambda+i}$, combining (4.19), (4.18), and (4.17), we obtain

$$Q^{\lambda+i-1} \rho_K^{\lambda+i} = c_\lambda \mathrm{R}_i^* g_\lambda, \qquad c_\lambda = \frac{(\lambda+i)\,\Gamma((n-i)/2)\,\Gamma((n-i-\lambda)/2)}{2\pi^{(n-1)/2}\,\Gamma((n-1)/2)}. \tag{4.20}$$

Owing to (1.8), analytic family Q^α includes the identity operator (for $\alpha = 0$). Hence analytic continuation (*a.c.*) of (4.20) at $\lambda = 1 - i$ reads

$$\rho_K = c\, \mathrm{R}_i^* \left[a.c.\, g_\lambda \big|_{\lambda=1-i} \right], \qquad c = \frac{\Gamma((n-i)/2)}{2\pi^{(n-1)/2}}. \tag{4.21}$$

We evaluate analytic continuation in the square brackets using (4.16). By the well-known formula from [GS, Chapter 1, Sec. 3], for $-\ell < Re\,\lambda < -\ell + 1$, $\ell \in \mathbb{N}$, we have

$$a.c. \int_0^\infty t^{\lambda-1} A_i(t,\xi) dt = \int_0^\infty t^{\lambda-1} \Big[A_i(t,\xi) - \sum_{j=0}^{\ell-1} \frac{t^j}{j!} A_i^{(j)}(0,\xi) \Big] dt. \tag{4.22}$$

Since all derivatives of $A_i(t,\xi)$ of odd order are zero at $t=0$, then for ℓ odd, the sum $\sum_{j=0}^{\ell-1}$ can be replaced by $\sum_{j=0}^{\ell}$, and (4.22) holds for $-\ell-1<Re\,\lambda<-\ell+1$. It follows that for i even one can set $\ell=i-1$ in (4.22) and obtain (4.13). On the other hand, the duplication formula for Γ-functions yields

$$g_\lambda(\xi)=\frac{2^{\lambda-1}\pi^{1/2}\,\sigma_{n-i-1}}{(\lambda+i)\cos(\lambda\pi/2)\,\Gamma((1-\lambda)/2)}\Big[\frac{1}{\Gamma(\lambda)}\int_0^\infty t^{\lambda-1}A_i(t,\xi)dt\Big],$$

and therefore [GS, Chapter 1],

$$a.c.\,g_\lambda(\xi)\big|_{\lambda=1-i}=\frac{(-1)^{(i-1)/2}\,\pi^{1/2}\,\sigma_{n-i-1}}{2^i\,\Gamma(i/2)}A_i^{(i-1)}(0,\xi).$$

This equality together with (4.21) imply (4.14). □

5. Positive answers

The following theorem gives a partial solution to Problem B.

Theorem 5.1. *Let $i=2$, or 3, $k>0$, and $K,L\in\mathcal{K}_e^n$. If $I_k(K,\xi)\le I_k(L,\xi)$ for any $\xi\in G_{n,i}$, then $J_{k+1}(K)\le J_{k+1}(L)$.*

Proof. Let K have a C^∞ boundary. Then

$$t\mapsto[\mathrm{vol}_i(K\cap\{\xi+tu\})]^{1/i},\qquad u\in\xi^\perp,$$

is a concave function and has maximum at $t=0$. It follows that $A_i''(0,\cdot)\le 0$ (the derivative exists by Lemma 4.2), and $A_2(0,\xi)-A_2(t,\xi)\ge 0\ \forall\xi$. Hence Theorem 4.3 implies that $K\in\mathcal{I}_{i,1}$ when $i=2,\text{or }3$. The desired result now follows by Lemma 2.4. If K is not smooth, one can pick a sequence $\{K_j\}$ specified by Lemma 3.2. Since $K_j\subset K$, then $I_k(K_j,\xi)\le I_k(K,\xi)\le I_k(L,\xi)$, and, by above, $J_{k+1}(K_j)\le J_{k+1}(L)$ for all j. It remains to pass to the limit in the last inequality, by taking into account that $J_{k+1}(K_j)\to J_{k+1}(K)$ as $\rho_{K_j}(u)\to\rho_K(u)$ uniformly on S^{n-1}. □

The following theorem gives a partial solution to Problem A and generalizes positive solutions to the Busemann-Petty problem in $\mathbb{R}^3$ and $\mathbb{R}^4$; cf. [G1], [GKS], [Z2].

Theorem 5.2. *Let $1\le k\le i, i=2$, or 3, $K,L\in\mathcal{S}_e^n$. If K is convex, and*

$$\tilde V_k(K\cap\xi)\le\tilde V_k(L\cap\xi)$$

for all $\xi\in G_{n,i}$, then

$$\tilde V_{k+1}(K)\le\tilde V_{k+1}(L).$$

Theorem 5.1 resolves Problems A and B only for $l=k+1$. If we wish to extend it to any $l>k$ then, by Lemma 2.4, we need representation

$$\rho_K^{l-k}=\mathrm{R}_i^*\varphi,\qquad\varphi\ge 0.\tag{5.1}$$

Theorem 4.5 does not provide it, but the proof of this theorem shows that one can arrive at (5.1) and express φ in the same manner as in (4.13), if we change the definition (4.1) of $A_i(t,\xi)$. Let us sketch the basic idea [R4]. The expression $\text{vol}_i(K \cap \{\xi + tu\})$ is the euclidean Radon i-plane transform of the characteristic function $\chi_K(x)$ of the body K [He]. If we replace $A_i(t,\xi)$ by the i-plane transform of the more general function $|x|^\lambda \chi_K(x)$ with appropriate λ and proceed as in the proof of Theorem 4.5 we arrive at (5.1). This way leads to another class of star bodies rather than just convex bodies.

Bodies of revolution. Theorems 5.1 and 5.2 can be essentially strengthened for bodies of revolution. To this end we use representation (1.5) the right hand side of which resembles the so-called Erdelyi-Kober fractional integral. Such integrals are well known in fractional calculus and arise in numerous applications; see [SKM, Section 18(1)] and references therein. We recall some basic facts.

For $\alpha > 0$ and $\eta \geq -1/2$, *the Erdelyi-Kober fractional integral* is defined by

$$I^\alpha_\eta f(r) = \frac{2r^{-2(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^r (r^2 - s^2)^{\alpha-1} s^{2\eta+1} f(s) ds. \tag{5.2}$$

For $\alpha = 0$, we set $I^\alpha_\eta f = f$. Operators I^α_η enjoy the composition law

$$I^\alpha_\eta I^\beta_{\eta+\alpha} = I^{\alpha+\beta}_\eta. \tag{5.3}$$

This can be easily checked by changing the order of integration. The left inverse of I^α_η has the form

$$(I^\alpha_\eta)^{-1} f(r) = r^{-2\eta} \left(\frac{1}{2r} \frac{d}{dr} \right)^m r^{2\eta+2m} I^{m-\alpha}_{\eta+\alpha} f(r), \quad \forall m \geq \alpha, \ m \in \mathbb{N}. \tag{5.4}$$

Owing to (1.5), we have

$$\mathrm{R}^*_i \varphi(u) = c\, I^\alpha_\eta \varphi_0(r), \tag{5.5}$$

$$c = \frac{\pi^{(i-1)/2} \sigma_{n-i-1}}{\sigma_{n-2}}, \qquad \alpha = \frac{i-1}{2}, \qquad \eta = \frac{n-i}{2} - 1, \qquad r = \sin d(e_n, u).$$

Formulae (5.2)-(5.5) will be repeatedly used in the following.

We recall that a body K is axially convex (with respect to the x_n-axis) if any segment $[A, B]$ parallel to the x_n-axis lies in K provided that $A, B \in K$.

Theorem 5.3. *Let $i = 2$ or 3, $k > 0$, and $0 < p \leq n - i$. If $K, L \in \mathcal{S}^n_e$ and K is an axially convex body of revolution satisfying $I_k(K,\xi) \leq I_k(L,\xi)$ for any $\xi \in G_{n,i}$, then $J_{k+p}(K) \leq J_{k+p}(L)$.*

Proof. It is enough to prove the result when $\rho_K \in C^\infty(S^{n-1})$. In this case $\rho^p_K \in \mathcal{C}_1$, and, by Lemma 1.3, $\rho^p_K = \mathrm{R}^*_i \varphi$, $\varphi \in \mathcal{C}_i$. According to Lemma 1.1, we set $\rho_K(u) \equiv \rho(r)$, $\varphi(\xi) \equiv \varphi_0(s)$, where $r = \sin d(e_n, u)$, $s = \sin d(e_n, \xi)$. By (5.5),

$$\rho(r)^p = c\, I^\alpha_\eta \varphi_0(r). \tag{5.6}$$

If $i = 3$, then $\alpha = 1$, $2\eta = n - 5$, and (5.4) (with $m = 1$) yields

$$c\,\varphi_0(r) = \frac{1}{2} r^{4-n} (r^{n-3}\rho(r)^p)'. \tag{5.7}$$

This expression is non-negative for $0 \le p \le n-3$ because $r\rho(r)$ is non-decreasing (owing to axial convexity of K). Therefore, $K \in \mathcal{I}_{3,p}$. If $i = 2$, then $\alpha = 1/2$, $2\eta = n - 4$, and by (5.4) we have

$$\begin{aligned} c\,\varphi_0(r) &= \frac{r^{3-n}}{2} \frac{d}{dr} \left(r^{n-2} I^{1/2}_{(n-3)/2} \rho^p \right)(r) \\ &= \frac{r^{3-n}}{\pi^{1/2}} \frac{d}{dr} \int_0^r (r^2 - s^2)^{-1/2} s^{n-2} \rho(s)^p \, ds \qquad (5.8) \\ &= \frac{r^{3-n}}{\pi^{1/2}} \frac{d}{dr} \int_0^1 (1 - t^2)^{-1/2} (rt)^{n-2} \rho(rt)^p \, dt. \qquad (5.9) \end{aligned}$$

If $0 \le p \le n - 2$, then $r^{n-2}\rho(r)^p = r^{n-2-p}(r\rho(r))^p$ is non-decreasing since $r\rho(r)$ is non-decreasing. It follows that the integral in (5.9) is a non-decreasing function of r. Hence $\varphi_0 \ge 0$, and therefore, $K \in \mathcal{I}_{2,p}$. The desired result now follows from Lemma 2.4. $\square$

For $n = 3, i = 2$, Theorem 5.3 was proved by Hadwiger [H].

In the context of Problem A we have the following

Corollary 5.4. *Let $i = 2$ or 3, $1 \le k \le i$, $k < l \le n + k - i$. If $K, L \in \mathcal{S}^n_e$ and K is an axially convex body of revolution satisfying*

$$\tilde{V}_k(K \cap \xi) \le \tilde{V}_k(L \cap \xi)$$

for any $\xi \in G_{n,i}$, then

$$\tilde{V}_l(K) \le \tilde{V}_l(L).$$

6. Negative answers

To prove negative results, we need to construct counterexamples. The counterexamples are perturbations of a cylinder, which are constructed as follows: We take a rectangle centered at the origin in the two-plane (x_1, x_n): $\{(x_1, x_n) : |x_1| \le 1,\ |x_n| \le h\}$, $h > 0$. Smoothen each corner of the rectangle to get a closed convex curve C that is symmetric about the axes. Specifically, for small $\varepsilon > 0$, around the corner $(1, h)$, connect the points $(1 - \varepsilon, h)$ and $(1, h - \varepsilon)$ with a C^∞ convex curve that is C^∞ tangent to the rectangle at both points. Do the same for other corners. Then the curve C is C^∞ and contains four line segments. As an explicit example, one can divide the closed convex curve $e^{-1/x_1^2} + e^{-1/x_n^2} = e^{-1/\varepsilon^2}$ into four symmetric parts and put them at the four corners of the rectangle. Rotating the curve C about the x_n-axis, we get a C^∞ convex body of revolution L_ε.

Theorem 6.1. *For all $k > 0$, in the cases*

$$\text{(a)} \quad 1 < i \le 3,\ p > n - i, \qquad \textit{and} \qquad \text{(b)} \quad i \ge 4,\ p > 0,$$

there exist origin-symmetric convex bodies of revolution K, L in $\mathbb{R}^n$ such that

$$I_k(K,\xi) \le I_k(L,\xi)$$

for any $\xi \in G_{n,i}$, but

$$J_{k+p}(K) > J_{k+p}(L).$$

Proof. By Lemma 2.5, it suffices to show that there exists a C^∞ origin-symmetric convex body of revolution with positive curvature that is not in $\mathcal{I}_{i,p}$. By Lemma 3.3, we only need to construct a C^∞ origin-symmetric convex body of revolution $L \notin \mathcal{I}_{i,p}$. That is, if

$$\rho_L^p = \mathrm{R}_i^* \varphi, \tag{6.1}$$

then φ is negative somewhere. Let $L = L_\varepsilon$. Note that for some $\omega_\varepsilon > 0$,

$$\rho_L(u) = \frac{1}{\sin\omega}, \qquad \omega_\varepsilon \le \omega \le \frac{\pi}{2}, \tag{6.2}$$

where ω is the geodesic distance between u and the north pole. Let $\rho(r) = \rho_L(u)$, $r = \sin\omega$, $r_\varepsilon = \sin\omega_\varepsilon$. Then $\rho(r) = 1/r\ \forall r \in [r_\varepsilon, 1]$.

Let us consider all cases step by step.

1°. Let $i = 2$, $p > n-2$. We choose a long smoothened cylinder L_ε (h is large). As in (5.8), we have

$$c\pi^{1/2} r^{n-3} \varphi_0(s) = \frac{d}{ds}\int_0^s (s^2 - r^2)^{-1/2} r^{n-2} \rho(r)^p\, dr = g_1(s) + g_2(s)$$

where

$$g_1(s) = \frac{d}{ds}\int_0^{r_\varepsilon} (s^2 - r^2)^{-1/2} r^{n-2} \rho(r)^p dr < 0,$$

and

$$\begin{aligned} g_2(s) &= \frac{d}{ds}\int_{r_\varepsilon}^s (s^2 - r^2)^{-1/2} r^{n-2-p} dr \\ &= \frac{d}{ds}\Big[s^{n-2-p}\int_{r_\varepsilon/s}^1 (1-t^2)^{-1/2}\, t^{n-2-p}\, dt\Big]. \end{aligned}$$

An elementary calculation gives

$$g_2(s) = s^{-2} r_\varepsilon^{n-1-p}[a(r_\varepsilon/s) + b(r_\varepsilon/s)],$$

$$a(x) = (n-2-p)x^{p+1-n} \int_x^1 (1-t^2)^{-1/2}\, t^{n-2-p}\, dt, \qquad b(x) = (1-x^2)^{-1/2},$$

$x = r_\varepsilon / s \in [r_\varepsilon, 1]$. Note that r_ε is very small when the height of L_ε is large. Therefore, the statement will be proved if we show that

$$\lim_{x\to+0}[a(x)+b(x)] < 0. \tag{6.4}$$

Indeed, in this case there exists $x_\varepsilon \in (0,1)$ such that $a(x_\varepsilon) + b(x_\varepsilon) < 0$. If we choose ω_ε in (6.2) so that $\sin\omega_\varepsilon = x_\varepsilon$, then $g_2(1) = x_\varepsilon^{n-1-p}[a(x_\varepsilon)+b(x_\varepsilon)] < 0$, and we are done.

Let us check (6.4). If $n-1-p \ge 0$ then $a(x) \to -\infty$, $b(x) \to 1$, and (6.4) follows. In the case $n-1-p<0$ we have

$$\lim_{x\to+0}[a(x)+b(x)] = \frac{n-2-p}{p+1-n} + 1 = \frac{1}{n-1-p} < 0.$$

2°. Let $i=3$, $p > n-3$. Then for $r \in [r_\varepsilon, 1]$, (5.7) yields

$$2c\varphi_0(r) = r^{4-n}(r^{n-3-p})' = (n-3-p)r^{-p} < 0.$$

3°. Let $i=4$. Then $\alpha = \frac{3}{2}$, and $\eta = \frac{n}{2} - 3$. We choose the example L_ε so that $h=1$. By (5.6),

$$\rho(r)^p = cI^{\frac{3}{2}}_{\frac{n}{2}-3}\varphi_0(r),$$

and therefore (5.4) and (5.2) yield

$$\begin{aligned} c\varphi_0(r) &= r^{6-n}\Big(\frac{1}{2r}\frac{d}{dr}\Big)^2 r^{n-2}\big(I^{\frac{1}{2}}_{\frac{n-3}{2}}\rho(\cdot)^p\big)(r) \\ &= \frac{2r^{6-n}}{\sqrt{\pi}}\Big(\frac{1}{2r}\frac{d}{dr}\Big)^2 \int_0^r (r^2-s^2)^{-\frac{1}{2}} s^{n-2}\rho(s)^p ds. \end{aligned}$$

Using integration by parts and differentiation twice, we have

$$c\varphi_0(r) = \frac{r^{6-n}}{2\sqrt{\pi}} \int_0^r (r^2-s^2)^{-\frac{1}{2}} [(s^{n-3}\rho^p)'/s]' ds.$$

Changing variable $s = \sin\theta$ gives

$$c\varphi_0\Big(\frac{1}{\sqrt{2}}\Big) = \frac{2^{\frac{n}{2}-4}}{\sqrt{\pi}} \int_0^{\frac{\pi}{4}} g(\theta)\Big(\frac{1}{2} - \sin^2\theta\Big)^{-\frac{1}{2}} d\theta,$$

where

$$g(\theta) = \frac{d}{d\theta}\Big(\frac{\frac{d}{d\theta}(\rho^p \sin^{n-3}\theta)}{\sin\theta\cos\theta}\Big) = a(\theta) - b(\theta)D(\theta)\sin^{n-4}\theta,$$

$$a(\theta) = (n-3)(n-5)\rho^p \sin^{n-6}\theta\cos\theta + \sin^{n-5}\theta\big(2n-8+\frac{1}{\cos^2\theta}\big)p\rho^{p-1}\frac{d\rho}{d\theta}$$
$$+\frac{\sin^{n-4}\theta}{\cos\theta}p\rho^{p-2}\big(\rho^2+(p+1)\big(\frac{d\rho}{d\theta}\big)^2\big),$$

$$b(\theta) = \frac{p\rho^{p-2}}{\cos\theta}\big(\rho^2+\big(\frac{d\rho}{d\theta}\big)^2\big),$$

$$D(\theta) = \kappa(\theta)\big(\rho^2+\big(\frac{d\rho}{d\theta}\big)^2\big)^{\frac{1}{2}},$$

$$\kappa(\theta) = \frac{\rho^2+2\big(\frac{d\rho}{d\theta}\big)^2-\rho\frac{d^2\rho}{d\theta^2}}{\big[\rho^2+\big(\frac{d\rho}{d\theta}\big)^2\big]^{3/2}}. \tag{6.5}$$

Since ρ and $\frac{d\rho}{d\theta}$ are uniformly bounded for any $0<\varepsilon<\frac{1}{4}$, there are constants b_0 and M so that

$$|a(\theta)| < M, \quad b(\theta) > b_0 > 0$$

for $0\le\theta\le\frac{\pi}{4}$. Let $M_1 = M\int_0^{\frac{\pi}{4}}\big(\frac{1}{2}-\sin^2\theta\big)^{-\frac{1}{2}}d\theta$. Then,

$$c\varphi_0\big(\frac{1}{\sqrt{2}}\big) < M_1 - b_0\int_0^{\frac{\pi}{4}} D(\theta)\frac{\sin^{n-4}\theta}{\big(\frac{1}{2}-\sin^2\theta\big)^{\frac{1}{2}}}d\theta. \tag{6.6}$$

Since the curve C converges to a square as $\varepsilon\to 0$, the integral in (6.6) can be arbitrarily large. Indeed, one can choose $\delta>0$ so that

$$\frac{\sin^{n-4}\theta}{\big(\frac{1}{2}-\sin^2\theta\big)^{\frac{1}{2}}} > \frac{8M_1}{\pi b_0}, \qquad \forall\theta\in\big(\frac{\pi}{4}-\delta,\frac{\pi}{4}\big).$$

Then we choose $\varepsilon>0$ of L_ε such that $\kappa(\theta)=0$ for $0\le\theta\le\frac{\pi}{4}-\delta$. By Lemma 3.4, we obtain

$$\int_0^{\frac{\pi}{4}} D(\theta)\frac{\sin^{n-4}\theta}{\big(\frac{1}{2}-\sin^2\theta\big)^{\frac{1}{2}}}d\theta > \frac{8M_1}{\pi b_0}\int_0^{\frac{\pi}{4}}D(\theta)d\theta = \frac{2M_1}{b_0}.$$

Therefore, by (6.6), $\varphi_0\big(\frac{1}{\sqrt{2}}\big) < -M_1/c < 0$ when ε is small enough.

4°. Let $i\ge 5$. Again, we choose the example L_ε so that $h=1$. By (5.6) and (5.3),

$$\rho(r)^p = cI_\eta^\alpha\varphi_0 = cI_\eta^2 I_{\eta+2}^{\alpha-2}\varphi_0 = I_\eta^2\psi,$$

where $\psi = cI_{\eta+2}^{\alpha-2}\varphi_0$. By (5.4),

$$\psi(r) = r^{i+2-n}\big(\frac{1}{2r}\frac{d}{dr}\big)^2 r^{n-i+2}\rho(r)^p.$$

Let $r = \sin\phi$. Note that $\frac{d\rho}{d\phi}\big|_{\phi=\frac{\pi}{4}} = 0$. An elementary calculation gives

$$\psi\Big(\frac{1}{\sqrt{2}}\Big) = \frac{1}{2}\left((n-i+2)(n-i)\rho^p + p\rho^{p-1}\frac{d^2\rho}{d\phi^2}\right)_{\phi=\frac{\pi}{4}}.$$

Since the curvature κ of the curve C at $\phi = \frac{\pi}{4}$ tends to $+\infty$ as $\varepsilon \to 0$, (6.5) implies that $\frac{d^2\rho}{d\phi^2}$ tends to $-\infty$ as $\varepsilon \to 0$. Therefore, $\psi\big(\frac{1}{\sqrt{2}}\big) < 0$ when ε is small enough. It follows that φ_0 is negative somewhere when ε is sufficiently small. $\square$

Remark 6.2. In the case $i \geq 4$, $p \geq n-i$, the statement of Theorem 6.1 can be proved more easily. Namely, by using notation of Lemma 1.2, from (1.5), (6.1) and (6.2), we have

$$r^{n-i-p} = c_2\int_0^r \left(1-\frac{s^2}{r^2}\right)^{(i-3)/2} s^{n-i-1}\varphi_0(s)\,ds, \qquad \forall r \in [r_\varepsilon, 1], \tag{6.7}$$

where $r = \sin\omega$, $\varphi_0(\sin d(e_n,\xi)) = \varphi(\xi)$. If $\varphi_0 \geq 0$ and $p \geq n-i$ then the left hand side of (6.7) does not increase on $[r_\varepsilon, 1]$, whereas the right hand side is increasing because its derivative is positive. This contradiction shows that φ_0 is negative somewhere.

For $p = n-k$, Theorem 6.1 or Remark 6.2 imply the following

Corollary 6.3. *If $1 \leq k \leq i$ and $4 \leq i < n$, then there exist origin-symmetric convex bodies of revolution K and L in $\mathbb{R}^n$ such that $I_k(K,\xi) \leq I_k(L,\xi)$ for any $\xi \in G_{n,i}$, but*

$$\mathrm{vol}_n(K) > \mathrm{vol}_n(L).$$

If $k = i$ this corollary gives the known negative answer to the generalized Busemann-Petty problem for i-dimensional sections of origin-symmetric convex bodies in $\mathbb{R}^n$ in the case $4 \leq i < n$. This result was established in [BZh] and [K2] by different methods.

The following table is a summary of solutions to Problem A when K is a body of revolution.

Table 1

i	**k**	**l**	**Answer**	
2	1	n	No	Theorem 6.1
2	1	$1 < l < n$	Yes	Corollary 5.4
2	2	$2 < l \leq n$	Yes	Corollary 5.4
3	$1 \leq k \leq 3$	$k < l \leq n+k-3$	Yes	Corollary 5.4
3	$1 \leq k \leq 3$	$n+k-3 < l \leq n$	No	Theorem 6.1
$i \geq 4$	$1 \leq k < n$	$k < l \leq n$	No	Theorem 6.1

Appendix

The following theorem is in [BZh].

Theorem A.1. (see [BZh], Theorem 1.3). *For $3 < i < n$, there exist origin-symmetric convex bodies of revolution K and L in $\mathbb{R}^n$ so that*

$$\mathrm{vol}_i(K \cap \xi) < \mathrm{vol}_i(L \cap \xi), \quad \forall \xi \in G_{n,i},$$

but

$$\mathrm{vol}_n(K) > \mathrm{vol}_n(L).$$

The proof of this theorem used the following

Lemma A.2. (see Lemma 3.2 in [BZh]). *For $3 < i < n$, there exist a C^∞ convex body K and a C^∞ function g so that*

$$\mathrm{R}_i\big(\rho_K^{i-4} g\big) < 0, \qquad \int_{S^{n-1}} \rho_K^{n-4} g > 0.$$

The proof of this lemma given in [BZh] has certain gap. We give a correct proof here.

Proof of Lemma A.2. We shall seek a C^∞ convex body K and a C^∞ function g that are $SO(n-1)$-invariant. One can take K to be a smoothened cylinder L_ε described in the biginning of Section 6. By Lemma 1.3,

$$\rho_K^{n-i} = \mathrm{R}_i^* \varphi, \tag{A.1}$$

and (1.5) yields

$$1 = c_2 \int_0^r \left(1 - \frac{s^2}{r^2}\right)^{(i-3)/2} s^{n-i-1} \varphi_0(s)\, ds, \qquad \forall r \in [r_\varepsilon, 1], \tag{A.2}$$

where $r = \sin\omega$, $\varphi_0(\sin d(e_n, \xi)) = \varphi(\xi)$. If $\varphi_0 \geq 0$ then the right hand side of (A.2) is increasing because its derivative is positive. This contradiction shows that the function $\varphi(\xi)$ in (A.1) is negative somewhere. Then we construct $g_1 \in \mathcal{C}_i$ so that $g_1(\xi) < 0\ \forall \xi \in G_{n,i}$, and $\int_{G_{n,i}} \varphi g_1 > 0$. To this end we take the absolute value of g_1 sufficiently small where $\varphi \geq 0$ and sufficiently large where $\varphi < 0$. By Lemma 1.3, there is $g \in \mathcal{C}_1$ satisfying $g_1 = \mathrm{R}_i(\rho_K^{i-4} g)$. Now by (A.1) and (1.2), we have

$$\int_{S^{n-1}} \rho_K^{n-4} g = \int_{S^{n-1}} \mathrm{R}_i^* \varphi\, \rho_K^{i-4} g = \int_{G_{n,i}} \varphi g_1 > 0,$$

and the lemma is proved. $\square$

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