CONVOLUTION-BACKPROJECTION METHOD FOR THE k-PLANE TRANSFORM, AND CALDERÓN'S IDENTITY FOR RIDGELET TRANSFORMS

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Abstract. We develop a convolution-backprojection method for the k-plane Radon transform $f \to \hat{f}, \quad f \in L^p(\mathbb{R}^n)$. A slight modification of this method gives an explicit inversion formula for $\hat{f}$ in terms of the corresponding wavelet-like transforms (or the k-plane ridgelet transforms), and a generalization of Calderón's reproducing formula.

1. Introduction

Let $\mathcal{G}_{n,k}$ be the manifold of all non-oriented k-dimensional planes τ in $\mathbb{R}^n$, $1 \le k < n$. The k-plane Radon transform of a function $f(x)$ on $\mathbb{R}^n$ is defined by $\hat{f}(\tau) = \int_\tau f(x) d_\tau x$ where $d_\tau x$ denotes the Lebesgue measure on τ. In the past few years an interest to this transform grew up owing to recent developments in approximation theory and computer sciences related to the so-called ridgelet transforms. This direction was stimulated by works of D. Donoho, E. Candès and their collaborators; see, e.g., [Do1, Do2, C1-C3]. Ridgelet transforms are particular cases (corresponding to $k = n-1$) of the scaled intertwining operators

$$(1.1) \qquad (U_t f)(\tau) = t^{k-n} \int\limits_{\mathbb{R}^n} f(x) u\Big(\frac{|x-\tau|}{t}\Big)\, dx,$$

$$(1.2) \qquad (V_t^* \varphi)(x) = t^{k-n} \int\limits_{\mathcal{G}_{n,k}} \varphi(\tau) v\Big(\frac{|x-\tau|}{t}\Big)\, d\tau, \qquad t > 0,$$

generated by appropriate "wavelet functions" $u(\cdot)$ and $v(\cdot)$ depending on the euclidean distance $|x-\tau|$ between the point x and the k-plane

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τ. Here dx and $d\tau$ stand for the corresponding canonical measures. One can call (1.1), (1.2) *the k-plane ridgelet transforms* or wavelet-like transforms associated to the k-plane Radon transform and its dual, respectively. For $k = 0$, integrals (1.1), (1.2) represent ordinary continuous wavelet transforms on $\mathbb{R}^n$ [Da]. The asterisk "*" in V_t^* indicates that the operator acts backwards, from $\mathcal{G}_{n,k}$ to $\mathbb{R}^n$. For $k = n-1$, the functions like $x \to u(|x-\tau|)$ are constant on hyperplanes parallel to τ and called *plane waves* or *ridge functions*.

In the present article we develop a general point of view on the convolution-backprojection method for the k-plane transform. A slight modification of this method will give an explicit inversion formula for $\hat{f}$, $f \in L^p$, in terms of the corresponding wavelet-like transforms, and a generalization of the Calderón reproducing formula [FJW], [R8] to the k-plane ridgelet transforms (1.1), (1.2). Our approach relies on some properties of intertwining operators, covers the maximal range of functions f, and can be carried out without using the Fourier transform technique (the latter leads to inevitable "$L^1 - L^2$" restrictions that may be artificial in some contexts).

Some historical comments are in order. Decomposition of functions on $\mathbb{R}^n$ into plane waves (as a substitute for the Fourier analysis) is widely used in PDE, integral geometry and related problems of approximation theory; see, e.g., [J], [H], [C3]. The relevant wavelet-like transforms were introduced independently at about the same time by E.J. Candès, N. Murata, and B. Rubin. In author's work [R1, R5-R7] transformations like (1.1)-(1.2) (with u and v replaced by finite Borel measures) were used for explicit inversion of the hyperplane Radon transform and the k-plane transform of L^p functions. This investigation was inspired by M. Holschneider [Ho] and author's previous research in fractional calculus; see [R2, Sections 10.7, 12-14, 17, 34]. Wavelet-like transforms (1.1), (1.2) can be used for analytic continuation of intertwining fractional integrals

$$(P^\alpha f)(\tau) = \int\limits_{\mathbb{R}^n} f(x)|x-\tau|^{\alpha-n+k} dx, \qquad \tau \in \mathcal{G}_{n,k},$$

$$(\overset{*}{P}{}^{\alpha}\varphi)(x) = \int\limits_{\mathcal{G}_{n,k}} \varphi(\tau)|x-\tau|^{\alpha-n+k} d\tau, \qquad x \in \mathbb{R}^n,$$

associated to the k-plane transform and its dual, respectively; see [R4, Section 3]. One should also mention the papers by C.A. Berenstein and D. Walnut [BW] and D. Walnut [W] devoted to investigation of Radon transforms by using wavelet transforms. A series of papers related to application of intertwining wavelet-like transforms to various problems of integral geometry on spaces of constant curvature were published by the author in 1998/9. The expository paper [R4] gives a flavor of these results.

The simplest inversion formula for the k-plane transform $\varphi = \hat{f}$ in terms of the corresponding wavelet-like transforms can be written as

$$\int\limits_0^\infty \frac{W_t^* \varphi}{t}\, dt = cf, \quad (W_t^*\varphi)(x) = \frac{1}{t^n} \int\limits_{\mathcal{G}_{n,k}} \varphi(\tau) w\Big(\frac{|x-\tau|}{t}\Big)\, d\tau. \tag{1.3}$$

Deconvolution and re-scaling of $w(\cdot)$ enable us to write (1.3) as

$$\int\limits_0^\infty \frac{V_t^* U_t f}{t^{1+k}}\, dt = cf \tag{1.4}$$

in terms of operators (1.1), (1.2). This equality generalizes the Calderón reproducing formula [FJW], [R8],

$$f = \int\limits_0^\infty \frac{f * u_t * v_t}{t}\, dt, \tag{1.5}$$

$u_t(x) = t^{-n}u(x/t)$, $v_t(x) = t^{-n}v(x/t)$, which can be obtained from (1.4) by setting $k = 0$.

Different ideas originated in computations on neural networks and approximation theory led Murata [M1], [M2], and Candès [C1], [C2], to an analog of (1.4) for $k = n-1$. Their argument is based on the Fourier transforms technique, and assumptions for functions are more restrictive than in [R1, Theorem 3.11] (for $k = n-1$) and [R6, Theorem 3.7] (for any $1 \le k \le n-1$). Operators (1.1), (1.2) in the case $k = n-1$ were written by Candès in a slightly different form and called *continuous ridgelet transform*. Their discrete modifications were also studied in [C1]-[C3]. For arbitrary $1 \le k \le n-1$, D. Donoho introduced wavelet-like tight frames, called *k-plane ridgelets* [Do1]. These can be regarded as discrete versions of (1.1), (1.2) for all such k.

The paper is organized as follows. Section 2 contains some properties of the k-plane transform and the corresponding intertwining operators. Section 3 is devoted to the convolution-backprojection method and justification of (1.3), (1.4). Main results are stated in Theorems 3.1, 3.2, and 3.5. Different account of the convolution-backprojection method for the k-plane transform of L^2 functions was presented in [K].

Notation. In the following $\mathbf{M}(n)$ is the group of isometries of $\mathbb{R}^n$; $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ denotes the area of the unit sphere S^{n-1}

in $\mathbb{R}^n$; $e_1, \dots, e_n$ are coordinate unit vectors;

$$\mathbb{R}^k = \mathbb{R}e_1 + \ldots + \mathbb{R}e_k, \qquad \mathbb{R}^{n-k} = \mathbb{R}e_{k+1} + \ldots + \mathbb{R}e_n.$$

The notation $C = C(\mathbb{R}^n)$ and $L^p = L^p(\mathbb{R}^n)$ for function spaces is standard; $C_0 = \{f \in C : \lim\limits_{|x|\to\infty} f(x) = 0\}$. The Riemann-Liouville (or Abel) fractional integral of a function g on $\mathbb{R}_+ = (0, \infty)$ is defined by

$$(I^\alpha_+ g)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{g(r)}{(t-r)^{1-\alpha}} \, dr, \qquad Re\,\alpha > 0. \tag{1.6}$$

2. Some properties of k-plane transforms

We start with basic definitions. Let $\mathcal{G}_{n,k}$ and $G_{n,k}$ be the *affine* Grassmann manifold of all non-oriented k-planes τ in $\mathbb{R}^n$, and the ordinary Grassmann manifold of k-dimensional subspaces ζ of $\mathbb{R}^n$, respectively. The group $\mathbf{M}(n)$ of isometries of $\mathbb{R}^n$ acts on $\mathcal{G}_{n,k}$ transitively. Given $\zeta \in G_{n,k}$, each vector $x \in \mathbb{R}^n$ can be written as $x = (x', x'') = x' + x''$ where $x' \in \zeta$ and $x'' \in \zeta^\perp$, $\zeta^\perp$ being the orthogonal complement to ζ in $\mathbb{R}^n$. Each k-plane τ is parameterized by the pair (ζ, x'') where $\zeta \in G_{n,k}$ and $x'' \in \zeta^\perp$. The manifold $\mathcal{G}_{n,k}$ will be endowed with the product measure $d\tau = d\zeta dx''$, where $d\zeta$ is the $SO(n)$-invariant measure on $G_{n,k}$ of total mass 1, and dx'' denotes the usual volume element on $\zeta^\perp$.

The k-plane transform $\hat{f}(\tau)$ of a function $f(x)$ and the dual k-plane transform $\check{\varphi}(x)$ of a function $\varphi(\tau) \equiv \varphi(\zeta, x'')$ are defined by

$$\hat{f}(\tau) = \int\limits_\zeta f(x' + x'') dx', \qquad \tau = (\zeta, x'') \in \mathcal{G}_{n,k}; \tag{2.1}$$

$$\check{\varphi}(x) = \int\limits_{SO(n)} \varphi(x + \gamma\zeta_0) d\gamma = \int\limits_{G_{n,k}} \varphi(\zeta, \mathrm{Pr}_{\zeta^\perp} x) d\zeta, \quad x \in \mathbb{R}^n. \tag{2.2}$$

Here $\mathrm{Pr}_{\zeta^\perp} x$ denotes the orthogonal projection of x onto $\zeta^\perp$, ζ_0 is an arbitrary fixed k-plane through the origin. We denote

$$(f_1, f_2) = \int\limits_{\mathbb{R}^n} f_1(x) f_2(x) dx, \qquad (\varphi_1, \varphi_2)^\sim = \int\limits_{\mathcal{G}_{n,k}} \varphi_1(\tau)\varphi_2(\tau) d\tau. \tag{2.3}$$

An important duality relation for (2.1) and (2.2) reads

$$(\hat{f}, \varphi)^\sim = (f, \check{\varphi}) \tag{2.4}$$

provided that either side is finite for f and φ replaced by $|f|$ and $|\varphi|$, respectively [H, So]. A simple proof of (2.4) is as follows:

$$\begin{aligned}(\hat{f}, \varphi)^{\sim} &= \int\limits_{G_{n,k}} d\zeta \int\limits_{\zeta^{\perp}} \varphi(\zeta, x'')dx'' \int\limits_{\zeta} f(x'+x'')dx' \\ &= \int\limits_{G_{n,k}} d\zeta \int\limits_{\mathbb{R}^n} \varphi(\zeta, \mathrm{Pr}_{\zeta^{\perp}} x) f(x)dx = (f, \check{\varphi}).\end{aligned}$$

Lemma 2.1. *For $x \in \mathbb{R}^n$ and $\tau \equiv (\zeta, x'') \in \mathcal{G}_{n,k}$, let*

$$r = |x| = \mathrm{dist}(o, x), \qquad s = |x''| = \mathrm{dist}(o, \tau) = |\tau| \tag{2.5}$$

denote the corresponding distances from the origin. If $f(x)$ and $\varphi(\tau)$ are radial, i.e. $f(x) \equiv f_0(r)$ and $\varphi(\tau) \equiv \varphi_0(s)$, then $\hat{f}(\tau)$ and $\check{\varphi}(x)$ are represented by Abel type integrals

$$\hat{f}(\tau) = \sigma_{k-1} \int\limits_s^{\infty} f_0(r)(r^2 - s^2)^{k/2-1} r dr, \tag{2.6}$$

$$\check{\varphi}(x) = \frac{\sigma_{k-1}\, \sigma_{n-k-1}}{\sigma_{n-1}} \int\limits_0^r \varphi_0(s)(r^2 - s^2)^{k/2-1} s^{n-k-1} ds, \tag{2.7}$$

provided the integrals $\int_s^{\infty}(...)$ and $\int_0^r(...)$ are absolutely convergent.

Proof. In bi-polar coordinates $x = t\omega + s\theta$; $t, s \ge 0$; $\omega \in \zeta \cap S^{n-1}$, $\theta \in \zeta^{\perp} \cap S^{n-1}$, (2.1) reads

$$\hat{f}(\tau) = \int\limits_0^{\infty} t^{k-1} dt \int\limits_{\zeta \cap S^{n-1}} f_0(|t\omega + s\theta|) d\omega = \sigma_{k-1} \int\limits_0^{\infty} t^{k-1} f_0(\sqrt{t^2 + s^2}) dt.$$

This gives (2.6). Furthermore,

$$\begin{aligned}\check{\varphi}(x) &= \int\limits_{G_{n,k}} \varphi_0(|\mathrm{Pr}_{\zeta^{\perp}} x|) d\zeta = \int\limits_{SO(n)} \varphi_0(|\mathrm{Pr}_{\gamma \mathbb{R}^{n-k}} x|) d\gamma \\ &= \frac{1}{\sigma_{n-1}} \int\limits_{S^{n-1}} \varphi_0(|\mathrm{Pr}_{\mathbb{R}^{n-k}} r\sigma|) d\sigma, \quad r = |x|.\end{aligned}$$

By passing to bi-spherical coordinates $\sigma = a \cos\psi + b \sin\psi$,

$$a \in S^{k-1} \subset \mathbb{R}^k, \qquad b \in S^{n-k-1} \subset \mathbb{R}^{n-k}, \qquad 0 < \psi < \pi/2,$$

$d\sigma = \sin^{n-k-1}\psi\cos^{k-1}\psi\, d\psi\, da\, db$ [VK, pp. 12, 22], we obtain

$$\check{\varphi}(x) = \frac{\sigma_{k-1}\,\sigma_{n-k-1}}{\sigma_{n-1}} \int\limits_0^{\pi/2} \varphi_0(r\sin\psi)\, \sin^{n-k-1}\psi\cos^{k-1}\psi\, d\psi.$$

This coincides with (2.7). □

Lemma 2.2. *If $f \in L^p$ then $\hat{f}(\tau)$ is finite a.e. provided $1 \le p < n/k$. If $p \ge n/k$ and $f(x) = (2+|x|)^{-n/p}(\log(2+|x|))^{-1}$ $(\in L^p)$ then $\hat{f}(\tau) \equiv \infty$.*

This result is due Solmon [So]. Note that the second statement follows immediately from (2.6).

Lemma 2.3. *For measurable functions $a(\cdot)$ and $b(\cdot)$ on $\mathbb{R}_+$, the following relations hold:*

$$\text{(i)} \quad \int\limits_{\mathcal{G}_{n,k}} \hat{f}(\tau) a(|x-\tau|) d\tau = \int\limits_{\mathbb{R}^n} f(y)\psi(|x-y|)dy, \tag{2.8}$$

$$\psi(r) = c r^{2-n} \int\limits_0^r s^{n-k-1} a(s)(r^2-s^2)^{k/2-1}\, ds, \qquad c = \frac{\sigma_{k-1}\,\sigma_{n-k-1}}{\sigma_{n-1}}. \tag{2.9}$$

$$\text{(ii)} \quad \int\limits_{\mathbb{R}^n} \check{\varphi}(y) b(|x-y|) dy = \int\limits_{\mathcal{G}_{n,k}} \varphi(\tau) h(|x-\tau|) d\tau, \tag{2.10}$$

$$h(s) = \sigma_{k-1} \int\limits_s^\infty b(r)(r^2-s^2)^{k/2-1} r dr, \tag{2.11}$$

(we assume that integrals in these formulae exist in the Lebesgue sense).

Proof. We fix $x \in \mathbb{R}^n$, and introduce functions $f_x(y) = f(y+x)$, $\varphi_x(\tau) = \varphi(\tau+x)$ of shifted argument. Let $a_1(\tau) = a(|\tau|)$, $b_1(y) = b(|y|)$. Owing to $\mathbf{M}(n)$-invariance, the left hand side of (2.8) reads

$$\int\limits_{\mathcal{G}_{n,k}} \hat{f}_x(\tau) a_1(\tau) d\tau \overset{(2.4)}{=} \int\limits_{\mathbb{R}^n} f_x(y)\check{a}_1(y)dy = \int\limits_{\mathbb{R}^n} f(x+y)\check{a}_1(y)\, dy.$$

By (2.7), $\check{a}_1(y) = \psi(|y|)$. Similarly,

$$\int\limits_{\mathbb{R}^n} \check{\varphi}_x(y) b_1(y) dy = \int\limits_{\mathcal{G}_{n,k}} \varphi_x(\tau)\hat{b}_1(\tau) d\tau \overset{(2.6)}{=} \int\limits_{\mathcal{G}_{n,k}} \varphi(x+\tau) h(|\tau|) d\tau.$$

□

3. The convolution-backprojection method and ridgelet transforms

For $t > 0$, let

$$(W_t^* \varphi)(x) = \frac{1}{t^n} \int\limits_{\mathcal{G}_{n,k}} \varphi(\tau) w\Big(\frac{|x-\tau|}{t}\Big)\, d\tau. \tag{3.1}$$

By Lemma 2.3(i),

$$(W_t^* \hat{f})(x) = \frac{1}{t^n} \int\limits_{\mathbb{R}^n} f(y) \psi\Big(\frac{|x-y|}{t}\Big)\, dy, \tag{3.2}$$

$$\psi(r) = c r^{2-n} \int\limits_0^r s^{n-k-1} w(s) (r^2 - s^2)^{k/2-1}\, ds, \qquad c = \frac{\sigma_{k-1}\, \sigma_{n-k-1}}{\sigma_{n-1}}. \tag{3.3}$$

Regarding (3.2) as an approximate identity [St, Chapter VIII], we obtain the following

Theorem 3.1. *Let $\psi(|x|)$ have an integrable (on $\mathbb{R}^n$) decreasing majorant, and let $w(s)$ obey (3.3). If $f \in L^p$, $1 \le p < n/k$, then*

$$\lim_{t \to 0} (W_t^* \hat{f})(x) = c f(x), \qquad c = \int\limits_{\mathbb{R}^n} \psi(|x|) dx, \tag{3.4}$$

in the L^p-norm and for almost all x. If $f \in C_0 \cap L^p$ for some $1 \le p < n/k$, then (3.4) holds uniformly on $\mathbb{R}^n$.

This theorem is a core of the convolution-backprojection method. The function ψ, realizing approximation to the identity, is in our disposal. The generating function w is determined by ψ as a solution of the Abel type integral equation (3.3).

Instead of passing to the limit in (3.4) as $t \to 0$, one can integrate in t from 0 to ∞ with respect to invariant measure dt/t. This formalism leads to the following

Theorem 3.2. *Let $f \in L^p$, $1 \le p < n/k$. If $\lambda(s)$ is a measurable function on $\mathbb{R}_+$ having an integrable decreasing majorant, and the function $\tilde{w}(s) = s^{(n-k)/2-1} w(\sqrt{s})$ satisfies the Abel integral equation*

$$(I_{0+}^{k/2+1} \tilde{w})(s^2) = s \lambda(s) \tag{3.5}$$

(see (1.6)), then

$$\int\limits_0^\infty (W_t^* \hat{f})(x) \frac{dt}{t} = c f(x), \qquad c = \frac{\pi^{n/2}}{\Gamma((n-k)/2)} \int\limits_0^\infty \lambda(s)\, ds, \tag{3.6}$$

where $\int_0^\infty (...)dt/t = \lim_{\varepsilon\to 0}\int_\varepsilon^\infty (...)dt/t$ *in the* L^p*-norm and in the a.e. sense. If* $f \in C_0 \cap L^p$*, this limit is uniform on* $\mathbb{R}^n$*.*

Proof. Owing to (3.2) and (3.3), we obtain

$$\int_\varepsilon^\infty (W_t^* \hat{f})(x)\frac{dt}{t} = \int_\varepsilon^\infty \frac{dt}{t^{1+n}} \int_{\mathbb{R}^n} f(y)\psi\Big(\frac{|x-y|}{t}\Big)\,dy$$

$$= c_1 \int_{\mathbb{R}^n} f(y)\tilde{\psi}_\varepsilon(x-y)\,dy, \qquad c_1 = \frac{\pi^{k/2}\sigma_{n-k-1}}{2\sigma_{n-1}}, \tag{3.7}$$

where

$$\tilde{\psi}_\varepsilon(x) = \int_\varepsilon^\infty \psi\Big(\frac{|x|}{t}\Big)\frac{dt}{t^{1+n}} = \frac{1}{\varepsilon^n}\tilde{\psi}\Big(\frac{x}{\varepsilon}\Big),$$

$\tilde{\psi}(x) = |x|^{-n}(I_{0+}^{k/2+1}\tilde{w})(|x|^2) = |x|^{1-n}\lambda(|x|)$ (we leave simple computations to the reader). If $\lambda(s)$ has an integrable decreasing majorant, the convolution (3.7) is an approximate identity, and we are done. □

Theorem 3.2 shows that assumptions for the wavelet function w are closely connected with following question: for which functions g on $\mathbb{R}_+$ does the Riemann-Liouville fractional integral $I_+^\alpha g$ (see (1.6)) belong to $L^1(\mathbb{R}_+)$? It is possible only if g enjoys some oscillation. This question was studied in [R2] in detail.

One can call $(W_t^*\varphi)(x)$ *the wavelet transform of* φ *associated to the dual* k*-plane transform and generated by the wavelet function* w. It can be shown [RR] that any function w satisfying

$$\int_0^\infty s^{j+n-k-1}w(s)ds = 0 \quad \forall j = 0, 2, 4, \dots, 2[k/2], \tag{3.8}$$

($[k/2]$ is the integral part of $k/2$), and

$$\int_1^\infty s^{\beta+n-k-1}|w(s)|ds < \infty \qquad \textit{for some } \beta > k, \tag{3.9}$$

is admissible in Theorem 3.2. If w obeys (3.8) and (3.9) then, by Lemma 2.4 from [R3], the constant c in (3.6) has the form

(3.10)

$$c = \frac{2\pi^{n/2}}{\Gamma((n-k)/2)} \begin{cases} \Gamma(-k/2)\displaystyle\int_0^\infty s^{n-1}w(s)ds & \text{for } k \text{ odd},\\[2ex] \dfrac{2(-1)^{1+k/2}}{(k/2)!}\displaystyle\int_0^\infty s^{n-1}w(s)\log s\,ds & \text{for } k \text{ even}. \end{cases}$$

Thus we have the following

Corollary 3.3. *Let $f \in L^p$, $1 \le p < n/k$. If the function w satisfies (3.8) and (3.9), then the inversion formula (3.6) holds with the constant c defined by (3.10).*

Examples of wavelet functions can be found in [R9, Example 5.4].

Let us pass to the ridgelet transforms. The Calderón reproducing formula (1.5) follows from the integral representation

$$f = \int_0^\infty \frac{f * w_t}{t}\, dt, \tag{3.11}$$

if we set $w = u * v$. Admissible pairs of functions u and v in (1.5) are determined by their convolution $w = u * v$. The latter must satisfy certain cancellation conditions [FJW], [R2, Section 12]. The same idea of deconvolution goes through smoothly if we start with the formula (3.6) as a substitute for (3.11). We recall the notation

$$(U_t f)(\tau) = t^{k-n} \int_{\mathbb{R}^n} f(x) u\Big(\frac{|x-\tau|}{t}\Big)\, dx, \tag{3.12}$$

$$(V_t^* \varphi)(x) = t^{k-n} \int_{\mathcal{G}_{n,k}} \varphi(\tau) v\Big(\frac{|x-\tau|}{t}\Big)\, d\tau, \qquad t > 0. \tag{3.13}$$

Note that for $\tau = (\zeta, x'')$, $x'' \in \zeta^\perp$,

$$(U_t f)(\tau) = t^{k-n} \int_{\zeta^\perp} \hat{f}(\zeta, y'')\, u\Big(\frac{|x''-y''|}{t}\Big)\, dy'', \tag{3.14}$$

and therefore, for $f \in L^p$, $p \ge 1$, the integral (3.12) can be well defined only if $p < n/k$ (see Lemma 2.2). The integral (3.14) represents the ordinary continuous wavelet transform of the function $y'' \to \hat{f}(\zeta, y'')$ on the $(n-k)$-plane $\zeta^\perp$.

Definition 3.4. A pair of functions $u(s)$ and $v(s)$, $s \in \mathbb{R}_+$, is called admissible if the function $w(s)$ defined by

$$w(|y|) = \int_{\mathbb{R}^{n-k}} u(|z|)\, v(|y-z|) dz, \qquad y \in \mathbb{R}^{n-k}, \tag{3.15}$$

satisfies cancellation conditions (3.8), (3.9).

Theorem 3.5. *Let $f \in L^p$, $1 \le p < n/k$. If the pair (u,v) is admissible, and c is a constant (3.10), then*

$$\int\limits_0^\infty (V_t U_t f)(x) \frac{dt}{t^{1+k}} = cf(x) \tag{3.16}$$

where $\int_0^\infty (...)dt/t = \lim\limits_{\varepsilon \to 0} \int_\varepsilon^\infty (...)dt/t$ in the L^p-norm and in the a.e. sense. If $f \in C_0 \cap L^p$ this limit is uniform on $\mathbb{R}^n$.

Proof. Let $\hat f(\tau) \equiv \hat f(\zeta, y)$, $\zeta \in G_{n,k}$, $y \in \zeta^\perp$. Abusing notation, for $u(y) \equiv u(|y|)$ and $v(y) \equiv v(|y|)$, we set $u_t(y) = t^{k-n}u(y/t)$, $v_t(y) = t^{k-n}v(y/t)$. By (3.15),

$$t^{k-n}w(y/t) = \int\limits_{\zeta^\perp} v_t(z)u_t(y-z)dz. \tag{3.17}$$

Consider the wavelet-like transform (3.1) and show that

$$W_t^* \hat f = V_t U_t f. \tag{3.18}$$

Denoting $f_x(\cdot) = f(\cdot + x)$, by $\mathbf{M}(n)$-invariance we have

$$\begin{aligned}(W_t^*\hat f)(x) &= \int\limits_{G_{n,k}} d\zeta \int\limits_{\zeta^\perp} \hat f_x(\zeta, y)dy \int\limits_{\zeta^\perp} v_t(z)u_t(y-z)dz \\ &= \int\limits_{G_{n,k}} d\zeta \int\limits_{\zeta^\perp} v_t(z)dz \int\limits_{\zeta^\perp} u_t(y-z)\hat f_x(\zeta, y)dy. \end{aligned} \tag{3.19}$$

The last integral in (3.19) reads

$$\int\limits_{\zeta^\perp} u_t(y-z)\hat f(\zeta, y + x_{\zeta^\perp})dy = \int\limits_{\zeta^\perp} \hat f(\zeta, y)u_t(y - z - x_{\zeta^\perp})dy,$$

where $x_{\zeta^\perp} = \mathrm{Pr}_{\zeta^\perp} x$. Replacing $z + x_{\zeta^\perp} = z_1$, we get

$$\begin{aligned}(W_t^*\hat f)(x) &= \int\limits_{G_{n,k}} d\zeta \int\limits_{\zeta^\perp} v_t(z_1 - x_{\zeta^\perp})dz_1 \int\limits_{\zeta^\perp} \hat f(\zeta, y)u_t(y - z_1)dy \\ &= \int\limits_{\mathcal{G}_{n,k}} v\Big(\frac{|x-\tau|}{t}\Big)\frac{d\tau}{t^{n-k}} \int\limits_{\mathbb{R}^n} f(x_0)u\Big(\frac{|x_0-\tau|}{t}\Big)\frac{dx_0}{t^{n-k}},\end{aligned}$$

which means (3.18). It remains to apply Corollary 3.3. □

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