RECONSTRUCTION OF FUNCTIONS FROM THEIR INTEGRALS OVER k-PLANES

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Abstract. The k-plane Radon transform assigns to a function $f(x)$ on $\mathbb{R}^n$ the collection of integrals $\hat{f}(\tau) = \int_\tau f$ over all k-dimensional planes τ. We give a systematic treatment of two inversion methods for this transform, namely, the method of Riesz potentials, and the method of spherical means. We develop new analytic tools which allow to invert $\hat{f}(\tau)$ under minimal assumptions for f. We assume that $f \in L^p$, $1 \le p < n/k$, or f is continuous with the minimal rate of decay at infinity. In the framework of the first method, our approach employs intertwining fractional integrals associated to the k-plane transform. Following the second method, we extend the original formula of Radon for continuous functions on $\mathbb{R}^2$ to $f \in L^p(\mathbb{R}^n)$ and all $1 \le k < n$. New integral formulae and estimates, generalizing those of Fuglede and Solmon, are obtained.

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1. Introduction

Let $\mathcal{G}_{n,k}$ be the manifold of all non-oriented k-dimensional planes τ in $\mathbb{R}^n$, $1 \le k < n$. The k-plane Radon transform of a function $f(x)$ on $\mathbb{R}^n$ is defined by $\hat{f}(\tau) = \int_\tau f(x) d_\tau x$ where $d_\tau x$ denotes the Lebesgue measure on τ. The present article is motivated by the following.

1^0. On a formal level traditional inversion formulae for $\hat{f}$ read

$$f = c_1(-\Delta_x)^{k/2}(\hat{f})^\vee, \qquad f = c_2((-\Delta_\tau)^{k/2}\hat{f})^\vee, \tag{1.1}$$

where Δ_x and Δ_τ denote the corresponding Laplace operators, and "$\vee$" designates the dual k-plane transform. The first formula was presented in [H2, p. 29] under the following assumptions

$$(a)\ f \in C^\infty(\mathbb{R}^n); \qquad (b)\ f(x) = O(|x|^{-a}) \quad \text{for some} \quad a > n.$$

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The second formula can be found in [H2, p. 18] (for $k = n-1$ and f belonging to the Schwartz space $S(\mathbb{R}^n)$), in [SSW, p. 1260] (for $k = n-1$, $f \in L^2(\mathbb{R}^n)$), and in [K, p. 287] (for $1 \le k \le n-1$ without rigorous justification). On the other hand, $\hat f(\tau)$ is well defined under much weaker assumptions. Namely, it exists *for all* τ if $f(x)$ is continuous and $O(|x|^{-a})$, $a > k$. Moreover, $\hat f(\tau)$ exists *for almost all* τ if $f \in L^p(\mathbb{R}^n)$, $1 \le p < n/k$. The restrictions $a > k$ and $p < n/k$ are minimal in the framework of the corresponding function spaces [So]; see also [Str]. Our aim is to study applicability of (1.1) under these mild assumptions. Some results in this direction were obtained by S.R. Jensen [J]. She studied applicability of the first formula in (1.1) to sufficiently smooth functions f by interpreting $(-\Delta_x)^{k/2}$ as analytic continuation of the corresponding Riesz potential (1.5).

2^0. In 1917 J. Radon [R] employed invariance of the hyperplane transform (the case $k = n-1$) under isometries of $\mathbb{R}^n$ and reduced the inversion problem for $\hat f$ to the one-dimensional Abel integral equation. The key idea is to average $\hat f(\tau)$ over all τ at distance $r > 0$ from x, and then apply the Riemann-Liouville fractional derivative in the r-variable. This gives the spherical mean of f which tends to f as $r \to 0$. The same idea was applied by S. Helgason to k-dimensional totally geodesic Radon transforms of compactly supported C^∞ functions on the unit sphere S^n and the hyperbolic space $\mathbb{H}^n$ [H1, H2]. B. Rubin [Ru3, Ru4] extended these results to continuous and L^p functions without any support restrictions. The celebrated Radon's formula for continuous functions on $\mathbb{R}^2$ reads

$$f(x) = -\frac{1}{\pi}\int\limits_0^\infty \frac{dF_x(r)}{r} = \frac{1}{\pi}\lim_{\varepsilon\to 0}\Big(\frac{F_x(\varepsilon)}{\varepsilon} - \int\limits_\varepsilon^\infty \frac{F_x(r)}{r^2}\,dr\Big), \tag{1.2}$$

(see [R, Proposition III]) where $F_x(r)$ is the average of $\hat f$ over all lines at distance r from x. We could not find in the literature any generalization of (1.2) to all $1 \le k < n$ and non-smooth f. The desired generalization is obtained in the present paper.

The plan of the paper and main results are as follows. Section 2 is of preliminary character. We derive new integral formulae, generalize some estimates of Solmon [So], and introduce important mean value operators. In Section 3 we introduce analytic families of intertwining fractional integrals $(P^\alpha f)(\tau)$, $(\overset{*}{P}{}^{\alpha}\varphi)(x)$, including (for $\alpha = 0$) the k-plane transform and its dual, respectively (see (3.4), (3.2)). For $k = n-1$, these families were introduced by Semyanisty [Se]. Similar families associated to totally geodesic Radon transforms on S^n and $\mathbb{H}^n$

were introduced in [Ru4, Ru5]. The main result of Section 3 is the following equality

$$\overset{*}{P}{}^{\alpha} P^{\beta} f = c_{k,n} I^{\alpha+\beta+k} f, \quad \text{(the Riesz potential of } f\text{)}, \tag{1.3}$$

which generalizes the well known formula of Fuglede $(\hat{f})^{\vee} = c_{k,n} I^k f$; see [F], [H2, p. 29]. Section 4 contains a series of inversion formulae related to (1.1) under minimal assumptions for f. The structure of these formulae is determined by (1.3).

Section 5 is devoted to the method of spherical means. Main results are stated in Theorem 5.4 and Corollaries 5.3, 5.6. In particular, for $k = 1$ (the X-ray case), we obtain the inversion formula

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \frac{\check{\varphi}(x) - \check{\varphi}_r(x)}{r^2} dr, \qquad \varphi = \hat{f}, \tag{1.4}$$

where $\check{\varphi}(x)$ (the dual k-plane transform of φ) is the integral of $\varphi(\tau)$ over all k-planes τ meeting x, and $\check{\varphi}_r(x)$ denotes the mean value of $\varphi(\tau)$ over all k-planes at distance r from x. The expression (1.4) can be written as a limit

$$\frac{1}{\pi} \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\infty} \frac{\check{\varphi}(x) - \check{\varphi}_r(x)}{r^2}\, dr = \frac{1}{\pi} \lim_{\varepsilon \to 0} \Big(\frac{\check{\varphi}(x)}{\varepsilon} - \int_{\varepsilon}^{\infty} \frac{\check{\varphi}_r(x)}{r^2}\, dr \Big)$$

in appropriate sense, and coincides with (1.2) because

$$\lim_{\varepsilon \to 0} \frac{\check{\varphi}_{\varepsilon}(x) - \check{\varphi}(x)}{\varepsilon} = \frac{\partial}{\partial r} \check{\varphi}_r(x) \Big|_{r=0} = 0.$$

We see that Radon's formula (1.2) remains unchanged for all n provided $k = 1$. Theorem 5.4 generalizes (1.2) and (1.4) to all $1 \le k < n$.

In the present paper we do not touch such important questions as the range characterization, support theorems, the Fourier transform approach, the convolution-backprojection method, and other related topics. More information and further references can be found in [H2]; see also papers by E.E. Petrov and a recent preprint [Ru6].

Notation. In the following $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere S^{n-1} in $\mathbb{R}^n$; $e_1, \ldots, e_n$ are coordinate unit vectors;

$$\mathbb{R}^k = \mathbb{R}e_1 + \ldots + \mathbb{R}e_k, \qquad \mathbb{R}^{n-k} = \mathbb{R}e_{k+1} + \ldots + \mathbb{R}e_n.$$

For the sake of convenience, we denote by $|x - \tau|$ the euclidean distance between the point $x \in \mathbb{R}^n$ and the k-plane τ. This notation is not confusing, and agrees with the usual definition $|x - y|$ for $x, y \in \mathbb{R}^n$.

The notation C, C^m, C^∞, L^p for spaces of functions on $\mathbb{R}^n$ is standard; $C_0 = \{f \in C(\mathbb{R}^n) : \lim_{|x|\to\infty} f(x) = 0\}$. $\Phi = \Phi(\mathbb{R}^n)$ is the Semyanisty-Lizorkin space of rapidly decreasing C^∞-functions which are orthogonal to all polynomials (see [Se], [SKM]). The Riesz potential $I^\alpha f$ on $\mathbb{R}^n$ is defined by

$$(1.5)\qquad (I^\alpha f)(x) = \frac{1}{\gamma_n(\alpha)} \int\limits_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \quad \gamma_{n,\alpha} = \frac{2^\alpha \pi^{n/2}\Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)},$$

$Re\,\alpha > 0$, $\alpha - n \neq 0, 2, 4, \ldots$. The operator I^α is an automorphism of Φ, and $F[I^\alpha f](x) = |x|^{-\alpha} F[f](x)$ for $f \in \Phi$ in the Fourier terms. The last relation extends $I^\alpha f$ to all $\alpha \in \mathbb{C}$ as an entire function of α. For α real and $f \in L^p$, the integral $I^\alpha f$ exists a.e. if and only if $1 \le p < n/\alpha$, and $||I^\alpha f||_q \le c\,||f||_p$ for $1 < p < q = np(n-\alpha p)^{-1}$ [St]. The Riemann-Liouville fractional integrals are defined by

$$(1.6)\quad (I^\alpha_+ u)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t \frac{u(r)}{(t-r)^{1-\alpha}} dr, \quad (I^\alpha_- u)(t) = \frac{1}{\Gamma(\alpha)} \int\limits_t^\infty \frac{u(r)}{(r-t)^{1-\alpha}} dr,$$

$Re\alpha > 0$. More information about Riesz potentials and fractional integrals can be found in [Ru1], [SKM]. The letter c stands for a constant that can be different at each occurrence. Given a real-valued expression A, we set $(A)^\lambda_+ = A^\lambda$ if $A > 0$ and 0 if $A \le 0$.

2. Some properties of k-plane transforms

We recall basic definitions. Let $\mathcal{G}_{n,k}$ and $G_{n,k}$ be the *affine* Grassmann manifold of all non-oriented k-planes τ in $\mathbb{R}^n$, and the ordinary Grassmann manifold of k-dimensional subspaces ζ of $\mathbb{R}^n$, respectively. Each subspace $\zeta \in G_{n,k}$ represents a k-plane passing through the origin. The group $\mathbf{M}(n)$ of isometries of $\mathbb{R}^n$ acts on $\mathcal{G}_{n,k}$ transitively. Each k-plane τ is parameterized by the pair (ζ, u) where $\zeta \in G_{n,k}$ and $u \in \zeta^\perp$ (the orthogonal complement to ζ in $\mathbb{R}^n$). The manifold $\mathcal{G}_{n,k}$ will be endowed with the product measure $d\tau = d\zeta du$, where $d\zeta$ is the $SO(n)$-invariant measure on $G_{n,k}$ of total mass 1, and du denotes the usual volume element on $\zeta^\perp$.

The k-plane transform $\hat{f}(\tau)$ of a function $f(x)$ and the dual k-plane transform $\check{\varphi}(x)$ of a function $\varphi(\tau) \equiv \varphi(\zeta, u)$ are defined by

$$\hat{f}(\tau) = \int\limits_{\zeta} f(u+v)dv, \qquad \tau = (\zeta, u) \in \mathcal{G}_{n,k}; \tag{2.1}$$

$$\check{\varphi}(x) = \int\limits_{SO(n)} \varphi(\gamma\zeta_0 + x)d\gamma = \int\limits_{G_{n,k}} \varphi(\zeta, \mathrm{Pr}_{\zeta^\perp}x)d\zeta, \quad x \in \mathbb{R}^n. \tag{2.2}$$

Here $\mathrm{Pr}_{\zeta^\perp}x$ denotes the orthogonal projection of x onto $\zeta^\perp$, ζ_0 is an arbitrary fixed k-plane through the origin. We denote

$$(f_1, f_2) = \int\limits_{\mathbb{R}^n} f_1(x)f_2(x)dx, \qquad (\varphi_1, \varphi_2)^\sim = \int\limits_{\mathcal{G}_{n,k}} \varphi_1(\tau)\varphi_2(\tau)d\tau. \tag{2.3}$$

An important duality relation for (2.1) and (2.2) reads

$$(\hat{f}, \varphi)^\sim = (f, \check{\varphi}) \tag{2.4}$$

provided that either side is finite for f and φ replaced by $|f|$ and $|\varphi|$, respectively [H2, So].

Lemma 2.1. *For $x \in \mathbb{R}^n$ and $\tau \equiv (\zeta, u) \in \mathcal{G}_{n,k}$, let*

$$r = |x| = \mathrm{dist}(o, x), \qquad s = |u| = \mathrm{dist}(o, \tau) = |\tau| \tag{2.5}$$

denote the corresponding distances from the origin. If $f(x)$ and $\varphi(\tau)$ are radial, i.e. $f(x) \equiv f_0(r)$ and $\varphi(\tau) \equiv \varphi_0(s)$, then $\hat{f}(\tau)$ and $\check{\varphi}(x)$ are represented by Abel type integrals

$$\hat{f}(\tau) = \sigma_{k-1} \int\limits_s^\infty f_0(r)(r^2 - s^2)^{k/2-1} r dr, \tag{2.6}$$

$$\check{\varphi}(x) = \frac{\sigma_{k-1}\, \sigma_{n-k-1}}{\sigma_{n-1}\, r^{n-2}} \int\limits_0^r \varphi_0(s)(r^2 - s^2)^{k/2-1} s^{n-k-1} ds, \tag{2.7}$$

provided that these integrals exist in the Lebesgue sense.

Proof. We set $x = t\omega + s\theta$; $t, s \geq 0$; $\omega \in \zeta \cap S^{n-1}$, $\theta \in \zeta^\perp \cap S^{n-1}$. Then (2.1) reads

$$\hat{f}(\tau) = \int\limits_0^\infty t^{k-1}dt \int\limits_{\zeta \cap S^{n-1}} f_0(|t\omega + s\theta|)d\omega = \sigma_{k-1} \int\limits_0^\infty t^{k-1} f_0(\sqrt{t^2 + s^2})dt.$$

This gives (2.6). Furthermore,

$$\begin{aligned}\check{\varphi}(x) &= \int\limits_{G_{n,k}} \varphi_0(|\mathrm{Pr}_{\zeta^\perp} x|)d\zeta = \int\limits_{SO(n)} \varphi_0(|\mathrm{Pr}_{\gamma\mathbb{R}^{n-k}} x|)d\gamma \\ &= \frac{1}{\sigma_{n-1}} \int\limits_{S^{n-1}} \varphi_0(|\mathrm{Pr}_{\mathbb{R}^{n-k}} r\sigma|)d\sigma, \quad r = |x|.\end{aligned}$$

By passing to bi-spherical coordinates $\sigma = a\cos\psi + b\sin\psi$,

$$a \in S^{k-1} \subset \mathbb{R}^k, \qquad b \in S^{n-k-1} \subset \mathbb{R}^{n-k}, \qquad 0 < \psi < \pi/2,$$

$d\sigma = \sin^{n-k-1}\psi \cos^{k-1}\psi \, d\psi \, da \, db$ [VK, pp. 12, 22], we obtain

$$\check{\varphi}(x) = \frac{\sigma_{k-1}\,\sigma_{n-k-1}}{\sigma_{n-1}} \int\limits_0^{\pi/2} \varphi_0(r\sin\psi)\, \sin^{n-k-1}\psi \cos^{k-1}\psi \, d\psi.$$

This coincides with (2.7). □

Example 2.2. The following useful formulae can be obtained from (2.6), (2.7) by elementary calculations. For $Re\alpha > 0$ and $a > 0$,

$$\begin{aligned}
&(2.8) & |x|^{-\alpha-k} &\overset{\wedge}{\to} \lambda_1|\tau|^{-\alpha}, \\
&& \lambda_1 &= \frac{\pi^{k/2}\,\Gamma(\alpha/2)}{\Gamma((\alpha+k)/2)}. \\
&(2.9) & (1+|x|^2)^{-(\alpha+k)/2} &\overset{\wedge}{\to} \lambda_2(1+|\tau|^2)^{-\alpha/2}, \\
&& \lambda_2 &= \lambda_1. \\
&(2.10) & (a^2-|x|^2)_+^{\alpha-1} &\overset{\wedge}{\to} \lambda_3(a^2-|\tau|^2)_+^{\alpha+k/2-1}, \\
&& \lambda_3 &= \frac{\pi^{k/2}\,\Gamma(\alpha)}{\Gamma(\alpha+k/2)}. \\
&(2.11) & |\tau|^{\alpha+k-n} &\overset{\vee}{\to} \lambda_4|x|^{\alpha+k-n}, \\
&& \lambda_4 &= \frac{\Gamma(\alpha/2)\,\Gamma(n/2)}{\Gamma((\alpha+k)/2)\,\Gamma((n-k)/2)}. \\
&(2.12) & \frac{(|\tau|^2-a^2)_+^{\alpha-1}}{|\tau|^{n-k-2}} &\overset{\vee}{\to} \lambda_5\frac{(|x|^2-a^2)_+^{\alpha+k/2-1}}{|x|^{n-2}}, \\
&& \lambda_5 &= \frac{\pi^{k/2}\,\sigma_{n-k-1}\,\Gamma(\alpha)}{\sigma_{n-1}\,\Gamma(\alpha+k/2)}.
\end{aligned}$$

$$\text{(2.13)} \qquad \begin{aligned} \frac{|\tau|^{\alpha+k-n}}{(1+|\tau|^2)^{(\alpha+k)/2}} &\overset{\vee}{\to} \lambda_6 \frac{|x|^{\alpha+k-n}}{(1+|x|^2)^{\alpha/2}}, \\ \lambda_6 &= \frac{\pi^{k/2}\,\sigma_{n-k-1}\,\Gamma(\alpha/2)}{\sigma_{n-1}\,\Gamma((\alpha+k)/2)}. \end{aligned}$$

The last equality is especially important, and we present its proof (all the rest are left to the reader). Let

$$\varphi(\tau)=\frac{|\tau|^{\alpha+k-n}}{(1+|\tau|^2)^{(\alpha+k)/2}}, \qquad c=\frac{\sigma_{k-1}\,\sigma_{n-k-1}}{\sigma_{n-1}}.$$

Then (2.7) yields

$$\begin{aligned} \check{\varphi}(x) &= \frac{c}{r^{n-2}}\int_0^r \frac{(r^2-s^2)^{k/2-1}\,s^{\alpha-1}}{(1+s^2)^{(\alpha+k)/2}}\,ds \\ &= \frac{c}{2r^{n-2}}\int_1^{1+r^2} \frac{(1+r^2-t)^{k/2-1}\,(t-1)^{\alpha/2-1}}{t^{(\alpha+k)/2}}\,dt \\ &= \frac{\pi^{k/2}\,\sigma_{n-k-1}\,\Gamma(\alpha/2)}{\sigma_{n-1}\,\Gamma((\alpha+k)/2)}\,\frac{r^{\alpha+k-n}}{(1+r^2)^{\alpha/2}}. \end{aligned}$$

Combining (2.8)-(2.13) with the duality (2.4), we obtain the following equalities that give precise information about behavior of $\hat{f}(\tau)$ and $\check{\varphi}(x)$.

Theorem 2.3. *For $Re\alpha>0$ and $a>0$,*

$$\text{(2.14)} \qquad \int_{\mathbb{R}^n} \check{\varphi}(x)\frac{dx}{|x|^{\alpha+k}} = \lambda_1 \int_{\mathcal{G}_{n,k}} \varphi(\tau)\frac{d\tau}{|\tau|^{\alpha}},$$

$$\text{(2.15)} \qquad \int_{\mathbb{R}^n} \check{\varphi}(x)\frac{dx}{(1+|x|^2)^{(\alpha+k)/2}} = \lambda_2 \int_{\mathcal{G}_{n,k}} \varphi(\tau)\frac{d\tau}{(1+|\tau|^2)^{\alpha/2}},$$

$$\text{(2.16)} \qquad \int_{|x|<a} \check{\varphi}(x)(a^2-|x|^2)^{\alpha-1}\,dx = \lambda_3 \int_{|\tau|<a} \varphi(\tau)(a^2-|\tau|^2)^{\alpha+k/2-1}d\tau,$$

$$\text{(2.17)} \qquad \int_{\mathcal{G}_{n,k}} \hat{f}(\tau)|\tau|^{\alpha+k-n}d\tau = \lambda_4 \int_{\mathbb{R}^n} f(x)|x|^{\alpha+k-n}dx,$$

$$\text{(2.18)} \qquad \int_{|\tau|>a} \hat{f}(\tau)\frac{(|\tau|^2-a^2)^{\alpha-1}}{|\tau|^{n-k-2}}\,d\tau = \lambda_5 \int_{|x|>a} f(x)\frac{(|x|^2-a^2)^{\alpha+k/2-1}}{|x|^{n-2}}\,dx,$$

$$\int_{\mathcal{G}_{n,k}} \hat{f}(\tau)\frac{|\tau|^{\alpha+k-n}}{(1+|\tau|^2)^{(\alpha+k)/2}}d\tau \;=\; \lambda_6 \int_{\mathbb{R}^n} f(x)\frac{|x|^{\alpha+k-n}}{(1+|x|^2)^{\alpha/2}}dx, \tag{2.19}$$

provided that either side of the corresponding equality exists in the Lebesgue sense.

Corollary 2.4. *If $f \in L^p$, $1 \le p < n/k$, then $\hat{f}(\tau)$ is finite for almost all $\tau \in \mathcal{G}_{n,k}$. If $p \ge n/k$ and $f(x) = (2+|x|)^{-n/p}(\log(2+|x|))^{-1}$ $(\in L^p)$, then $\hat{f}(\tau) \equiv \infty$.*

Proof. By Hölder's inequality, the right hand side of (2.19) does not exceed $A\lambda_6||f||_p$ where

$$A^{p'} = \int_{\mathbb{R}^n} \frac{|x|^{(\alpha+k-n)p'}}{(1+|x|^2)^{\alpha p'/2}}\, dx = \sigma_{n-1}\int_0^\infty \frac{r^{(\alpha+k-n)p'+n-1}}{(1+r^2)^{\alpha p'/2}}\, dr,$$

$(1/p+1/p'=1)$. For $1 \le p < n/k$ and $\alpha > n/p-k$, this integral is finite, and therefore the left hand side of (2.19) is finite too. It follows that the Radon transform $\hat{f}(\tau)$ is finite for almost all $\tau \in \mathcal{G}_{n,k}$. The second statement follows from (2.6). $\square$

Remark 2.5. The statement of Corollary 2.4 is due to Solmon [So]. His proof is different and based on the estimate

$$\int_{\mathcal{G}_{n,k}} \frac{|\hat{f}(\tau)|\, d\tau}{(1+|\tau|)^{n-k+\delta}} \le c\int_{\mathbb{R}^n} \frac{|f(x)|\, dx}{(1+|x|)^{n-k}}, \quad \forall\delta > 0. \tag{2.20}$$

Below we obtain more informative inequalities. Let $\alpha > 0,\ \beta \in \mathbb{R}$,

$$u(\tau) = |\tau|^{\alpha+k-n}(1+|\tau|)^{-\beta}, \quad v(x) = |x|^{\beta-k-\alpha}(1+|x|)^{-\beta},$$

$$\tilde{u}(x) = \begin{cases} (1+|x|)^{-\alpha} & \text{if } \alpha < \beta, \\ (1+|x|)^{-\beta} & \text{if } \alpha > \beta, \\ (1+|x|)^{-\beta}\log(2+|x|) & \text{if } \alpha = \beta, \end{cases}$$

$$\tilde{v}(\tau) = \begin{cases} (1+|\tau|)^{-\alpha} & \text{if } \alpha < \beta, \\ |\tau|^{\beta-\alpha}(1+|\tau|)^{-\beta} & \text{if } \alpha > \beta, \\ (1+|\tau|)^{-\beta}\log(2+1/|\tau|) & \text{if } \alpha = \beta. \end{cases}$$

Lemma 2.6. *For nonnegative functions f and φ,*

$$\int_{\mathcal{G}_{n,k}} \hat{f}(\tau)u(\tau)\, d\tau \le c\int_{\mathbb{R}^n} f(x)\tilde{u}(x)\, dx, \tag{2.21}$$

$$\int_{\mathbb{R}^n} \check{\varphi}(x) v(x)\, dx \le c \int_{\mathcal{G}_{n,k}} \varphi(\tau)\tilde{v}(\tau)\, d\tau. \tag{2.22}$$

Note that (2.21) implies Solmon's estimate (2.20) if $\beta > \alpha = n-k$.

Proof. Let us prove (2.21). We replace $\varphi(\tau)$ in (2.4) by the weight function $u(\tau)$, and make use of (2.7). This gives

$$\check{\varphi}(x) = c\,|x|^{\alpha+k-n}\psi(|x|), \quad \psi(r) = \int_0^1 \frac{t^{\alpha-1}(1-t^2)^{k/2-1}}{(1+rt)^\beta}\, dt. \tag{2.23}$$

If $r \to 0$ then $\psi(r) \to \text{const} \ne 0$. For sufficiently large r, the desired estimate follows from known properties of hypergeometric functions, or can be easily obtained by setting

$$\psi(r) = \Big(\int_0^{1/r} + \int_{1/r}^{1/2} + \int_{1/2}^{1}\Big)(\ldots), \qquad r > 2,$$

and estimating each integral. To prove (2.22) we set $f(x) = v(x)$ in (2.4) and make use of (2.6). We get

$$\hat{f}(\tau) = c \int_s^\infty r^{\beta-k-\alpha}(1+r)^{-\beta}(r^2-s^2)^{k/2-1} r dr = cs^{-\alpha}\psi(1/s),$$

ψ being the same as in (2.23). This gives what was required. □

Let us introduce important mean value operators.

Definition 2.7. For $r \ge 0$, $x \in \mathbb{R}^n$, $\tau = (\zeta, u) \in \mathcal{G}_{n,k}$, $\zeta \in G_{n,k}$, $u \in \zeta^\perp$, we define

$$\begin{aligned} \hat{f}_r(\tau) &= \frac{1}{\sigma_{n-k-1}} \int_{\zeta^\perp \cap S^{n-1}} d\omega \int_\zeta f(r\omega + u + v)\, dv \\ &= \frac{1}{\sigma_{n-k-1}} \int_{\zeta^\perp \cap S^{n-1}} \hat{f}(\zeta, u + r\omega)\, d\omega, \end{aligned} \tag{2.24}$$

$$\check{\varphi}_r(x) = \int_{SO(n)} \varphi(\gamma\mathbb{R}^k + x + r\gamma e_n)\, d\gamma = \int_{SO(n)} \varphi(\gamma\tau_r + x)\, d\gamma, \tag{2.25}$$

τ_r being an arbitrary fixed k-plane at distance r from the origin.

The integral (2.24) can be regarded as a mean value of $f(x)$ over all x at distance r from the k-plane τ. If $r = 0$ then $\hat{f}_r(\tau)$ coincides with the k-plane transform $\hat{f}(\tau)$. The integral (2.25) averages $\varphi(\tau)$ over all τ at distance r from x, and coincides with the dual k-plane transform $\check{\varphi}(x)$ if $r=0$. Clearly, operators $f(x) \to \hat{f}_r(\tau)$, $\varphi(\tau) \to \check{\varphi}_r(x)$ commute with the group $\mathbf{M}(n)$ of isometries of $\mathbb{R}^n$.

Let us consider intertwining operators of the form

$$(Wf)(\tau) = \int\limits_{\mathbb{R}^n} f(x) w(|x-\tau|)\, dx, \tag{2.26}$$

$$(W^*\varphi)(x) = \int\limits_{\mathcal{G}_{n,k}} \varphi(\tau) w(|x-\tau|)\, d\tau, \tag{2.27}$$

where $w(\cdot)$ is assumed to be sufficiently good. If $\tau=(\zeta,u)$, $u \in \zeta^\perp$, then

$$(Wf)(\tau) = \int\limits_{\zeta^\perp} \hat{f}(\zeta, v)\, w(|u-v|) dv,$$

and therefore, for $f \in L^p$, $p \ge 1$, the integral (2.26) is well defined only if $p < n/k$; cf. Corollary 2.4. In (2.27) it suffices to assume $\varphi \in L^1_{loc}(\mathcal{G}_{n,k})$.

Lemma 2.8. *The following representations hold:*

$$(Wf)(\tau) = \sigma_{n-k-1} \int\limits_0^\infty r^{n-k-1} w(r) \hat{f}_r(\tau)\, dr, \tag{2.28}$$

$$(W^*\varphi)(x) = \sigma_{n-k-1} \int\limits_0^\infty r^{n-k-1} w(r) \check{\varphi}_r(x)\, dr. \tag{2.29}$$

It is assumed that either side of the corresponding equality exists in the Lebesgue sense.

Proof. For $\tau=(\zeta,u) \in \mathcal{G}_{n,k}$, we have

$$(Wf)(\tau) = \int\limits_{\zeta^\perp} w(|u-v|) \hat{f}(\zeta, v) dv = \int\limits_0^\infty w(r) r^{n-k-1} dr \int\limits_{S^{n-k-1}} \hat{f}(\zeta, u-r\sigma) d\sigma.$$

By (2.24), this gives (2.28). In order to prove (2.29), let $\tau_0 = \mathbb{R}^k$, $\varphi_x(\tau) = \varphi(\tau + x)$, $b(\tau) = \varphi_x(\tau)w(|\tau|)$, $b(\tau) \equiv b(\zeta, u)$. Then

$$(W^*\varphi)(x) = \int\limits_{G_{n,k}} d\zeta \int\limits_{\zeta^\perp} b(\zeta, u)du = \int\limits_{SO(n)} d\gamma \int\limits_{\gamma\mathbb{R}^{n-k}} b(\gamma\tau_0, u)du$$

$$= \int\limits_{\mathbb{R}^{n-k}} du \int\limits_{SO(n)} b(\gamma\tau_0, \gamma u)d\gamma = \int\limits_0^\infty r^{n-k-1}dr \int\limits_{S^{n-k-1}} d\omega \int\limits_{SO(n)} b(\gamma\tau_0, r\gamma\omega)d\gamma$$

$$= \sigma_{n-k-1}\int\limits_0^\infty r^{n-k-1}dr \int\limits_{SO(n)} b(\gamma\tau_0 + r\gamma e_n)d\gamma = \sigma_{n-k-1}\int\limits_0^\infty r^{n-k-1}\check{b}_r(o)\, dr,$$

o being the origin of $\mathbb{R}^n$. Since

$$\check{b}_r(o) = \int\limits_{SO(n)} \varphi_x(\gamma\tau_0 + r\gamma e_n)w(|\gamma\tau_0 + r\gamma e_n|)\, d\gamma = w(r)\check{\varphi}_r(x)$$

we are done. □

3. Analytic families associated to the k-plane transform

Example 2.2 and duality (2.4) give rise to six equalities (2.14)-(2.19). Let us focus on (2.17). We replace f by the shifted function $f_x(y) = f(x + y)$ and get

$$\frac{1}{\Gamma(\alpha/2)} \int\limits_{\mathcal{G}_{n,k}} \hat{f}(\tau)|x - \tau|^{\alpha+k-n}\, d\tau \tag{3.1}$$

$$= \frac{\Gamma(n/2)}{\Gamma((n-k)/2)\,\Gamma((\alpha+k)/2)} \int\limits_{\mathbb{R}^n} f(y)|x - y|^{\alpha+k-n}\, dy, \quad Re\,\alpha > 0.$$

The right hand side resembles the Riesz potential (1.5). Denoting

$$(\overset{*}{P}{}^{\alpha}\varphi)(x) = \frac{1}{\gamma_{n-k}(\alpha)} \int\limits_{\mathcal{G}_{n,k}} \varphi(\tau)|x - \tau|^{\alpha+k-n}\, d\tau, \tag{3.2}$$

$Re\alpha > 0$, $\quad \alpha + k - n \neq 0, 2, 4, \ldots$, from (3.1) and (1.5) we obtain

$$\overset{*}{P}{}^{\alpha}\hat{f} = c_{k,n}I^{\alpha+k}f, \qquad c_{k,n} = (2\pi)^k\sigma_{n-k-1}/\sigma_{n-1}, \tag{3.3}$$

provided that either side of (3.3) exists in the Lebesgue sense (e.g., for $f \in L^p(\mathbb{R}^n)$, $1 \le p < n(\alpha + k)^{-1}$). By duality we define

$$(P^\alpha f)(\tau) = \frac{1}{\gamma_{n-k}(\alpha)} \int\limits_{\mathbb{R}^n} f(x)|x - \tau|^{\alpha+k-n}\, dx. \tag{3.4}$$

Operators (3.4) and (3.2) can be represented as

$$P^\alpha f = I^\alpha_{n-k} \hat f, \qquad \overset{*}{P}{}^\alpha \varphi = (I^\alpha_{n-k}\varphi)^\vee, \tag{3.5}$$

where for $\tau = (\zeta, u)$, I^α_{n-k} denotes the Riesz potential on $\zeta^\perp$ in the u-variable. For sufficiently good f and φ,

$$\lim_{\alpha\to 0} P^\alpha f = \hat f, \qquad \lim_{\alpha\to 0} \overset{*}{P}{}^\alpha \varphi = \check\varphi. \tag{3.6}$$

This can be easily seen if we represent $P^\alpha f$ and $\overset{*}{P}{}^\alpha\varphi$ according to (2.28) and (2.29), respectively. Thus we can extend definitions (3.4) and (3.2) to $\alpha = 0$ by setting $P^0 f = \hat f$, $\overset{*}{P}{}^0\varphi = \check\varphi$, and obtain analytic families $\{P^\alpha\}$ and $\{\overset{*}{P}{}^\alpha\}$ which include the k-plane transform and its dual. The equality (3.3) generalizes the known formula of Fuglede

$$(\hat f)^\vee = c_{k,n} I^k f \tag{3.7}$$

[F], [H2, p. 29] to $Re\,\alpha > 0$.

Theorem 3.1. *Let $f \in L^p$, $1 \le p < n(\alpha+\beta+k)^{-1}$, $\alpha \ge 0$, $\beta \ge 0$. Then*

$$\overset{*}{P}{}^\alpha P^\beta f = c_{k,n} I^{\alpha+\beta+k} f, \qquad c_{k,n} = (2\pi)^k \sigma_{n-k-1}/\sigma_{n-1}. \tag{3.8}$$

Proof. By (3.3) and (3.5),

$$c_{k,n} I^{\alpha+\beta+k} f = \overset{*}{P}{}^{\alpha+\beta} \hat f = (I^{\alpha+\beta}_{n-k} \hat f)^\vee = (I^\alpha_{n-k} I^\beta_{n-k} \hat f)^\vee = \overset{*}{P}{}^\alpha P^\beta f.$$

□

Remark 3.2. If f belongs to the Semyanisty-Lizorkin space Φ (see Notation), then (3.8) extends to all complex α, β. This follows from (3.5) and the equality $(I^{\alpha-k}_{n-k}\hat f)^\vee = c_{k,n} I^\alpha f$, $\alpha \in \mathbb{C}$, which was proved in [Ru2, Theorem 2.6] using the Fourier transform technique.

4. Inversion of k-plane transforms. The method of Riesz potentials

Throughout this section

$$c_{k,n} = (2\pi)^k \sigma_{n-k-1}/\sigma_{n-1}.$$

Equalities (3.8) and (3.5) give a family of inversion formulae:

$$c_{k,n} f = I^{-\alpha-\beta-k} \overset{*}{P}{}^\alpha I^\beta_{n-k} \hat f \qquad \forall\, \alpha, \beta \in \mathbb{C} \tag{4.1}$$

(at least formally). For $f \in \Phi$, (4.1) is well justified (see Remark 3.2). In the general case we are faced with the following questions. What choice of α and β is preferable? How to represent operators in

(4.1) constructively and recover $f(x)$ pointwise for all or almost all x? To answer these questions we employ appropriate tools of fractional calculus and singular integrals.

4.1. **The case $\alpha = \beta = 0$.** In this case (4.1) reads

$$c_{k,n} f = D^k \check{\varphi}, \qquad \varphi = \hat{f}, \tag{4.2}$$

where $D^k = I^{-k} = (-\Delta)^{k/2}$ denotes the Riesz fractional derivative, Δ being the Laplace operator. Thus the problem is how to invert the Riesz potential $g = I^k f$ (in our case $g = c_{k,n}^{-1} \check{\varphi}$)? Numerous investigations are devoted to this question; see [Ru1, SKM] and references therein.

4.1.1. *Hypersingular integrals.* Below we review some results in the context of their application to the k-plane transform. Let us consider finite differences

$$(\Delta_y^\ell g)(x) = \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j g(x - jy),$$

$$(\tilde{\Delta}_y^m g)(x) = \sum_{j=0}^{m} \binom{m}{j} (-1)^j g(x - \sqrt{j} y),$$

and normalizing constants

$$d_{n,\ell}(k) = \int_{\mathbb{R}^n} \frac{(1 - e^{iy_1})^\ell}{|y|^{n+k}} dy \quad (y_1 \text{ is the first coordinate of } y), \tag{4.3}$$

$$\tilde{d}_{n,m}(k) = \frac{\pi^{n/2}}{2^k\, \Gamma((n+k)/2)} \int_0^\infty \frac{(1 - e^{-t})^m}{t^{1+k/2}} dt. \tag{4.4}$$

We assume $\ell = k$ if k is odd , and any $\ell > k$ if k is even; $m > k/2$. Integrals (4.3), (4.4) can be evaluated explicitly, and the following statement holds [Ru1, pp. 238, 239], [SKM, Section 26]:

Theorem 4.1. *Let $g = I^k f$, $f \in L^p$, $1 \le p < n/k$. Then*

$$f(x) = \frac{1}{d_{n,\ell}(k)} \int_{\mathbb{R}^n} \frac{(\Delta_y^\ell g)(x)}{|y|^{n+k}} dy = \frac{1}{\tilde{d}_{n,m}(k)} \int_{\mathbb{R}^n} \frac{(\tilde{\Delta}_y^m g)(x)}{|y|^{n+k}} dy \tag{4.5}$$

where $\int_{\mathbb{R}^n} = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon}$. This limit exists in the L^p-norm and in the a.e. sense. For $f \in C_0 \cap L^p$, it exists in the sup*-norm.*

Corollary 4.2. *In assumptions of Theorem 4.1, the k-plane transform $\varphi = \hat{f}$ can be inverted by*

$$c_{k,n} f(x) = \frac{1}{d_{n,\ell}(k)} \int\limits_{\mathbb{R}^n} \frac{(\Delta^\ell_y \check{\varphi})(x)}{|y|^{n+k}} dy = \frac{1}{\tilde{d}_{n,m}(k)} \int\limits_{\mathbb{R}^n} \frac{(\tilde{\Delta}^m_y \check{\varphi})(x)}{|y|^{n+k}} dy. \tag{4.6}$$

Remark 4.3. Let us compare (4.6) with the known formula

$$f = c\Lambda^k (\hat{f})^\vee, \tag{4.7}$$

(see formula (3.12) in [So]) where

$$\Lambda = \sum_{j=1}^n R_j \partial_j, \quad (R_j\psi)(x) = \frac{2}{\sigma_n} \text{P.V.} \int\limits_{\mathbb{R}^n} \frac{y_j}{|y|^{n+1}} \psi(x-y)\, dy, \tag{4.8}$$

$\partial_j = \frac{\partial}{\partial x_j}$. Operators R_j are called *the Riesz transforms.* They are understood in the Cauchy principal value sense and bounded on L^p for $1 < p < \infty$ [Ne, p. 101].

The following advantages of (4.6) are worth to be noted. The function f is expressed by (4.6) through the only one singular integral which is understood in the usual sense for sufficiently good f. The formula (4.7), unlike (4.6) contains (apart from derivatives) nk singular integral operators R_j, the L^p-theory of which is much more sophisticated than that of (4.5), and does not include the L^1 case.

Remark 4.4. (i) The set of continuous functions

$$C_a = \{f : \ f \in C(\mathbb{R}^n), \ f(x) = O(|x|^{-a})\}, \quad a > 0, \tag{4.9}$$

is contained in L^p for $n/a < p < n/k$. Hence (4.5) and (4.6) are applicable to $f \in C_a, \ a > k$.

(ii) Instead of (4.5) one can use many other inversion formulae for Riesz potentials which can be found in [Ru1]. If $f(x) \equiv 0$ for $|x| > R > 0$, it suffices to determine $\check{\varphi}(x)$ for $|x| < R$ only. Then we get

$$\frac{c_{k,n}}{\gamma_n(k)} \int\limits_{|y|<R} \frac{f(y)dy}{|x-y|^{n-k}} = \check{\varphi}(x), \qquad |x| < R.$$

Equations of this type play an important role in mixed boundary value problems of mathematical physics (in particular, in mechanics). They can be solved explicitly, but inversion formulae are more complicated than those for potentials on $\mathbb{R}^n$. The interested reader is addressed to [Ru1, Chapter 7] for details.

4.1.2. *Powers of "minus Laplacian".* Another series of inversion formulae can be obtained using integer powers of "minus Laplacian".

Definition 4.5. For $\lambda \in (0,1)$, let $\mathrm{Lip}_\lambda^{loc}$ be the space of functions $f(x)$ on $\mathbb{R}^n$ having the following property: for each finite domain $\Omega \subset \mathbb{R}^n$, there is a constant $A > 0$ such that

$$|f(x) - f(y)| \le A|x-y|^\lambda \qquad \forall x, y \in \bar{\Omega} \quad \text{(the closure of } \Omega). \tag{4.10}$$

We denote

$$C_a^* = \{f : \ f \in C_a \cap \mathrm{Lip}_\lambda^{loc} \quad \text{for some } \lambda \in (0,1)\}. \tag{4.11}$$

Theorem 4.6. *Let $\varphi = \hat{f}$, $1 \le k \le n-1$.*
(i) *For k even, $a > k$, and $f \in C_a^*$, we have*

$$c_{k,n} f(x) = (-\Delta)^{k/2} \check{\varphi}(x). \tag{4.12}$$

(ii) *For k odd, the following statements hold.*
(a) If $f \in C_a$, $a > k$, then

$$c_{k,n} f(x) = \frac{2}{\sigma_n} \int\limits_{\mathbb{R}^n} \frac{(-\Delta)^{(k-1)/2} \check{\varphi}(x) - (-\Delta)^{(k-1)/2} \check{\varphi}(x-y)}{|y|^{n+1}} \, dy \tag{4.13}$$

where $\int_{\mathbb{R}^n} = \lim\limits_{\varepsilon \to 0} \int_{|y|>\varepsilon}$ uniformly in $x \in \mathbb{R}^n$.
(b) If $f \in C_a^*$, $a > k$, and Λ is the operator (4.8), then

$$c_{k,n} f(x) = (\Lambda (-\Delta)^{(k-1)/2} \check{\varphi})(x). \tag{4.14}$$

Furthermore,

$$c_{k,n} f(x) = -(-\Delta)^{(k-1)/2} (I^1 \Delta \check{\varphi})(x) \tag{4.15}$$

if $3 \le k \le n-1$, $f \in C_a^*$, $a > k$, and

$$c_{k,n} f(x) = (-\Delta)^{(k+1)/2} (I^1 \check{\varphi})(x) \tag{4.16}$$

if $1 \le k \le n-2$, $f \in C_a^*$, $a > k+1$.
All derivatives in (4.12)-(4.16) exist in the classical sense.

Proof. These statements are consequences of known facts for potentials and singular integrals. In the following, according to (3.7), we denote $g = c_{k,n}^{-1} \check{\varphi}$ so that $g = I^k f$.

(i) To "localize" the problem, let $x \in B_R = \{x : \ |x| < R\}$ and choose $\chi(x) \in C^\infty$ so that

$$0 \le \chi(x) \le 1, \ \chi(x) \equiv 0 \text{ if } |x| \le R+1, \text{ and } \chi(x) \equiv 1 \text{ if } |x| \ge R+2.$$

We have $f = f_1 + f_2,\ f_1 = \chi f,\ f_2 = (1-\chi)f,$

$$f_1(x) = \begin{cases} 0 & \text{if } |x| \le R+1, \\ f(x) & \text{if } |x| \ge R+2, \end{cases} \qquad f_2(x) = \begin{cases} f(x) & \text{if } |x| \le R+1, \\ 0 & \text{if } |x| \ge R+2. \end{cases} \tag{4.17}$$

Let $g = g_1 + g_2,\ g_1 = I^k f_1,\ g_2 = I^k f_2$. Then $g_1 \in C^\infty(B_R)$, and for all multi-indices γ,

$$\partial^\gamma g_1(x) = \frac{1}{\gamma_n(k)} \int\limits_{|y|>R+1} f_1(y)\, \partial^\gamma |x-y|^{k-n} dy.$$

In particular, for k even, we get $(-\Delta)^{k/2} g_1(x) = 0$. The function g_2 belongs at least to $C^{k-1}(B_R)$, and differentiation is possible under the sign of integration; see, e.g. [Vl, Section 1(6)]. Hence, for k even, $(-\Delta)^{(k-2)/2} g_2 = I^2 f_2$ (the Newtonian potential over a finite domain), and (i) follows by Theorem 11.6.3 from [Mi2, p. 231].

(ii) Consider the case k odd. By reasoning from above,

$$(-\Delta)^{(k-1)/2} g(x) = (I^1 f)(x), \tag{4.18}$$

and (4.13) holds owing to Remark 4.4(i). In order to prove (4.14) we note that $\varphi_j \equiv \partial_j I^1 f = R_j f$ (see (4.8)) where $R_j f \in \text{Lip}_\lambda^{loc}$ for some $\lambda \in (0,1)$ [Mi1, pp. 59, 46]. Since $f \in L^p$ for $\max(1, n/a) < p < n/k$, and R_j is bounded on L^p, then $\varphi_j \in \text{Lip}_\lambda^{loc} \cap L^p$. Let us consider $R_j \varphi_j$. As in (4.17), we define $\varphi_{j,1}$ and $\varphi_{j,2}$ so that $\varphi_j = \varphi_{j,1} + \varphi_{j,2}$,

$$\begin{aligned} (R_j \varphi_j)(x) &= \frac{2}{\sigma_n} \int\limits_{|y|>R+1} \varphi_{j,1}(y) \frac{x_j - y_j}{|x-y|^{n+1}}\, dy \\ &+ \frac{2}{\sigma_n} \text{P.V.} \int\limits_{|y|<R+2} \varphi_{j,2}(y) \frac{x_j - y_j}{|x-y|^{n+1}}\, dy. \end{aligned}$$

The first term $\in C^\infty(B_R)$ while the second one is Lip_λ in B_R (use Theorem 1.6 from [Mi1, p. 46]). Since R can be arbitrary large and R_j is bounded on L^p, then $R_j \varphi_j \equiv R_j^2 f \in \text{Lip}_\lambda^{loc} \cap L^p$. By taking into account that $\sum_j R_j^2 f = f$ [Ne], owing to (4.18), we obtain

$$f(x) = \sum_j (R_j \varphi_j)(x) = \sum_j (R_j \partial_j (-\Delta)^{(k-1)/2} g)(x) \quad \forall x \in \mathbb{R}^n.$$

This gives (4.14).

If $k \ge 3$ then, as in (i), we have $-\Delta g = I^{k-2} f$. Hence $-I^1 \Delta g = I^{k-1} f$, and (4.15) follows. To prove (4.16) we note that for $f \in C_a,\ a > k+1$ and $k+1 < n$, one can write $I^1 g = I^1 I^k f = I^{k+1} f$. Since f satisfies some Lipschitz condition the argument from (i) is applicable, and we are done. □

Remark 4.7. If $f \in L^p$, $1 < p < n/k$, all formulae (4.12)-(4.16) remain true with the following changes: (a) The corresponding derivatives are understood in the sense of $\mathcal{S}'$ or Φ' distributions. They also exist in a certain L^q-norm for almost all x; see [St, Chapter VIII], about this notion of differentiation. (b) In (4.16) we have to assume $1 < p < n/(k+1)$ (otherwise $I^1 g$ may be divergent). (c) Convergence of the hypersingular integral (4.13) is interpreted in the L^p-norm or in the a.e. sense.

4.2. **The case $\alpha = 0$, $\beta = -k$.** In this case (4.1) reads

$$c_{k,n} f = (I^{-k}_{n-k} \varphi)^{\vee}, \qquad \varphi(\tau) = \hat{f}(\tau) \equiv \hat{f}(\zeta, u), \tag{4.19}$$

and one has to give precise sense to the operator I^{-k}_{n-k} acting in the u variable. The first way to do this is to use hypersingular integrals like (4.5) in the $(n-k)$-plane $\zeta^{\perp}$. Let, for example,

$$(\Delta^{\ell}_{v} \varphi)(\zeta, x) = \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j \varphi(\zeta, \mathrm{Pr}_{\zeta^{\perp}} x - jv),$$

$$x \in \mathbb{R}^n, \quad v \in \zeta^{\perp}, \qquad d_{n-k,\ell}(k) = \int\limits_{\mathbb{R}^{n-k}} \frac{(1 - e^{iy_1})^{\ell}}{|y|^n} \, dy,$$

where $\ell = k$ for k odd, and $\forall\, \ell > k$ for k even; cf. (4.3).

Theorem 4.8. *If $\varphi = \hat{f}$, $f \in L^p$, $1 \le p < n/k$, then*

$$c_{k,n} f(x) = \frac{1}{d_{n-k,\ell}(k)} \int\limits_{\mathcal{G}_{n,k}} \frac{(\Delta^{\ell}_{v} \varphi)(\zeta, x)}{|v|^n} \, d\zeta dv \tag{4.20}$$

$$\equiv \lim_{\varepsilon \to 0} \frac{1}{d_{n-k,\ell}(k)} \int\limits_{G_{n,k}} d\zeta \int\limits_{\{v:\, v \in \zeta^{\perp},\, |v| > \varepsilon\}} \frac{(\Delta^{\ell}_{v} \varphi)(\zeta, x)}{|v|^n} \, dv. \tag{4.21}$$

The limit (4.21) exists in the L^p-norm and in the a.e. sense. If $f \in C_0 \cap L^p$ for some $1 \le p < n/k$, this limit is uniform in $x \in \mathbb{R}^n$.

This statement was obtained in [Ru2, Theorem 3.6] as a particular case of a more general result. Theorem 4.8 gives precise sense to the second formula in (1.4) for $f \in L^p$. In order to interpret this formula in terms of pointwise laplacians, one has to impose extra smoothness conditions on f (which are redundant for existence of $\hat{f}$), and proceed as in Section 4.1.2.

5. Inversion of k-plane transforms. The method of spherical means

The method of spherical means is alternative to that of Section 4. It is based on the definition (2.25) and the following

Lemma 5.1. *Let*

$$(\mathcal{M}_t f)(x) = \frac{1}{\sigma_{n-1}} \int\limits_{S^{n-1}} f(x+t\theta)d\theta, \qquad t>0, \tag{5.1}$$

be the spherical mean of f. If $f \in L^p$, $1 \le p < n/k$, then

$$(\hat{f})^\vee_r(x) = \sigma_{k-1} \int\limits_r^\infty (\mathcal{M}_t f)(x)(t^2-r^2)^{k/2-1} t\,dt. \tag{5.2}$$

Proof. Let $f_x(y) = f(x+y)$. For any fixed $\tau \in \mathcal{G}_{n,k}$ such that $|\tau| = r$, we have

$$(\hat{f})^\vee_r(x) = \int\limits_{SO(n)} \hat{f}(\gamma\tau + x)d\gamma = \int\limits_{SO(n)} (f_x \circ \gamma)^\wedge(\tau)d\gamma = \hat{F}(\tau),$$

$$F(y) = \int\limits_{SO(n)} f_x(\gamma y)d\gamma = \int\limits_{SO(n)} f(x+\gamma y)d\gamma = (\mathcal{M}_{|y|} f)(x).$$

It remains to make use of the Abel type representation (2.6). $\square$

For $\varphi = \hat{f}$, we denote

$$g_x(s) = (\mathcal{M}_{\sqrt{s}} f)(x), \qquad \psi_x(s) = \pi^{-k/2} \check{\varphi}_{\sqrt{s}}(x). \tag{5.3}$$

Then (5.2) reads

$$(I^{k/2}_{-} g_x)(s) = \psi_x(s) \tag{5.4}$$

(see notation (1.6)). If f is continuous and decays sufficiently fast at infinity then (5.4) can be easily inverted, and we get

$$f(x) = \Big(-\frac{d}{ds}\Big)^m (I^{m-k/2}_{-} \psi_x)(s)\Big|_{s=0}, \qquad \forall m \in \mathbb{N}, \quad m > k/2. \tag{5.5}$$

This formula is inapplicable for generic $f \in L^p$ because the integral

$$\begin{aligned}(I^{m-k/2}_{-} \psi_x)(s) &= (I^m_{-} g_x)(s) \\ &= \frac{2}{\Gamma(m)\,\sigma_{n-1}} \int\limits_{|y|^2>s} f(x-y)(|y|^2-s)^{m-1} \frac{dy}{|y|^{n-2}}\end{aligned} \tag{5.6}$$

can be divergent for $n/2m \le p < n/k$. Thus the main difficulties are connected with behavior of functions at infinity, and the inversion procedure should not increase the order of the fractional integral (5.4). For $k > 1$, the order can be reduced by differentiation in the s-variable according to the following

Lemma 5.2. *Let* $g_x(s) = (\mathcal{M}_{\sqrt{s}} f)(x)$, $f \in L^p$.

(i) *If* $1 \le p < n-1$ *then* $-\frac{d}{ds}(I^1_- g_x)(s)|_{s=0} = f(x)$, *the derivative being well defined in the* L^p*-norm and for almost all* x.

(ii) *If* $\alpha > 1$ *then for each* $s > 0$, $-\frac{d}{ds}(I^\alpha_- g_x)(s) = (I^{\alpha-1}_- g_x)(s)$ *where differentiation is understood for almost all* x *or in the* L^q*-norm,* $0 \le 1/q < 1/p - 2(\alpha-1)/n$.

(iii) *If* $f \in C_0 \cap L^p$ *then derivatives in* (i) *and* (ii) *exist for all* x *in the classical sense.*

Proof. (i) A standard machinery of approximation to the identity [St, Chapter III, Sec. 2] yields

$$
\begin{aligned}
- \quad \frac{(I^1_- g_x)(\delta) - (I^1_- g_x)(0)}{\delta} &= \frac{1}{\delta}\int_0^\delta (\mathcal{M}_{\sqrt{s}} f)(x)\,ds \\
= \quad \frac{2}{\sigma_{n-1}} \int\limits_{|y|<1} f(x - \sqrt{\delta}y)\frac{dy}{|y|^{n-2}} &\to f(x) \quad \text{as} \quad \delta \to 0
\end{aligned}
$$

in the required sense. The condition $p < n-1$ is necessary for the existence of $I^1_- g_x$; cf. (5.6).

(ii) We note that $(I^{\alpha-1}_- g_x)(s)$, $\alpha > 1$, exists in the Lebesgue sense if and only if $1/p > 2(\alpha-1)/n$. Furthermore, for each $s > 0$,

$$
(5.7) \qquad ||(I^{\alpha-1}_- g_{(\cdot)})(s)||_q \le c_s\,||f||_p, \qquad 0 \le \frac{1}{q} < \frac{1}{p} - \frac{2(\alpha-1)}{n}.
$$

To see this one should replace m by $\alpha - 1$ in (5.6) and make use of Young's inequality. Our aim is to show that

$$
\frac{(I^\alpha_- g_x)(s) - (I^\alpha_- g_x)(s+\delta)}{\delta} - (I^{\alpha-1}_- g_x)(s)
$$

tends to 0 as $\delta \to 0$ in the required sense. This expression can be written as a convolution $f * h_{\delta,s}$ where

$$
h_{\delta,s}(x) = \lambda_s(x)\, h\Big(\frac{\delta}{|x|^2 - s}\Big),
$$

$$
\lambda_s(x) = \frac{2}{\sigma_{n-1}\,\Gamma(\alpha-1)} \frac{(|x|^2 - s)_+^{\alpha-2}}{|x|^{n-2}}, \qquad h(t) = \frac{1 - (1-t)_+^{\alpha-1}}{t(\alpha-1)} - 1.
$$

The function $h(t)$ is bounded and $\lim\limits_{t\to 0} h(t) = 0$. Since $|f * h_{\delta,s}| \le ||h||_\infty |f| * \lambda_s|$ and the convolution $|f| * \lambda_s$ obeys the same estimate

(5.7), by the Lebesgue theorem on dominated convergence we have

$$\overset{\text{a.e.}}{\lim_{\delta\to 0}}(f * h_{\delta,s})(x) = 0, \qquad \lim_{\delta\to 0} ||f * h_{\delta,s}||_q = 0$$

for each $s > 0$ and q satisfying (5.7).

The proof of (iii) follows the same lines. □

Application of Lemma 5.2 to (5.4) gives the following

Corollary 5.3. *Let $\varphi(\tau) = \hat{f}(\tau)$, $\tau \in \mathcal{G}_{n,k}$. If $f \in L^p$, $1 \le p < n/k$, then for k even,*

$$f(x) \overset{\text{a.e.}}{=} \pi^{-k/2}\Big(-\frac{1}{2r}\frac{d}{dr}\Big)^{k/2} \check{\varphi}_r(x)\Big|_{r=0} \tag{5.8}$$

where $\check{\varphi}_r(x)$ is the average of $\varphi(\tau)$ over all k-planes τ at distance r from x. If $f \in C_0 \cap L^p$ then (5.8) holds for all $x \in \mathbb{R}^n$.

Let us consider arbitrary $1 \le k \le n-1$. As we have already seen, fractional differentiation of (5.4) in the Riemann-Liouville sense blows up. To resolve the problem we use the Marchaud fractional derivative

$$(\mathbb{D}^\alpha_- \psi)(s) = \frac{1}{\kappa_\ell(\alpha)} \int_0^\infty \Big[\sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j \psi(s+jt)\Big] \frac{dt}{t^{1+\alpha}}, \quad \ell > \alpha, \tag{5.9}$$

see [Ru1, SKM]. Here

$$\begin{aligned}
\kappa_\ell(\alpha) &= \int_0^\infty (1-e^{-t})^\ell t^{-1-\alpha} dt \\
&= \begin{cases} \Gamma(-\alpha)\displaystyle\sum_{j=1}^{\ell} \binom{\ell}{j}(-1)^j j^\alpha, & \alpha \ne 1,2,\dots,\ell-1, \\ \dfrac{(-1)^{1+\alpha}}{\alpha!} \displaystyle\sum_{j=1}^{\ell} \binom{\ell}{j}(-1)^j j^\alpha \log j, & \alpha = 1,2,\dots,\ell-1. \end{cases}
\end{aligned}$$

Owing to normalization, $\mathbb{D}^\alpha_- \psi$ is independent of $\ell > \alpha$. The right hand side of (5.9) is understood as a limit of the truncated integral

$$(\mathbb{D}^\alpha_{-,\,\varepsilon} \psi)(s) = \frac{1}{\kappa_\ell(\alpha)} \int_\varepsilon^\infty \Big[\sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j \psi(s+jt)\Big] \frac{dt}{t^{1+\alpha}} \tag{5.10}$$

as $\varepsilon \to 0$ in the appropriate sense.

Theorem 5.4. *Let $\varphi = \hat{f},\ f \in L^p,\ 1 \le p < n/k$. For any $\ell > k/2$,*

$$f(x) = \frac{\pi^{-k/2}}{\kappa_\ell(k/2)} \int_0^\infty \Big[\sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j \check{\varphi}_{\sqrt{jt}}(x) \Big] \frac{dt}{t^{1+k/2}} \tag{5.11}$$

where $\int_0^\infty = \lim\limits_{\varepsilon \to 0} \int_\varepsilon^\infty$ in the L^p-norm and in the a.e. sense. If $f \in C_0 \cap L^p$ this limit exists in the sup*-norm.*

Remark 5.5. The right hand side of (5.11) represents the Marchaud derivative of order $k/2$ of the function $\psi_x(s)$ (see (5.3)) evaluated at $s = 0$. The formula (5.11) is applicable to all $1 \le k \le n-1$. For $k = 1$ (the X-ray case), (5.11) has an especially simple form

$$f(x) = \frac{1}{\pi} \int_0^\infty \frac{\check{\varphi}(x) - \check{\varphi}_r(x)}{r^2} dr. \tag{5.12}$$

Proof. For $\alpha = k/2$, according to (5.4) we have

$$\sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j (I_-^\alpha g_x)(jt) = t^\alpha \int_0^\infty k(u) g_x(ut) du,$$

$$k(u) = \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j (u-j)_+^{\alpha-1}.$$

This gives

$$(\mathbb{D}_{-,\,\varepsilon}^\alpha \psi_x)(0) = (\mathbb{D}_{-,\,\varepsilon}^\alpha I_-^\alpha g_x)(0) = \int_0^\infty \lambda_{\ell,\alpha}(\eta) g_x(\varepsilon\eta) d\eta, \tag{5.13}$$

$$\lambda_{\ell,\alpha}(\eta) = [\kappa_\ell(\alpha)\Gamma(1+\alpha)\eta]^{-1} \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j (\eta - j)_+^\alpha.$$

It is known [Ru1, Lemma 10.17] that

$$\int_0^\infty \lambda_{\ell,\alpha}(\eta) d\eta = 1, \qquad \lambda_{\ell,\alpha}(\eta) = \begin{cases} O(\eta^{\alpha-1}) & \text{if } \eta < 1, \\ O(\eta^{\alpha-\ell-1}) & \text{if } \eta > 1. \end{cases} \tag{5.14}$$

Since $g_x(s) = (\mathcal{M}_{\sqrt{s}} f)(x)$, then

$$\begin{aligned} (\mathbb{D}_{-,\,\varepsilon}^\alpha \psi_x)(0) &= \frac{1}{\sigma_{n-1}} \int_0^\infty \lambda_{\ell,\alpha}(\eta) d\eta \int_{S^{n-1}} f(x + \sqrt{\varepsilon\eta}\theta)\, d\theta \\ &= \int_{\mathbb{R}^n} f(x + \sqrt{\varepsilon} y) \Lambda_{\ell,\alpha}(y)\, dy, \qquad \Lambda_{\ell,\alpha}(y) = \frac{2\lambda_{\ell,\alpha}(|y|^2)}{\sigma_{n-1}\, |y|^{n-2}}. \end{aligned} \tag{5.15}$$

By (5.14), this is an approximate identity, and the result follows. □

Corollary 5.6. *Let $\varphi = \hat{f}$, $f \in L^p$, $1 \le p < n/k$. If k is odd and $m = (k-1)/2$ then the derivative*

$$h_x(r) = \Big(-\frac{1}{2r}\frac{d}{dr}\Big)^m \check{\varphi}_r(x)$$

exists for almost all x, and all $r \ge 0$. The function f can be recovered by the formula

$$f(x) = \frac{1}{\pi^{(k+1)/2}} \int\limits_0^\infty \frac{h_x(0) - h_x(r)}{r^2} dr, \qquad \int\limits_0^\infty = \overset{\text{a.e.}}{\lim_{\varepsilon \to 0}} \int\limits_\varepsilon^\infty . \tag{5.16}$$

If $f \in C_0 \cap L^p$ the integral (5.16) converges uniformly in $x \in \mathbb{R}^n$.

Proof. By Lemma 5.2(ii), the equality (5.4) yields

$$(I_-^{1/2} g_x)(s) \overset{\text{a.e.}}{=} \pi^{-k/2}\Big(-\frac{d}{ds}\Big)^m \check{\varphi}_{\sqrt{s}}(x) \overset{\text{def}}{=} \tilde{h}_x(s),$$

and therefore

$$(\mathbb{D}_{-,\,\varepsilon}^{1/2} \tilde{h}_x)(0) \equiv \frac{1}{2\pi^{1/2}} \int\limits_\varepsilon^\infty \frac{\tilde{h}_x(0) - \tilde{h}_x(s)}{s^{3/2}} ds = \int\limits_{\mathbb{R}^n} f(x + \sqrt{\varepsilon} y) \Lambda_{1,1/2}(y)\, dy,$$

cf. (5.15). This implies (5.16). $\square$

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