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RADON INVERSION ON GRASSMANNIANS VIA GÅRDING-GINDIKIN FRACTIONAL INTEGRALS

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ABSTRACT. The Radon transform $\mathcal{R}f$ of functions on Stiefel and Grassmann manifolds is studied. Connection between $\mathcal{R}f$ and Gårding-Gindikin fractional integrals associated to the cone of positive definite matrices is established. By using this connection, we obtain Abel type representations and explicit inversion formulae for $\mathcal{R}f$ provided that f belongs to L^p and the space of continuous functions.

1. INTRODUCTION

Let $G_{n,k}, G_{n,k'}$ be the pair of Grassmann manifolds of linear k-dimensional and k'-dimensional subspaces of $\mathbb{R}^n$ respectively. Suppose that $1 \le k < k' \le n-1$. A function $\varphi(\xi)$ on $G_{n,k'}$ is called the Radon transform of a sufficiently good function $f(\eta)$ on $G_{n,k}$ if $\varphi(\xi) = \int_{\eta\subset\xi} f(\eta) d_\xi m(\eta)$, $d_\xi m(\eta)$ being a suitable measure on the space of planes η in ξ. The basic questions are whether the mapping $f \to \varphi$ is injective and how to invert it. By taking into account that $\dim G_{n,k} = k(n-k)$, we conclude that the condition $\dim G_{n,k'} \ge \dim G_{n,k}$, which is necessary for injectivity, is equivalent to $k+k' \le n$ (for $k<k'$). Thus the natural framework for the inversion problem is

$$1 \le k < k' \le n-1, \qquad k+k' \le n. \tag{1.1}$$

In the present paper we focus on this problem, leaving aside such important topics as range characterization, affine Grassmannians, the complex case, geometrical applications, and further possible generalizations. Concerning these topics, the reader is addressed to a series of fundamental papers by I.M. Gel'fand (and collaborators), F. Gonzalez, P. Goodey, E.L. Grinberg, S. Helgason, T. Kakehi, E.E. Petrov, R.S. Strichartz, and others.

The following aproaches to the inversion problem are known. One can apply a certain differential operator to the composition of the Radon transform and its dual [Go], [Gr1], [K]. The idea of this method belongs to S. Helgason. This method works only for smooth f and $k'-k$ even. Modification of this approach including both even and odd cases, was developed in [Ru2] for $k=1$ by embedding the Radon transform into a certain analytic family of intertwining operators. We believe this idea can be extended to $k>1$. Note that for $k>1$, the differential operator resolving the problem is very complicated, and the corresponding harmonic analysis on Grassmannians is rather difficult.

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Another approach, based on the use of differential forms was developed by Gel'fand, Graev and Šapiro [GGŠ]. As the previous one, it deals with smooth functions and $k'-k$ even.

The third method employs modification of the plane waves decomposition and belongs to Petrov. His inversion formulae (see (11), (11') in [P1]) are presented without proof for the special case $k'=n-k$. The class of admissible functions is not specified, and the divergent integral in (11') should be understood somehow in a regularized sense. In contrast to the previous two, Petrov's method is applicable both to even and odd cases; see also [P2].

Below we develop a new alternative aproach. It differs from aforementioned, covers the full range (1.1), and gives transparent and readable inversion formulae for any integrable f. Along the way we derive a series of integral formulae which can be used in different occurrences. In future publications we plan to present various consequences, further developments and geometrical applications of our results.

Let us describe the basic idea of the method. It is convenient to define the Radon transform in a slightly different form as follows. Let $V_{n,k}$ be the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$. Given a sufficiently good function $f(x)$ on $V_{n,k}$, we set

$$(\mathcal{R}f)(\xi)=\int\limits_{\xi} f(x)dm(x), \quad \xi\in G_{n,k'}, \quad k'>k. \tag{1.2}$$

It means that $f(x)$ is integrated over all k-frames x in ξ with respect to the relevant normalized measure. In the special case $k=1$, $V_{n,1}$ is the unit sphere S^{n-1} in $\mathbb{R}^n$, $G_{n,k'}$ can be identified with the set of $(k'-1)$-dimensional totally geodesic submanifolds of S^{n-1}, and (1.2) becomes the classical spherical Radon transform of the Minkowski-Funk-Helgason type [H2]. The latter was inverted by Helgason [H1], [H2, p. 99] (see also [Ru2]). In our notation his formula reads as follows:

$$f(x)=c\Big[\Big(\frac{d}{d(u^2)}\Big)^{k'-1}\int\limits_0^u(\mathcal{R}^*_{\cos^{-1}(v)}\mathcal{R}f)(x)v^{k'-1}(u^2-v^2)^{(k'-3)/2}dv\Big]_{u=1}, \tag{1.3}$$

provided that f is an even function on S^{n-1}, $c=2^{k'-1}/(k'-2)!\sigma_{k'-1}$, $\sigma_{k'-1}$ is the area of the unit sphere $S^{k'-1}$, $\mathcal{R}^*_{\cos^{-1}(v)}\mathcal{R}f$ is the average of $\mathcal{R}f$ over all $(k'-1)$-geodesics at distance $p=\cos^{-1}(v)$ from x. The expression in square brackets represents an obvious modification of the Riemann-Liouville fractional derivative [Ru1], so that fractional calculus plays a key role in derivation of (1.3).

We extend (1.3) to the higher rank case $k>1$. The assumption of evenness of f is replaced in a natural way by right invariance under the group $O(k)$ of orthogonal transformations so that $f(x)$ becomes a function on the Grassmann manifold $G_{n,k}$. The one-dimensional Riemann-Liouville integral, arising in Helgason's scheme and leading to (1.3), is substituted for its higher rank counterpart

$$(I^\alpha_+w)(r)=\frac{1}{\Gamma_k(\alpha)}\int\limits_0^r w(s)\,(\det(r-s))^{\alpha-(k+1)/2}ds, \quad Re\,\alpha>(k-1)/2, \tag{1.4}$$

associated to the cone $\mathcal{P}_k$ of symmetric positive definite $k\times k$ matrices (see Section 2.2). Integrals (1.4) were introduced by Gårding [Gå], who was inspired by Riesz [R1], Siegel [S], and Bochner [B1], [B2]. The function $\Gamma_k(\alpha)$ in (1.4) is the celebrated

Siegel Gamma function (see (2.4), (2.5) below). Substantial profound generalizations of Riesz-Gårding integrals are due to Gindikin [Gi]. Useful information on this subject can also be found in [FK], [M], [T].

Main results. We need some notation. Note that for $r \in \mathcal{P}_k$, integration in (1.4) is performed over the "interval" $\{s \in \mathcal{P}_k : r - s \in \mathcal{P}_k\}$. In the following $\mathrm{Pr}_\xi x$ denotes the orthogonal projection of $x \in V_{n,k}$ onto the subspace ξ; A' is the transpose of the matrix A, $|A| = \det(A)$. For $x \in V_{n,k}$, dx designates a measure on $V_{n,k}$ which is $O(n)$ left-invariant, $O(k)$ right-invariant, and normalized by

$$\sigma_{n,k} \equiv \mathrm{vol}(V_{n,k}) = \int\limits_{V_{n,k}} dx = \frac{2^k \pi^{nk/2}}{\Gamma_k(n/2)} \tag{1.5}$$

[M, p. 70], [J, p. 57]. Each frame $x \in V_{n,k}$, generated by vectors $x^{(1)}, \ldots, x^{(k)}$, is identified with the matrix whose columns are $x^{(1)}, \ldots, x^{(k)}$ (in the same order).

To the analytic family (1.4) we associate a differential operator

$$D_+ = \det\left(\eta_{i,j}\,\frac{\partial}{\partial s_{i,j}}\right), \qquad \eta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 1/2 & \text{if } i \neq j, \end{cases} \qquad s = (s_{i,j})_{k\times k}, \tag{1.6}$$

so that $D_+ I_+^\alpha = I_+^{\alpha-1}$ [Gå] (see Sec. 2.2). Our first result concerns the so-called ℓ-zonal case, when $f(x)$ is invariant under the left action of all transformations $g \in O(n)$ preserving the coordinate frame $\sigma_\ell = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} \in V_{n,\ell}$, I_ℓ being the $\ell \times \ell$ identity matrix. In this case the restriction $k + k' \le n$ can be reduced.

Theorem 1.1. *Let $f(x) \in L^1(V_{n,k})$, $\varphi(\xi) = (\mathcal{R}f)(\xi)$, $\xi \in G_{n,k'}, 1 \le k < k' \le n-1$. Suppose that f is ℓ-zonal and $O(k)$ right-invariant. The following statements hold.*

(i) *For $1 \le \ell \le \min(k, n-k)$ and almost all x, one can write $f(x) = f_0(s)$, $s = \sigma_\ell' x x' \sigma_\ell \in \mathcal{P}_\ell$, where*

$$\frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(x)dx = \frac{\Gamma_\ell(n/2)}{\Gamma_\ell(k/2)\,\Gamma_\ell((n-k)/2)} \int\limits_0^{I_\ell} f_0(s)d\mu(s), \tag{1.7}$$

$$d\mu(s) = |s|^{(k-\ell-1)/2} |I_l - s|^{(n-k-\ell-1)/2} ds. \tag{1.8}$$

(ii) *If $\ell \le \min(k, n-k')$, then $(\mathcal{R}f)(\xi) = F_0(S)$, $S = \sigma_\ell' \mathrm{Pr}_\xi \sigma_\ell$, and $f \equiv f_0(s)$ can be recovered by*

$$f_0(s) = \frac{\Gamma_\ell(k/2)}{\Gamma_\ell(k'/2)} |s|^{-(k-\ell-1)/2} D_+^m I_+^{m-\alpha} [|S|^{(k'-\ell-1)/2} F_0(S)](s). \tag{1.9}$$

Here $\alpha = (k'-k)/2$, $m > (k'-1)/2$ is an otherwise arbitrary integer, and D_+^m is the m^{th} power of the operator (1.6) which is understood in the sense of distributions (see Sec. 2.2).

This theorem follows from the relevant Abel type representation of the Radon transform of ℓ-zonal functions. We also prove an analogue of Theorem 1.1 for the dual Radon transform $(\mathcal{R}^*\varphi)(x)$ that averages $\varphi(\xi)$ over all $\xi \in G_{n,k'}$ containing $x \in V_{n,k}$. For $k = 1$, these results are known; see, e.g. [Ru2]. Unlike the case $k = 1$, the treatment of D_+^m in the sense of distributions is essential in the framework of this method (even for smooth f), because of convergence of integrals involved.

Now we eliminate the left-invariance assumption and exhibit a higher rank generalization of Helgason's formula (1.3). To this end an analogue of the averaging operator $\mathcal{R}^*_{\cos^{-1}(v)}$ is needed. Given a positive definite matrix r in the "interval" $[0, I_k]$, we set

$$x_r = \begin{bmatrix} 0_{(k'-k)\times k} \\ r^{1/2} \\ 0_{(n-k'-k)\times k} \\ (I_k - r)^{1/2} \end{bmatrix} \in V_{n,k}, \tag{1.10}$$

where 0 (with subscripts) designates zero entries. Let $g_r \in O(n)$ be an orthogonal matrix which transforms $x_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix} \in V_{n,k}$ into x_r and preserves coordinate unit vectors $e_{k'+1}, \dots, e_{n-k}$. This matrix has the form

$$g_r = \begin{bmatrix} I_{k'-k} & 0 & 0 & 0 \\ 0 & (I_k - r)^{1/2} & 0 & r^{1/2} \\ 0 & 0 & I_{n-k'-k} & 0 \\ 0 & -r^{1/2} & 0 & (I_k - r)^{1/2} \end{bmatrix}. \tag{1.11}$$

Let $\xi_0 = \mathbb{R}e_1 + \ldots + \mathbb{R}e_{k'}$ be the coordinate k'-plane and let $K \subset SO(n)$ be the isotropy subgroup of rotations preserving x_0 (one can identify K with $SO(n-k)$). Given a function $\varphi(\xi)$ on $G_{n,k'}$, the required averaging operator is defined by

$$(M^*_r \varphi)(x) = \int_K \varphi(g_x \rho g_r^{-1} \xi_0) d\rho, \tag{1.12}$$

$g_x (\in SO(n))$ being an arbitrary rotation such that $g_x : x_0 \to x$. In the case $k = 1$ the operator (1.12) coincides with its prototype from [Ru2, formula (2.2)].

Theorem 1.2. *Let f be an $O(k)$ right-invariant function belonging to $L^p(V_{n,k})$, $1 \le p \le \infty$ (we identify $L^\infty(V_{n,k})$ with the space $C(V_{n,k})$ of continuous functions). Suppose that $\varphi(\xi) = (\mathcal{R}f)(\xi)$, $\xi \in G_{n,k'}$, $1 \le k < k' \le n-1$, $k + k' \le n$. Then for any integer $m > (k'-1)/2$,*

$$f = \frac{\Gamma_k(k/2)}{\Gamma_k(k'/2)} \overset{(L^p)}{\lim_{s \to I_k}} (D^m_+ I^{m-\alpha}_+ [|r|^{\alpha - 1/2} M^*_r \varphi])(s), \qquad \alpha = (k'-k)/2, \tag{1.13}$$

the differentiation being understood in the sense of distributions. In particular, for $k' - k = 2\ell$, $\ell \in \mathbb{N}$,

$$f = \frac{\Gamma_k(k/2)}{\Gamma_k(k'/2)} \overset{(L^p)}{\lim_{s \to I_k}} (D^\ell_+ [|r|^{\ell - 1/2} M^*_r \varphi])(s). \tag{1.14}$$

This theorem generalizes (1.3) to the higher rank case and $f \in L^p$. Theorems 1.1 and 1.2 can be reformulated for $f(x)$ replaced by the corresponding function on $G_{n,k}$.

A few words about technical tools are in order. We were inspired by the papers of Herz [Herz] and Petrov [P2] (unfortunately the latter was not translated into English). The key role in our argument belongs to Lemma 2.2 which generalizes Lemma 3.7 of Herz [Herz, p. 495]. It extends the notion of bispherical coordinates [VK, pp. 12, 22] to Stiefel manifolds.

It would be interesting to derive pointwise analogues of inversion formulae (1.9), (1.13). The problem reduces to pointwise inversion of Gårding-Gindikin fractional integrals. We plan to study this problem in forthcoming papers following some ideas developed in [Ru1]-[Ru3].

The paper is organized as follows. Section 2 contains preliminaries and derivation of basic integral formulae which are of independent interest. In Section 3 we introduce averaging operators, which serve the inversion problem for Radon transforms, and study their basic properties. Section 4 is devoted to Radon transforms. By making use of the formulae from Section 2, we obtain Abel type representations of Radon transforms of ℓ-zonal functions in terms of Gårding-Gindikin fractional integrals and prove Theorems 1.1, 1.2.

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2. Preliminaries

2.1. **Matrix spaces and Gamma functions.** We recall some definitions and facts from [M], [T], [Herz]. Let $\mathfrak{M}_{n,k}$, $n \ge k$, be the space of real matrices having n rows and k columns. We identify $\mathfrak{M}_{n,k}$ with the real Euclidean space $\mathbb{R}^{nk}$ so that for $x = (x_{i,j})$ the volume element is $dx = \prod_{i=1}^{n} \prod_{j=1}^{k} dx_{i,j}$. In the following x' denotes the transpose of x, and $|x| = \det(x)$ (for square matrices). Let S_k be the space of $k \times k$ real symmetric matrices $r = (r_{i,j})$, $r_{i,j} = r_{j,i}$, which is the Euclidean space of $k(k+1)/2$ dimensions with the volume element $dr = \prod_{i \le j} dr_{i,j}$. In S_k we consider a convex cone $\mathcal{P}_k$ of positive definite matrices r, so that $a'ra > 0$ for all vectors $a \neq 0$ in $\mathbb{R}^k$. For $r \in \mathcal{P}_k$ we write $r > 0$. Given r_1 and r_2 in S_k, the inequality $r_1 > r_2$ means $r_1 - r_2 > 0$. The function space $L^p(\Omega)$, $\Omega \subset \mathcal{P}_k$, is defined in a usual way with respect to the measure dr. The cone $\mathcal{P}_k$ is a symmetric space of the group $GL(k, \mathbb{R})$ of non-singular $k \times k$ real matrices. The action of $g \in GL(k, \mathbb{R})$ on $r \in \mathcal{P}_k$ is given by $r \to g'rg$. This action is transitive (but not simply transitive). The relevant invariant measure on $\mathcal{P}_k$ has the form

$$d\mu(r) = |r|^{-d} \prod_{1 \le i \le j \le k} dr_{i,j}, \qquad d = (k+1)/2 \tag{2.1}$$

[T, p. 18]. Let T_k be the group of upper triangular matrices t of the form

$$t = \begin{bmatrix} t_1 & & & t_{i,j} \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & t_k \end{bmatrix}, \qquad t_i > 0, \quad t_{i,j} \in \mathbb{R}. \tag{2.2}$$

Each $r \in \mathcal{P}_k$ has a unique representation $r = t't$, $t \in T_k$, so that

$$\int_{\mathcal{P}_k} f(r) dr = \int_0^\infty t_1^k dt_1 \int_0^\infty t_2^{k-1} dt_2 \ldots \int_0^\infty t_k \tilde{f}(t_1, \ldots, t_k)\, dt_k, \tag{2.3}$$

$$\tilde{f}(t_1, \ldots, t_k) = 2^k \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(t't) \prod_{i<j} dt_{i,j}$$

[T, p. 22], [M, p. 592]. To the cone $\mathcal{P}_k$ one can associate the Siegel Gamma function

$$\Gamma_k(\alpha) = \int\limits_{\mathcal{P}_k} e^{-\mathrm{tr}(r)} |r|^{\alpha-d} dr, \qquad \mathrm{tr}(r) = \text{trace of } r. \tag{2.4}$$

Using (2.3), it is easy to check that this integral converges absolutely for $Re\,\alpha > d-1$, and represents the product of usual Γ-functions:

$$\Gamma_k(\alpha) = \pi^{k(k-1)/4}\Gamma(\alpha)\Gamma(\alpha - \frac{1}{2}) \dots \Gamma(\alpha - \frac{k-1}{2}). \tag{2.5}$$

For the corresponding Beta function we have [Herz, p. 480]

$$\int\limits_0^R |r|^{\alpha-d}|R-r|^{\beta-d} dr = B_k(\alpha,\beta)|R|^{\alpha+\beta-d}, \tag{2.6}$$

$$B_k(\alpha,\beta) = \frac{\Gamma_k(\alpha)\,\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}; \qquad Re\,\alpha, Re\,\beta > d-1.$$

2.2. **Gårding-Gindikin fractional integrals.** Let $Q = \{r \in \mathcal{P}_k : 0 < r < I_k\}$ be the "unit interval" in $\mathcal{P}_k$, $f \in L^1(Q)$. The Gårding-Gindikin fractional integrals of f of order α are defined by

(2.7)

$$(I_+^\alpha f)(r) = \frac{1}{\Gamma_k(\alpha)} \int\limits_0^r f(s)|r-s|^{\alpha-d} ds, \quad (I_-^\alpha f)(r) = \frac{1}{\Gamma_k(\alpha)} \int\limits_r^{I_k} f(s)|s-r|^{\alpha-d} ds,$$

where $r \in Q$, $d = (k+1)/2$, $Re\,\alpha > d-1$. By (2.6),

$$\int\limits_0^{I_k} (I_\pm^\alpha f)(r) dr = \frac{\Gamma_k(d)}{\Gamma_k(\alpha+d)} \int\limits_0^{I_k} f(s) \left\{ \begin{array}{c} |I_k - s|^\alpha \\ |s|^\alpha \end{array} \right\} ds, \tag{2.8}$$

and therefore $(I_\pm^\alpha f)(r)$ are well defined for almost all $r \in Q$. The equality (2.6) also implies

$$I_\pm^\alpha I_\pm^\beta f = I_\pm^{\alpha+\beta} f, \qquad f \in L^1(Q), \qquad Re\,\alpha,\, Re\,\beta > d-1, \tag{2.9}$$

(the semigroup property). Denote

$$D_+ = \det\left(\eta_{i,j}\frac{\partial}{\partial s_{i,j}}\right), \quad \eta_{i,j} = \left\{ \begin{array}{ll} 1 & \text{if } i = j \\ 1/2 & \text{if } i \neq j, \end{array} \right. \qquad D_- = (-1)^{k^2} D_+. \tag{2.10}$$

For $m \in \mathbb{N}$ and sufficiently good f,

$$D_\pm^m I_\pm^\alpha f = I_\pm^{\alpha-m} f, \qquad Re\,\alpha > m+d-1 \tag{2.11}$$

(see, e.g., [Gå]). Let $\mathcal{D}(Q)$ be the space of infinitely differentiable functions supported in Q. For $w \in \mathcal{D}(Q)$, the integrals $I_\pm^\alpha w$ can be extended to all $\alpha \in \mathbb{C}$ as entire functions of α, so that $I_\pm^\alpha I_\pm^\beta w = I_\pm^{\alpha+\beta} w$ and $D_\pm^m I_\pm^\alpha w = I_\pm^\alpha D_\pm^m w = I_\pm^{\alpha-m} w$ for all $\alpha, \beta \in \mathbb{C}$ and all $m \in \mathbb{N}$ [Gi]. This enables us to define $I_\pm^\alpha f$ for $f \in L^1(Q)$ and $Re\,\alpha \le d-1$ in the sense of distributions by setting

$$(I_\pm^\alpha f, w) = \int\limits_Q (I_\pm^\alpha f)(r)\overline{w(r)} dr = (f, \overline{I_\mp^\alpha \overline{w}}), \quad w \in \mathcal{D}(Q).$$

Note that explicit construction of analytic continuation of $I^\alpha_\pm w$ is rather complicated if w does not vanish near the boundary of Q (cf. [Gå], [R1], [R2]). In order to invert $\varphi = I^\alpha_+ f$ for $f \in L^1(Q)$ and $Re\,\alpha > d-1$ in the sense of distributions, let $m \in \mathbb{N}$, $m - Re\,\alpha > d-1$. By (2.9), $I^m_+ f = I^{m-\alpha}_+ \varphi$, and therefore

$$(f,w) \equiv \int\limits_Q f(r)\overline{w(r)}dr = (D^m_+ I^{m-\alpha}_+ \varphi, w) \equiv (\varphi, I^{m-\alpha}_- D^m_- w). \tag{2.12}$$

2.3. **Stiefel manifolds.** Let $V_{n,k} = \{x \in \mathfrak{M}_{n,k} : x'x = I_k\}$ be the Stiefel manifold of orthonormal k-frames in $\mathbb{R}^n$. For $n = k$, $V_{n,n} = O(n)$ represents the orthogonal group in $\mathbb{R}^n$. The Stiefel manifold is a homogeneous space with respect to the action $V_{n,k} \ni x \to \gamma x \in V_{n,k}$, $\gamma \in O(n)$, so that $V_{n,k} = O(n)/O(n-k)$. The groups $O(n)$ and $SO(n) = \{\gamma \in O(n) : \det(\gamma) = 1\}$ act on $V_{n,k}$ transitively. It is known that $\dim V_{n,k} = k(2n-k-1)/2$. We fix invariant measures dx on $V_{n,k}$ and $d\gamma$ on $SO(n)$ normalized by $\int_{V_{n,k}} dx = \sigma_{n,k}$ (see (1.5)) and $\int_{SO(n)} d\gamma = 1$.

Lemma 2.1. ([Herz, p. 482], [GK, p. 93], [M, p. 66]). *Almost all $x \in \mathfrak{M}_{n,k}$, $n \geq k$, can be decomposed as*

$$x = vr^{1/2}, \quad v \in V_{n,k}, \quad r = x'x \in \mathcal{P}_k \quad \text{so that} \quad dx = 2^{-k}|r|^{(n-k-1)/2} dr dv.$$

Lemma 2.2. *Let k and ℓ be arbitrary integers satisfying $1 \leq k \leq \ell \leq n-1$, $k+\ell \leq n$. Almost all $x \in V_{n,k}$ can be represented in the form*

$$x = \begin{bmatrix} ur^{1/2} \\ v(I_k - r)^{1/2} \end{bmatrix}, \qquad u \in V_{\ell,k}, \quad v \in V_{n-\ell,k}, \quad r \in \mathcal{P}_k, \tag{2.13}$$

so that

$$\int\limits_{V_{n,k}} f(x)dx = \int\limits_0^{I_k} d\nu(r) \int\limits_{V_{\ell,k}} du \int\limits_{V_{n-\ell,k}} f\left(\begin{bmatrix} ur^{1/2} \\ v(I_k - r)^{1/2} \end{bmatrix}\right) dv, \tag{2.14}$$

$$d\nu(r) = 2^{-k}|r|^\gamma |I_k - r|^\delta dr \qquad \gamma = \frac{\ell-k-1}{2}, \quad \delta = \frac{n-\ell-k-1}{2}. \tag{2.15}$$

Proof. Let us check (2.13). If $x = \begin{bmatrix} a \\ b \end{bmatrix} \in V_{n,k}$, $a \in \mathfrak{M}_{\ell,k}$, $b \in \mathfrak{M}_{n-\ell,k}$, then $I_k = x'x = a'a + b'b$. By Lemma 2.1, $a = ur^{1/2}$. Hence $b'b = I_k - r$, and therefore $b = v(I_k - r)^{1/2}$. In order to prove (2.14) we write it in the form

$$\int\limits_{V_{n,k}} f(x)dx = \int\limits_{0<a'a<I_k} |I_k - a'a|^\delta da \int\limits_{V_{n-\ell,k}} f\left(\begin{bmatrix} a \\ v(I_k - a'a)^{1/2} \end{bmatrix}\right) dv \tag{2.16}$$

and show the coincidence of two measures, dx and $\tilde{d}x = |I_k - a'a|^\delta da dv$. Consider the Fourier transforms

$$F_1(s) = \int\limits_{V_{n,k}} \text{etr}(is'x)dx \quad \text{and} \quad F_2(s) = \int\limits_{V_{n,k}} \text{etr}(is'x)\tilde{d}x,$$

$s \in \mathfrak{M}_{n,k}$, $\text{etr}(\Lambda) = e^{\text{tr}(\Lambda)}$. Following [Herz], it suffices to show that $F_1 = F_2$. To this end we employ Bessel functions $A_\lambda(r)$ of Herz for which

$$\int\limits_{V_{n,k}} \text{etr}(is'x)dx = 2^k \pi^{nk/2} A_{(n-k-1)/2}(\frac{1}{4}s's). \tag{2.17}$$

Let

$$s = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}, \quad x = \begin{bmatrix} a \\ v(I_k - a'a)^{1/2} \end{bmatrix}; \quad s_1 \in \mathfrak{M}_{\ell,k},\; s_2 \in \mathfrak{M}_{n-\ell,k};\; a \in \mathfrak{M}_{\ell,k}.$$

Then $s'x = s_1'a + s_2'v(I_k - a'a)^{1/2}$, and we have

$$F_2(s) = \int\limits_{a'a<I_k} \mathrm{etr}(is_1'a)|I_k - a'a|^\delta da \int\limits_{V_{n-\ell,k}} \mathrm{etr}(is_2'v(I_k - a'a)^{1/2})dv.$$

By (2.17) (use the equality $\mathrm{tr}(is_2'vR) = \mathrm{tr}(iR^{-1}Rs_2'vR) = \mathrm{tr}(iRs_2'v)$ with $R = (I_k - a'a)^{1/2}$) the inner integral is evaluated as

$$2^k \pi^{(n-\ell)k/2} A_\delta(\frac{1}{4}Rs_2's_2R) = 2^k \pi^{(n-\ell)k/2} A_\delta(\frac{1}{4}s_2's_2R^2)$$

(the last passage holds due to invariance $A_\delta(R^{-1}rR) = A_\delta(r)$). Thus

$$F_2(s) = \int\limits_{a'a<I_k} \mathrm{etr}(is_1'a)\varphi(a'a)da, \quad \varphi(r) = 2^k\pi^{(n-\ell)k/2}|I_k - a'a|^\delta A_\delta(\frac{1}{4}s_2's_2(I_k - r)).$$

The function $F_2(s)$ can be transformed by the generalized Bochner formula

$$\int\limits_{\mathfrak{M}_{\ell,k}} \mathrm{etr}(iy'a)\varphi(a'a)da = \pi^{\ell k/2} g(\frac{1}{4}y'y),$$

$$g(\Lambda) = \int\limits_{\mathcal{P}_k} A_\lambda(\Lambda r)|r|^\lambda \varphi(r)dr, \qquad \lambda = \frac{\ell - k - 1}{2}, \quad y \in \mathfrak{M}_{\ell,k}$$

[Herz, p. 493], that yields

$$F_2(s) = 2^k\pi^{nk/2} \int\limits_0^{I_k} A_\gamma(\frac{1}{4}s_1's_1r)|r|^\gamma A_\delta(\frac{1}{4}s_2's_2(I_k - r))|I_k - r|^\delta dr,$$

γ, δ being defined by (2.15). This integral can be evaluated using the formula (2.6) from [Herz, p. 487]. The result is

$$F_2(s) = 2^k\pi^{nk/2} A_{(n-k-1)/2}(\frac{1}{4}(s_1's_1 + s_2's_2)) = 2^k\pi^{nk/2} A_{(n-k-1)/2}(\frac{1}{4}s's).$$

By (2.17), the latter coincides with $F_1(s)$. □

Remark 2.3. The assumptions $k + \ell \le n$ and $k \le \ell$ in Lemma 2.2 are necessary for absolute convergence of the integral $\int_0^{I_k}$ in the right hand side of (2.14). It would be interesting to prove this lemma directly, without using the Fourier transform.

Lemma 2.4. *Let $x \in V_{n,k}$, $y \in V_{n,\ell}$; $1 \le k, \ell \le n$. Then*

$$\frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(y'x)dx = \frac{1}{\sigma_{n,\ell}} \int\limits_{V_{n,\ell}} f(y'x)dy. \tag{2.18}$$

Proof. We should observe that formally the left hand side is a function of y, while the right hand side is a function of x. In fact, both are constant (see Lemma 2.5). Let $G = SO(n)$, $g \in G$, $g_1 = g^{-1}$. The left hand side is

$$\int\limits_G f(y'gx)dg = \int\limits_G f((g_1y)'x)dg_1$$

which equals the right hand side. □

We shall need a "lower-dimensional" representation of integrals of the form

$$I_f = \int\limits_{V_{n,k}} f(A'x)dx, \qquad A \in \mathfrak{M}_{n,\ell}; \quad 1 \le k, \ell \le n-1. \tag{2.19}$$

For $k = \ell = 1$ such a representation is well known.

Lemma 2.5. *Let $A \in \mathfrak{M}_{n,\ell}$, $R = A'A \in \mathcal{P}_\ell$, $k+\ell \le n$, $\gamma = (|k-\ell|-1)/2$, $\delta = (n-k-\ell-1)/2$. If $k \le \ell$, then*

$$I_f = \frac{\sigma_{n-\ell,k}}{2^k} \int\limits_0^{I_k} |r|^\gamma |I_k - r|^\delta dr \int\limits_{V_{\ell,k}} f(R^{1/2} u r^{1/2}) du. \tag{2.20}$$

If $k \ge \ell$, $c = 2^{-\ell}\sigma_{n,k}\sigma_{n-k,\ell}/\sigma_{n,\ell}$, then

$$I_f = c \int\limits_0^{I_\ell} |r|^\gamma |I_\ell - r|^\delta dr \int\limits_{V_{k,\ell}} f(R^{1/2} r^{1/2} u') du \tag{2.21}$$

$$= c|R|^{-\delta-k/2} \int\limits_0^{R} |r|^\gamma |R - r|^\delta dr \int\limits_{V_{k,\ell}} f(r^{1/2} u') du. \tag{2.22}$$

Proof. By Lemma 2.1, $a = vR^{1/2}$, $v \in V_{n,\ell}$, and

$$I_f = \int\limits_{V_{n,k}} f(R^{1/2} v' x) dx = \int\limits_{V_{n,k}} f(R^{1/2} \omega_\ell' x) dx, \qquad \omega_\ell = \begin{bmatrix} I_\ell \\ 0 \end{bmatrix}.$$

Now (2.20) follows by Lemma 2.2. If $k \ge \ell$, then (2.18) yields

$$I_f = \frac{\sigma_{n,k}}{\sigma_{n,\ell}} \int\limits_{V_{n,\ell}} f(R^{1/2} v' x) dv = \frac{\sigma_{n,k}}{\sigma_{n,\ell}} \int\limits_{V_{n,\ell}} f(R^{1/2} (\omega_k' v)') dv, \qquad \omega_k = \begin{bmatrix} I_k \\ 0 \end{bmatrix}.$$

We apply Lemma 2.2 again, but with k and ℓ interchanged. This gives (2.21). The proof of (2.22) is as follows.

$$\begin{aligned}
I_f &= \frac{\sigma_{n,k}}{\sigma_{n,\ell}} \int\limits_{V_{n,\ell}} f(R^{1/2}(\omega_k' v)') dv \\
&\overset{(2.16)}{=} \frac{\sigma_{n,k}}{\sigma_{n,\ell}} \int\limits_{o<a'a<I_\ell} |I_\ell - a'a|^\delta da \int\limits_{V_{n-k,\ell}} f\left(R^{1/2}\left(\omega_k' \begin{bmatrix} a \\ v(I_\ell - a'a)^{1/2} \end{bmatrix}\right)'\right) dv \\
&= \frac{\sigma_{n,k}\,\sigma_{n-k,\ell}}{\sigma_{n,\ell}} \int\limits_{o<a'a<I_\ell} |I_\ell - a'a|^\delta f(R^{1/2} a') da \\
&\quad \text{(set } s = aR^{1/2} \in \mathfrak{M}_{k,\ell} \text{ so that } ds = |R|^{k/2} da \quad \text{[M, p. 58])} \\
&= \frac{\sigma_{n,k}\,\sigma_{n-k,\ell}}{\sigma_{n,\ell}\,|R|^{\delta+k/2}} \int\limits_{o<s's<R} |R - s's|^\delta f(s') ds.
\end{aligned}$$

It remains to apply Lemma 2.1. □

2.4. **The Grassmann manifolds.** Analysis on the Stiefel manifold $V_{n,k}$ includes functions on the Grassmann manifold $G_{n,k} = V_{n,k}/O(k)$. The "points" of $G_{n,k}$ are k-dimensional subspaces of $\mathbb{R}^n$. Given $x \in V_{n,k}$, we denote by $\{x\}$ the subspace for which columns of x serve as a basis. Note that $\eta = \{x\} \in G_{n,k}$. In the following $\eta^\perp \in G_{n,n-k}$ designates the orthogonal complement to $\eta \in G_{n,k}$. A function $f(x)$ on $V_{n,k}$ is $O(k)$ right-invariant if and only if there is a function $F(\eta)$ on $G_{n,k}$ so that $f(x) = F(\{x\})$. We endow $G_{n,k}$ with the normalized $O(n)$ left-invariant measure $d\eta$ so that

$$\frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(x)dx = \int\limits_{G_{n,k}} F(\eta)d\eta. \tag{2.23}$$

2.5. **Invariant functions.**

Definition 2.6. Let $\rho \in O(n-\ell)$, $g_\rho = \begin{bmatrix} \rho & 0 \\ 0 & I_\ell \end{bmatrix} \in O(n)$. A function $f(x)$ on $V_{n,k}$ $(F(\eta)$ on $G_{n,k})$ is called *ℓ-zonal* if $f(g_\rho x) = f(x)$ $(F(g_\rho \eta) = F(\eta))$ for all $\rho \in O(n-\ell)$ and $x \in V_{n,k}$ $(\eta \in G_{n,k})$.

Lemma 2.7. *For $k+\ell \le n$ the following statements hold.*

(a) A function $f(x)$ on $V_{n,k}$ is ℓ-zonal if and only if there is a function f_1 on $\mathfrak{M}_{\ell,k}$ such that $f(x) \overset{\text{a.e.}}{=} f_1(\sigma'_\ell x)$, $\sigma_\ell = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} \in V_{n,\ell}$.

(b) Let $k \ge \ell$. A function $f(x)$ on $V_{n,k}$ is ℓ-zonal and $O(k)$ right-invariant (simultaneously) if and only if there is a function f_0 on $\mathcal{P}_\ell$ such that $f(x) \overset{\text{a.e.}}{=} f_0(s)$, $s = \sigma'_\ell x x' \sigma_\ell = \sigma'_\ell \mathrm{Pr}_{\{x\}} \sigma_\ell$. One can write $f_0(s) = f_1(s^{1/2}u'_0)$, $u'_0 = [0_{\ell\times(k-\ell)}, I_\ell]$, where f_1 is the function from (a).

(c) Let $k \ge \ell$. A function $F(\eta)$ on $G_{n,k}$ is ℓ-zonal if and only if there is a function F_0 (or $F_0^\perp$) on $\mathcal{P}_\ell$ such that $F(\eta) \overset{\text{a.e.}}{=} F_0(s)$, $s = \sigma'_\ell \mathrm{Pr}_\eta \sigma_\ell$ (or $F(\eta) \overset{\text{a.e.}}{=} F_0^\perp(r)$, $r = \sigma'_\ell \mathrm{Pr}_{\eta^\perp} \sigma_\ell$, $r + s = I_\ell$).

Proof. (a) Let f be ℓ-zonal. We write $x = \begin{bmatrix} a \\ b \end{bmatrix}$, $a \in \mathfrak{M}_{n-\ell,k}$, $b \in \mathfrak{M}_{\ell,k}$. By taking into account that $k+\ell \le n$, one can apply Lemma 2.1 which gives $a = vs^{1/2}$, $v \in V_{n-\ell,k}$, $s = a'a = I_k - b'b$, so that $\rho a = \rho v s^{1/2}$. Let

$$r_v \in SO(n-\ell) \quad \text{so that} \quad r_v : v_0 = \begin{bmatrix} I_k \\ 0_{(n-k-\ell)\times k} \end{bmatrix} \to v.$$

We set $\rho = r_v^{-1}$. Then

$$f(x) = f\left(\begin{bmatrix} v_0 s^{1/2} \\ b \end{bmatrix}\right) = f\left(\begin{bmatrix} v_0 (I_\ell - b'b)^{1/2} \\ b \end{bmatrix}\right) = f_1(b) = f_1(\sigma'_\ell x).$$

The converse statement in (a) is obvious.

(b) By (a), $f(x) = f_1(\sigma'_\ell x) = f_1((x'\sigma_\ell)')$, and Lemma 2.1 yields $x'\sigma_\ell = us^{1/2}$, $u \in V_{k,\ell}$, $s = \sigma'_\ell x x' \sigma_\ell$. Let $u_0 = \begin{bmatrix} 0_{(k-\ell)\times\ell} \\ I_\ell \end{bmatrix}$, $r_u \in O(k)$, so that $r_u u_0 = u$. Since f is $O(k)$ right-invariant, then

$$\begin{aligned} f(x) &= f(xr_u) = f_1(\sigma'_\ell x r_u) = f_1((r'_u x' \sigma_\ell)') = f_1((r'_u u s^{1/2})') \\ &= f_1((u_0 s^{1/2})') = f_1(s^{1/2} u'_0) = f_0(s). \end{aligned}$$

The converse statement is obvious.

(c) Let $x \in V_{n,k}$ and $y \in V_{n,n-k}$ be orthogonal bases of η and $\eta^\perp$ respectively, i.e. $\eta = \{x\} = \{y\}^\perp$. The functions $\psi(x) = F(\{x\})$ and $\psi^\perp(y) = F(\{y\}^\perp)$ are ℓ-zonal. Moreover, ψ $(\psi^\perp)$ is $O(k)$ $(\ O(n-k))$ right-invariant. Hence the result follows from (b). □

Lemmas 2.7, 2.5, and the equality (1.5) imply the following

Lemma 2.8. *Let* $1 \le k \le n-1, \quad 1 \le \ell \le \min(k, n-k)$. *Denote*

$$d\mu(s) = |s|^\gamma |I_\ell - s|^\delta ds, \quad \gamma = (k-\ell-1)/2, \quad \delta = (n-k-\ell-1)/2, \tag{2.24}$$

$$c = \Gamma_\ell(n/2)/\Gamma_\ell(k/2)\Gamma_\ell((n-k)/2).$$

(i) *If* $f \in L^1(V_{n,k})$ *is* ℓ*-zonal and* $O(k)$ *right-invariant, then for almost all* x, $f(x) = f_0(s),\ s = \sigma'_\ell x x' \sigma_\ell$, *and*

$$\frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(x)dx = c \int\limits_0^{I_\ell} f_0(s) d\mu(s). \tag{2.25}$$

(ii) *If* $F \in L^1(G_{n,k})$ *is* ℓ*-zonal, then for almost all* η, $F(\eta) = F_0(s),\ s = \sigma'_\ell \mathrm{Pr}_\eta \sigma_\ell$, *and*

$$\int\limits_{G_{n,k}} F(\eta)d\eta = c \int\limits_0^{I_\ell} F_0(s) d\mu(s). \tag{2.26}$$

This lemma proves part (i) in Theorem 1.1.

3. Averaging operators

Suppose that $1 \le k \le k' \le n-1,\ k+k' \le n$. We set $G = SO(n)$,

$$K = \left\{ \rho \in G : \rho = \begin{bmatrix} \gamma & 0 \\ 0 & I_k \end{bmatrix}, \quad \gamma \in SO(n-k) \right\},$$
$$K' = \left\{ \tau \in G : \tau = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad \alpha \in SO(k'),\ \beta \in SO(n-k') \right\},$$

so that K and K' are isotropy subgroups of the coordinate frame $x_0 = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$ and the plane $\xi_0 = \mathbb{R}e_1 + \ldots + \mathbb{R}e_{k'}$ respectively. The corresponding normalized invariant measures will be denoted by $d\rho$ and $d\tau$. Given $x \in V_{n,k},\ \xi \in G_{n,k'}$, let $g_x \in G$ and $g_\xi \in G$ be such that $g_x x_0 = x$ and $g_\xi \xi_0 = \xi$. For $f : V_{n,k} \to \mathbb{C}$ and $\varphi : G_{n,k'} \to \mathbb{C}$, we denote $f_\xi(x) = f(g_\xi x),\ \varphi_x(\xi) = \varphi(g_x \xi)$.

Let $r \in [0, I_k] = \{r \in \mathcal{P}_k : 0 < r < I_k\} \cup \{0\} \cup \{I_k\},\ x_r \in V_{n,k}$ be the frame (1.10) and $g_r : x_0 \to x_r$ be the matrix (1.11). Note that $g_r \in O(n)$. Indeed, by taking into account that matrices $r^{1/2}$ and $(I_k - r)^{1/2}$ commute (they are diagonalizable by the same orthogonal transformation; see the proof of Theorem A9.3 in [M, p. 588]), we get $g'_r g_r = I_n$.

We introduce the following averaging operators:

$$(M_r f)(\xi) = \int\limits_{K'} f_\xi(\tau x_r) d\tau, \quad (M^*_r \varphi)(x) = \int\limits_K \varphi_x(\rho g_r^{-1} \xi_0) d\rho. \tag{3.1}$$

Lemma 3.1. *For $1 \le k \le k' \le n-1,\ k+k' \le n$,*

$$\int\limits_{G_{n,k'}} \varphi(\xi)(M_r f)(\xi) d\xi = \frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(x)(M_r^* \varphi)(x) dx \tag{3.2}$$

provided that either of these integrals converges for f and φ replaced by $|f|$ and $|\varphi|$ respectively.

Proof. The left hand side is

$$\begin{aligned} & \int\limits_G \varphi(g\xi_o)(M_r f)(g\xi_o) dg = \int\limits_{K'} d\tau \int\limits_G \varphi(g\xi_o) f(g\tau g_r x_o) dg \\ = & \int\limits_K d\rho \int\limits_{K'} d\tau \int\limits_G \varphi(g\xi_o) f(g\tau g_r \rho^{-1} x_o) dg \\ = & \int\limits_G f(\lambda x_0) d\lambda \int\limits_K \varphi(\lambda \rho g_r^{-1} \xi_o) d\rho \end{aligned}$$

as desired. □

Lemma 3.2. *Suppose that $1 \le k \le k' \le n-1,\ k+k' \le n$, and let $d\nu(r)$ be the measure (2.15) with ℓ replaced by k'. Then*

$$\int\limits_{V_{n,k}} f(x) dx = \sigma_{k',k}\, \sigma_{n-k',k} \int\limits_0^{I_k} (M_r f)(\xi) d\nu(r), \quad \forall \xi \in G_{n,k'}; \tag{3.3}$$

$$\int\limits_{G_{n,k'}} \varphi(\xi) d\xi = \frac{\sigma_{k',k}\, \sigma_{n-k',k}}{\sigma_{n,k}} \int\limits_0^{I_k} (M_r^* \varphi)(x) d\nu(r), \quad \forall x \in V_{n,k}. \tag{3.4}$$

Proof. Replace in (2.14) ℓ by k', f by f_ξ and set

$$u = \alpha \begin{bmatrix} 0_{(k'-k)\times k} \\ I_k \end{bmatrix}, \qquad v = \beta \begin{bmatrix} 0_{(n-k'-k)\times k} \\ I_k \end{bmatrix},$$

$\alpha \in SO(k'),\ \beta \in SO(n-k')$. Integration with respect to $d\alpha d\beta$ (instead of $dudv$) gives (3.3). Let us prove (3.4). Denote the left hand side of (3.4) by I and write it as $I = \int_G \tilde{\varphi}(g) dg$ where $\tilde{\varphi}(g) = \int_K \varphi_x(\rho g^{-1} \xi_0) d\rho$. Since $\tilde{\varphi}$ is K right-invariant, one can identify it with a certain function ψ on $V_{n,k} = G/K$ so that $\tilde{\varphi}(g) = \psi(g x_0)$. By (3.3),

$$I = \frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} \psi(x) dx = \frac{\sigma_{k',k}\, \sigma_{n-k',k}}{\sigma_{n,k}} \int\limits_0^{I_k} d\nu(r) \int\limits_{K'} \psi(\tau g_r x_0) d\tau$$

where the inner integral reads

$$\int\limits_{K'} \tilde{\varphi}(\tau g_r) d\tau = \int\limits_{K'} d\tau \int\limits_K \varphi_x(\rho g_r^{-1} \tau^{-1} \xi_0) d\rho = \int\limits_K \varphi_x(\rho g_r^{-1} \xi_0) d\rho.$$

Thus we are done. □

Let us introduce another averaging operator on $V_{n,k}$ which serves as an analogue of the spherical mean. Assuming $2k \le n$, given a $k \times k$ matrix a such that $a'a \in [0, I_k]$, we set

$$(3.5)\qquad (\mathcal{M}_a f)(x) = \frac{1}{\sigma_{n-k,k}} \int\limits_{V_{n-k,k}} f_x\left(\left[\begin{array}{c} u(I_k - a'a)^{1/2} \\ a \end{array}\right]\right) du, \qquad x \in V_{n,k}.$$

This can be written as $\int_{x'y=a} f(y) d\sigma_a(y)$ where $d\sigma_a(y)$ designates the corresponding normalized measure on the "section" $\{y \in V_{n,k} : x'y = a\}$.

Lemma 3.3. (i) *For $x, z \in V_{n,k}$,*

$$(3.6)\qquad \int\limits_K f_x(\rho z) d\rho = (\mathcal{M}_{x_0' z} f)(x).$$

(ii) *Let $f^\gamma(x) = f(x\gamma), \quad \gamma \in O(k)$. Then*

$$(3.7)\qquad \mathcal{M}_{a\gamma} f = \mathcal{M}_a f^\gamma.$$

(iii) *If f is $O(k)$ right-invariant, then*

$$(3.8)\qquad \mathcal{M}_{x_0' z} f = \mathcal{M}_r f, \qquad r^2 = x_0' z z' x_0 \in [0, I_k].$$

Proof. (i) We write

$$z = \left[\begin{array}{c} z_1 \\ z_2 \end{array}\right], \qquad z_2 = x_0' z, \quad z_1 = u(I_k - z_2' z_2)^{1/2}, \quad u \in V_{n-k,k}.$$

Then

$$\int\limits_K f_x(\rho z) d\rho = \int\limits_K f_x\left(\rho \left[\begin{array}{c} u(I_k - z_2' z_2)^{1/2} \\ z_2 \end{array}\right]\right) d\rho$$

which gives (3.6).

(ii) We have

(3.9)

$$\int\limits_{V_{n-k,k}} f_x\left(\left[\begin{array}{c} u(I_k - a'a)^{1/2} \\ a \end{array}\right]\gamma\right) du = \int\limits_{V_{n-k,k}} f_x\left(\left[\begin{array}{c} v(\gamma'(I_k - a'a)^{1/2}\gamma) \\ a\gamma \end{array}\right]\right) dv,$$

$v = u\gamma$. Since $\gamma'(I_k - a'a)^{1/2}\gamma = (I_k - \gamma' a' a \gamma)^{1/2}$, (3.9) imples $\mathcal{M}_a f^\gamma = \mathcal{M}_{a\gamma} f$.

(iii) By Lemma 2.1, $x_0' z = (z' x_0)' = (vr)' = rv'$, $v \in O(k)$, and (3.8) follows from (3.7). □

Lemma 3.4. *Let $f \in L^p(V_{n,k})$, $||\cdot||_p = ||\cdot||_{L^p(V_{n,k})}$, $2k \le n$.*

(*a*) *If $1 \le p \le \infty$, then* $\sup_{0<a'a<I_k} ||\mathcal{M}_a f||_p \le ||f||_p$.

(*b*) *If $1 \le p < \infty$, then* $\lim\limits_{a \to I_k} ||\mathcal{M}_a f - f||_p = 0$.

(*c*) *If $f \in C(V_{n,k})$, then $\mathcal{M}_a f \to f$ as $a \to I_k$ uniformly on $V_{n,k}$.*

Proof. For $G = SO(n)$, we have $||f||_p^p = \sigma_{n,k}||f(gz)||^p_{L^p(G)}, \forall z \in V_{n,k}$. Hence, by the generalized Minkowski inequality,

$$\begin{aligned} ||\mathcal{M}_a f||_p &= \Big(\sigma_{n,k} \int\limits_G |(\mathcal{M}_a f)(gx_0)|^p dg\Big)^{1/p} \\ &= \frac{\sigma_{n,k}^{1/p}}{\sigma_{n-k,k}} \Big(\int\limits_G \Big| \int\limits_{V_{n-k,k}} f\left(g \begin{bmatrix} u(I_k - a'a)^{1/2} \\ a \end{bmatrix}\right) du \Big|^p dg\Big)^{1/p} \\ &\le \frac{\sigma_{n,k}^{1/p}}{\sigma_{n-k,k}} \int\limits_{V_{n-k,k}} \Big(\int\limits_G |f(g(...))|^p dg\Big)^{1/p} du = ||f||_p. \end{aligned}$$

Let us prove (b). Denote $z_u = \begin{bmatrix} u(I_k - a'a)^{1/2} \\ a \end{bmatrix}$. As above,

$$||\mathcal{M}_a f - f||_p \le \frac{\sigma_{n,k}^{1/p}}{\sigma_{n-k,k}} \int\limits_{V_{n-k,k}} ||f(gz_u) - f(gx_0)||_{L^p(G)} du$$

$$= \sigma_{n,k}^{1/p} \int\limits_{SO(n-k)} ||f(g\gamma z_\omega) - f(gx_0)||_{L^p(G)} d\gamma, \qquad \omega = \begin{bmatrix} I_k \\ 0_{(n-2k)\times k} \end{bmatrix}. \tag{3.10}$$

Replace $g\gamma \to g$ under the sign of the norm and denote

$$A_a = \begin{bmatrix} a & 0 & (I_k - a'a)^{1/2} \\ 0 & I_{n-2k} & 0 \\ -(I_k - a'a)^{1/2} & 0 & a \end{bmatrix}, \qquad \tilde{f} = f(gx_0). \tag{3.11}$$

Then $z_\omega = A_a x_0$, and the integral in (3.10) can be written as $||\tilde{f}(gA_a) - \tilde{f}(g)||_{L^p(G)}$. The latter tends to 0 as $a \to I_k$ (see [HR, Chapter 5, Sec. 20.4). The statement (c) follows directly from (3.5). □

Lemma 3.5. *Let* $1 \le k \le n-1$, $2k \le n$, $\lambda = (n-2k-1)/2$. *For any* $x \in V_{n,k}$,

$$\int\limits_{V_{n,k}} f(y)dy = \frac{\sigma_{n-k,k}}{2^k} \int\limits_0^{I_k} |I_k - r|^\lambda |r|^{-1/2} dr \int\limits_{O(k)} (\mathcal{M}_{vr^{1/2}} f)(x) dv. \tag{3.12}$$

If f is $O(k)$ right-invariant, then

$$\int\limits_{V_{n,k}} f(y)dy = \frac{\sigma_{n-k,k}\,\sigma_{k,k}}{2^k} \int\limits_0^{I_k} |I_k - r|^\lambda |r|^{-1/2} (\mathcal{M}_{r^{1/2}} f)(x) dr. \tag{3.13}$$

Proof. By making use of (2.14) with $\ell = n-k$ and r replaced by $I_k - r$, the integral $I = \int_{V_{n,k}} f(y)dy$ can written as

$$I = 2^{-k} \int\limits_0^{I_k} |I_k - r|^\lambda |r|^{-1/2} \int\limits_{V_{k,k}} dv \int\limits_{V_{n-k,k}} f_x \left(\begin{bmatrix} u(I_k - r)^{1/2} \\ vr^{1/2} \end{bmatrix} \right) du.$$

This coincides with (3.12). In order to derive (3.13) from (3.12), we make use of (3.8) and write $\mathcal{M}_{vr^{1/2}}f$ as $\mathcal{M}_{(vrv')^{1/2}}f$. Then we interchange integrals and replace $vrv' \to r$. □

Remark 3.6. The case $a=0$ in (3.5) is worth mentioning separately. In this case $(\mathcal{M}_0 f)(x) = \sigma_{n-k,k}^{-1} \int_{V_{n-k,k}} f_x \left(\begin{bmatrix} u \\ 0 \end{bmatrix} \right) du$ averages f over the set of all k-frames lying in the $(n-k)$-plane $\{x\}^{\perp}$. Thus $(\mathcal{M}_0 f)(x)$ is the Radon transform of the form $(\mathcal{R}f)(\{x\}^{\perp})$ (see definition in the next section).

4. Radon transforms

Definition 4.1. Let $x \in V_{n,k}$, $\xi \in G_{n,k'}$, $1 \le k < k' \le n-1$. For sufficiently good functions $f(x)$ and $\varphi(\xi)$, the Radon transform $(\mathcal{R}f)(\xi)$ and its dual $(\mathcal{R}^*\varphi)(x)$ are defined by (3.1) with $r = I_k$. A simple calculation gives

$$(\mathcal{R}f)(\xi) = \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} f_\xi \left(\begin{bmatrix} u \\ 0 \end{bmatrix} \right) du, \tag{4.1}$$

$$(\mathcal{R}^*\varphi)(x) = \int\limits_K \varphi_x(\rho g_k^{-1} \xi_0) d\rho, \qquad K = SO(n-k), \tag{4.2}$$

where

$$g_k = g_r|_{r=I_k} = \begin{bmatrix} I_{k'-k} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_k \\ 0 & 0 & I_{n-k'-k} & 0 \\ 0 & -I_k & 0 & 0 \end{bmatrix}, \tag{4.3}$$

$$g_k' g_k = I_n \quad \text{and} \quad g_k : x_0 = \begin{bmatrix} o \\ I_k \end{bmatrix} \to \begin{bmatrix} 0_{(k'-k)\times k} \\ I_k \\ 0_{(n-k')\times k} \end{bmatrix}.$$

In (4.1) f is integrated over all k-frames lying in the fixed k'-subspace ξ. The dual transform (4.2) averages φ over the set of all k'-subspaces containing the k-frame x.

The Radon transforms (4.1), (4.2) can be represented in terms of functions on Grassmannians. Let $\xi \in G_{n,k'}$, $\eta \in G_{n,k}$; $\varphi_\eta(\xi) = \varphi(g_\eta \xi)$, $F_\xi(\eta) = F(g_\xi \eta)$ where $g_\eta \in G$ and $g_\xi \in G$ satisfy $g_\eta : \eta_0 = \{x_0\} \to \eta$, $g_\xi : \xi_0 \to \xi$ (see Sec. 2.4). Assuming $f(x) = F(\{x\})$, from (4.1) and (4.2) we get the following traditional definitions:

$$(\tilde{\mathcal{R}}F)(\xi) = \int\limits_{\mathbb{R}^{k'}} F_\xi(\eta) d\eta = \int\limits_{\eta \subset \xi} F(\eta) dm_1(\eta), \tag{4.4}$$

$$(\tilde{\mathcal{R}}^*\varphi)(\eta) = \int\limits_K \varphi_\eta(\rho g_k^{-1} \xi_0) d\rho = \int\limits_{\xi \supset \eta} \varphi(\xi) dm_2(\xi), \tag{4.5}$$

$dm_1(\eta)$ and $dm_2(\xi)$ being the relevant normalized measures. Clearly,

$$(\tilde{\mathcal{R}}F)(\xi) = (\mathcal{R}f)(\xi) \quad \text{and} \quad (\tilde{\mathcal{R}}^*\varphi)(\{x\}) = (\mathcal{R}^*\varphi)(x). \tag{4.6}$$

By Lemma 3.1,

$$\int\limits_{G_{n,k'}} \varphi(\xi)(\mathcal{R}f)(\xi)d\xi = \frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(x)(\mathcal{R}^*\varphi)(x)dx, \tag{4.7}$$

$$\int\limits_{G_{n,k'}} \varphi(\xi)(\tilde{\mathcal{R}}F)(\xi)d\xi = \int\limits_{G_{n,k}} F(\eta)(\tilde{\mathcal{R}}^*\varphi)(\eta)d\eta. \tag{4.8}$$

These imply

$$\int\limits_{G_{n,k'}} (\mathcal{R}f)(\xi)d\xi = \frac{1}{\sigma_{n,k}} \int\limits_{V_{n,k}} f(x)dx, \qquad \int\limits_{V_{n,k}} (\mathcal{R}^*\varphi)(x)dx = \sigma_{n,k} \int\limits_{G_{n,k'}} \varphi(\xi)d\xi, \tag{4.9}$$

$$\int\limits_{G_{n,k'}} (\tilde{\mathcal{R}}F)(\xi)d\xi = \int\limits_{G_{n,k}} F(\eta)d\eta, \qquad \int\limits_{G_{n,k}} (\tilde{\mathcal{R}}^*\varphi)(\eta)d\eta = \int\limits_{G_{n,k'}} \varphi(\xi)d\xi. \tag{4.10}$$

Thus the Radon transforms (4.1), (4.2), (4.4) and (4.5) are well defined almost everywhere for all integrable f, F and φ.

Consider an important special case when f, F and φ are ℓ-zonal (see Definition 2.6). As we shall see, in this case the Radon transforms and their duals enjoy Abel type representation in terms of Gårding-Gindikin fractional integrals associated to the cone $\mathcal{P}_\ell$. Assuming $x \in V_{n,k}$, $\xi \in G_{n,k'}$, $1 \le k < k' \le n-1$, we denote

$$\sigma_\ell = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} \in V_{n,\ell}, \qquad \alpha = (k'-k)/2,$$

$$\gamma = \frac{|k-\ell|-1}{2}, \quad \tilde{\gamma} = \frac{|n-k'-\ell|-1}{2}, \quad \delta = \frac{k'-k-\ell-1}{2};$$

$$s = \sigma'_\ell x x' \sigma_\ell, \quad S = \sigma'_\ell \mathrm{Pr}_\xi \sigma_\ell; \qquad r = I_\ell - s; \quad R = \sigma'_\ell \mathrm{Pr}_{\xi^\perp} \sigma_\ell = I_\ell - S.$$

Theorem 4.2. (i) *Let* $f(x) = f_0(s)$, $\tilde{f}_0(s) = |s|^\gamma f_0(s)$, $1 \le \ell \le k'-k$. *Then*

$$(\mathcal{R}f)(\xi) = \begin{cases} c_1 |S|^{-\delta-k/2} (I_+^\alpha \tilde{f}_0)(S), & \text{if } \ell \le k, \\ c_2 \int\limits_0^{I_k} |I_k - t|^\delta |t|^\gamma dt \int\limits_{V_{\ell,k}} f_0(S^{1/2} u t u' S^{1/2}) du & \text{if } \ell > k; \end{cases} \tag{4.11}$$

$$c_1 = \frac{\Gamma_\ell(k'/2)}{\Gamma_\ell(k/2)}, \qquad c_2 = 2^{-k} \pi^{-\ell k/2} \frac{\Gamma_k(k'/2)}{\Gamma_k((k'-\ell)/2)}.$$

(ii) *Let* $\varphi(\xi) = \varphi_0(R)$, $\tilde{\varphi}_0(R) = |R|^{\tilde{\gamma}} \varphi_0(R)$, $1 \le \ell \le \min(k, k'-k)$. *Then*

$$(\mathcal{R}^*\varphi)(x) = \begin{cases} \tilde{c}_1 |r|^{-\delta-(n-k')/2} (I_+^\alpha \tilde{\varphi}_0)(R), & \text{if } \ell \le n-k', \\ \tilde{c}_2 \int\limits_0^{I_{n-k'}} |I_{n-k'} - R|^\delta |R|^{\tilde{\gamma}} dR \int\limits_{V_{\ell,n-k'}} \varphi_0(r^{1/2} u R u' r^{1/2}) du & \text{if } \ell > n-k'; \end{cases} \tag{4.12}$$

$$\tilde{c}_1 = \frac{\Gamma_\ell((n-k)/2)}{\Gamma_\ell((n-k')/2)}, \quad \tilde{c}_2 = 2^{k'-n} \pi^{\ell(k'-n)/2} \frac{\Gamma_{n-k'}((n-k)/2)}{\Gamma_{n-k'}((n-k-\ell)/2)}.$$

Proof. (i) By (4.1),

$$\mathcal{R}f)(\xi) = \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} f\left(g_\xi \begin{bmatrix} u \\ 0 \end{bmatrix}\right) du = \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} f_0(z_u z_u')du, \quad z_u = \sigma_\ell' g_\xi \begin{bmatrix} u \\ 0 \end{bmatrix}.$$

Denote

$$a = g_\xi^{-1}\sigma_\ell = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \qquad a_1 \in \mathfrak{M}_{k',\ell}, \quad a_2 \in \mathfrak{M}_{n-k',\ell}.$$

Then $z_u = a_1' u$, and one can write (use Lemma 2.1)

$$(\mathcal{R}f)(\xi) = \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} f_0(a_1' u u' a_1)du = \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} f_0(S^{1/2} v' u u' v S^{1/2})du \tag{4.13}$$

where

$$a_1' a_1 = \sigma_\ell' g_\xi \begin{bmatrix} I_{k'} & 0 \\ 0 & 0 \end{bmatrix} g_\xi^{-1} \sigma_\ell = \sigma_\ell' \mathrm{Pr}_\xi \sigma_\ell = S.$$

Now Lemma 2.5 yields the following equalities. In the case $\ell > k$:

$$(\mathcal{R}f)(\xi) = \frac{\sigma_{k'-\ell,k}}{2^k \sigma_{k',k}} \int\limits_0^{I_k} |I_k - t|^\delta |t|^\gamma dt \int\limits_{V_{\ell,k}} f_0(S^{1/2} u t u' S^{1/2})du.$$

In the case $\ell \le k$:

$$(\mathcal{R}f)(\xi) = \frac{\sigma_{k'-k,\ell}\, \sigma_{k,\ell}}{2^\ell\, \sigma_{k',\ell}\, |S|^{\delta+k/2}} \int\limits_0^S |S - s|^\delta |s|^\gamma f_0(s)ds.$$

By (1.5) and (2.7), these equalities imply (4.11).

(ii) By (4.2), $(\mathcal{R}^*\varphi)(x) = \int_K \varphi(g_x \rho^{-1} g_k^{-1} \xi_0)d\rho$. In our case $\varphi(g_x \rho^{-1} g_k^{-1} \xi_0) = \varphi_0(\sigma_\ell' \mathrm{Pr}_\tau \sigma_\ell)$, $\tau = (g_x \rho^{-1} g_k^{-1} \xi_0)^\perp$, and therefore

$$(\mathcal{R}^*\varphi)(x) = \int\limits_K \varphi_0(\sigma_\ell' g_x \rho^{-1} g_k^{-1} \mathrm{Pr}_{\xi_0^\perp} g_k \rho g_x^{-1} \sigma_\ell)d\rho, \quad \mathrm{Pr}_{\xi_0^\perp} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k'} \end{bmatrix}. \tag{4.14}$$

Multiplying matrices, we get

$$g_k^{-1} \mathrm{Pr}_{\xi_0^\perp} g_k = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & I_{n-k'-k} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \stackrel{\text{def}}{=} \Lambda,$$

so that the expression in brackets in (4.14) reads $y'\rho^{-1}\Lambda\rho y$ where $y = g_x^{-1}\sigma_\ell \in V_{n,\ell}$. Let us represent y in the form (2.13). To this end we make use of Lemma 2.2 with ℓ replaced by $n-k$ and k replaced by ℓ. In new notation the assumptions of Lemma 2.2 read $1 \le \ell \le \min(k, n-k)$, and we get

$$y = \begin{bmatrix} v_1 r^{1/2} \\ v_2 (I_k - r)^{1/2} \end{bmatrix}, \qquad v_1 \in V_{n-k,\ell}, \quad v_2 \in V_{k,\ell},$$

$$\begin{aligned} r &= y' \begin{bmatrix} I_{n-k} & 0 \\ 0 & 0 \end{bmatrix} y = I_\ell - y' \begin{bmatrix} 0 & 0 \\ 0 & I_k \end{bmatrix} y = I_\ell - \sigma_\ell' g_x x_0 x_0' g_x^{-1} \sigma_\ell \\ &= I_\ell - \sigma_\ell' x x' \sigma_\ell = I_\ell - s. \end{aligned}$$

Let

$$\tilde\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k'} \end{bmatrix}_{(n-k)\times(n-k)} = \tilde x_0 \tilde x_0', \quad \tilde x_0 = \begin{bmatrix} 0_{(k'-k)\times(n-k')} \\ I_{n-k'} \end{bmatrix} \in V_{n-k,n-k'}.$$

Then $y'\rho^{-1}\Lambda\rho y = r^{1/2}v_1'\rho'\tilde\Lambda\rho v_1 r^{1/2} = r^{1/2}v_1'\rho'\tilde x_0\tilde x_0'\rho v_1 r^{1/2}$, and (4.14) yields

$$(\mathcal{R}^*\varphi)(x) = \int\limits_K \varphi_0(r^{1/2}v_1'\rho'\tilde x_0\tilde x_0'\rho v_1 r^{1/2})d\rho$$

$$= \frac{1}{\sigma_{n-k,\ell}} \int\limits_{V_{n-k,\ell}} \varphi_0(r^{1/2}v_1'\tilde x_0\tilde x_0' v_1 r^{1/2})dv_1 \tag{4.15}$$

$$\overset{(2.18)}{=} \frac{1}{\sigma_{n-k,n-k'}} \int\limits_{V_{n-k,n-k'}} \varphi_0(A'\tilde x\tilde x' A)d\tilde x, \quad A = v_1 r^{1/2} \in \mathfrak{M}_{n-k,\ell}.$$

This integral has the form (2.19), and Lemma 2.5 gives the following equalities. In the case $n-k'<\ell$:

$$(\mathcal{R}^*\varphi)(x) = \frac{\sigma_{n-k-\ell,n-k'}}{2^{n-k'}\sigma_{n-k,n-k'}} \int\limits_0^{I_{n-k'}} |R|^{\tilde\gamma}|I_{n-k'}-R|^{\delta}dR \int\limits_{V_{\ell,n-k'}} \varphi_0(r^{1/2}uRu'r^{1/2})du.$$

In the case $n-k'\ge\ell$:

$$(\mathcal{R}^*\varphi)(x) = \frac{\sigma_{n-k',\ell}\,\sigma_{k'-k,\ell}}{2^\ell\,\sigma_{n-k,\ell}\,|r|^{\delta+(n-k')/2}} \int\limits_0^r |R|^{\tilde\gamma}|r-R|^{\delta}\varphi_0(R)dR.$$

Owing to (1.5) and (2.7), these coincide with (4.12). □

Let us reformulate Theorem 4.2 for functions on Grassmannians (see (4.4), (4.5)) by taking into account (4.6) and Lemma 2.7(c). We suppose that F and φ are ℓ-zonal integrable functions on $G_{n,k}$ and $G_{n,k'}$ respectively, $1\le k<k'\le n-1$, and keep on the same notation as in Theorem 4.2.

Theorem 4.3. (i) *If $1\le\ell\le\min(k,k'-k)$, then there is a function $F_0(s)$ on $(0,I_\ell)$ such that $F(\eta)\overset{\text{a.e.}}{=}F_0(\sigma_\ell'\mathrm{Pr}_\eta\sigma_\ell)$ and*

$$(\tilde{\mathcal{R}}F)(\xi) = c_1|S|^{-\delta-k/2}I_+^\alpha[|s|^\gamma F_0(s)](S), \quad S=\sigma_\ell'\mathrm{Pr}_\xi\sigma_\ell. \tag{4.16}$$

(ii) *If $1\le\ell\le\min(n-k',k,k'-k)$, then there is a function $\varphi_0^\perp(R)$ on $(0,I_\ell)$ such that $\varphi(\xi)\overset{\text{a.e.}}{=}\varphi_0^\perp(\sigma_\ell'\mathrm{Pr}_\xi\sigma_\ell)$ and*

$$(\tilde{\mathcal{R}}^*\varphi)(\eta) = \tilde c_1|r|^{-\delta-(n-k')/2}I_+^\alpha[|R|^{\tilde\gamma}\varphi_0^\perp(R)](r), \quad r=\sigma_\ell'\mathrm{Pr}_{\eta^\perp}\sigma_\ell. \tag{4.17}$$

Now we consider the general (not necessarily zonal) case. Let M_r^* and $\mathcal{M}_a$ be averaging operators defined by (3.1) and (3.5). For fixed $x\in V_{n,k}$, we denote

$$\psi(s) = \sigma_{k,k}^{-1}|s|^{-1/2}\int\limits_{O(k)} \mathcal{M}_{s^{1/2}}f^\gamma d\gamma, \qquad f^\gamma(x)=f(x\gamma). \tag{4.18}$$

If f is $O(k)$ right-invariant, then $\psi(s)=|s|^{-1/2}\mathcal{M}_{s^{1/2}}f$.

Lemma 4.4. *Let $1 \le k < k' \le n-1$, $\alpha = (k'-k)/2$. If $2k \le k'$, then*

$$|r|^{\alpha-1/2} M_r^* \mathcal{R} f = \frac{\Gamma_k(k'/2)}{\Gamma_k(k/2)} (I_+^\alpha \psi)(r), \qquad r \in (0, I_k), \tag{4.19}$$

where $I_+^\alpha \psi$ is the Gårding-Gindikin fractional integral associated to the cone $\mathcal{P}_k$.

Proof. By (3.1), (4.1) and (3.6),

$$\begin{aligned}(M_r^* \mathcal{R} f)(x) &= \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} du \int\limits_K f_x \left(\rho g_r^{-1} \left[\begin{array}{c} u \\ 0 \end{array} \right] \right) d\rho \\ &= \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} (\mathcal{M}_{a(u)} f)(x) du,\end{aligned} \tag{4.20}$$

$$a(u) = x_0' g_r^{-1} \left[\begin{array}{c} u \\ 0 \end{array} \right] = x_r' \left[\begin{array}{c} u \\ 0 \end{array} \right] = a_1' u, \quad a_1 = \left[\begin{array}{c} 0_{(k'-k) \times k} \\ r^{1/2} \end{array} \right] \in \mathfrak{M}_{k',k}. \tag{4.21}$$

If $2k \le k'$, Lemma 2.5 yields

$$M_r^* \mathcal{R} f = \frac{\sigma_{k'-k,k}}{2^k \sigma_{k',k}} |R|^{-(k'-k-1)/2} \int\limits_0^R |R-s|^{(k'-2k-1)/2} |s|^{-1/2} ds \int\limits_{O(k)} \mathcal{M}_{s^{1/2}\gamma} f \, d\gamma.$$

By (3.7), (4.16) and (1.5), this gives (4.19). □

Remark 4.5. For $k = 1$, the equality (4.19) is due to Helgason [H1], [H2]. The proof presented above generalizes the argument from [Ru2, Lemma 2.8(i)] to the matrix case. The obvious assumption $k' \ge 2$ in [H1], [H2] transforms to $k' \ge 2k$. For $k' < 2k$, the fractional integral in (4.19) diverges.

The equality (4.19) points up the way to inversion formulae. First we have to eliminate the artificial restriction $k' \ge 2k$.

Lemma 4.6. *Let $f \in L^1(V_{n,k})$, $\varphi(\xi) = (\mathcal{R} f)(\xi)$, $\xi \in G_{n,k'}$, $1 \le k < k' \le n-1$; $k + k' \le n$. Then*

$$I_+^{(n-k')/2} [|s|^{(k'-k-1)/2} M_s^* \varphi] = \frac{\Gamma_k(k'/2)}{\Gamma_k(k/2)} I_+^{(n-k)/2} \psi. \tag{4.22}$$

Proof. For $k' \ge 2k$, (4.22) follows immediately from (4.19) due to the semigroup property of fractional integrals. Once the result is known, we shall prove it in the maximal range of parameters. The idea is to use (2.2) twice, from right to left and from left to right, with different parameters. Owing to (4.20) and (4.21), by Lemma 2.1 we have

$$\begin{aligned} M_r^* \mathcal{R} f \quad &= \quad \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} \mathcal{M}_{a_1' u} f \, du = \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} \mathcal{M}_{s^{1/2} v' u} f \, du \\ &\qquad (s = a_1' a, \quad v \in V_{k',k}) \\ &\overset{(2.18)}{=} \frac{1}{\sigma_{k',k}} \int\limits_{V_{k',k}} \mathcal{M}_{s^{1/2} v' u_0} f \, dv, \qquad u_0 = \left[\begin{array}{c} I_k \\ 0 \end{array} \right] \in V_{k',k}. \end{aligned}$$

Thus the left hand side of (4.22) reads

$$\frac{1}{\sigma_{k',k}\,\Gamma_k((n-k')/2)}\int\limits_0^r |r-s|^{(n-k'-k-1)/2}|s|^{(k'-k-1)/2}ds\int\limits_{V_{k',k}} \mathcal{M}_{s^{1/2}v'u_0}f\,dv.$$

By (2.22) (with $k\sim k'$ and $\ell\sim k$), this expression can be written as

$$\frac{2^k\,\sigma_{n,k}\,|r|^{(n-k-1)/2}}{\sigma_{k',k}\,\sigma_{n,k'}\,\sigma_{n-k',k}\,\Gamma_k((n-k')/2)}\int\limits_{V_{n,k'}} \mathcal{M}_{A'yu_0}f\,dy, \tag{4.23}$$

with $A\in\mathfrak{M}_{n,k}$ such that $A'A=r$. Then we set

$$y=\gamma y_0,\qquad \gamma\in SO(n),\qquad y_0=\left[\begin{array}{c} I_{k'}\\ 0\end{array}\right]\in V_{n,k'},$$

and integrate in γ. Since $y_0u_0\in V_{n,k}$ we have

$$\int\limits_{V_{n,k'}} \mathcal{M}_{A'yu_0}f\,dy=\sigma_{n,k'}\int\limits_{SO(n)} \mathcal{M}_{A'\gamma y_0u_0}f\,d\gamma=\frac{\sigma_{n,k'}}{\sigma_{n,k}}\int\limits_{V_{n,k}} \mathcal{M}_{A'z}f\,dz. \tag{4.24}$$

Now we plug (4.24) in (4.23) and apply (2.22) again (with $\ell=k$). This gives (4.22). The conditions for k,k', and n in the lemma agree with those needed for application of (2.22). □

In a similar way one can extend the range of parameters in Theorems 4.2, 4.3.

Theorem 4.7. *Let $f(x)$ and $\varphi(\xi)$ be ℓ-zonal integrable functions on $V_{n,k}$ and $G_{n,k'}$ respectively; $1\le k<k'\le n-1,\ 1\le\ell\le\min(k,n-k')$. Suppose that f is $O(k)$ right-invariant, and set*

$$s=\sigma'_\ell xx'\sigma_\ell,\quad S=\sigma'_\ell\mathrm{Pr}_\xi\sigma_\ell,\quad r=I_\ell-s,\quad R=I_\ell-S,\quad \sigma_\ell=\left[\begin{array}{c} 0\\ I_\ell\end{array}\right].$$

(i) *There exist functions $f_0(s)$ and $F_0(S)$ on $(0,I_\ell)$ such that $f(x)\overset{\text{a.e.}}{=}f_0(s)$, $(\mathcal{R}f)(\xi)\overset{\text{a.e.}}{=}F_0(S)$, and*

$$I_+^{(n-k')/2}[|S|^{(k'-\ell-1)/2}F_0(S)]=\frac{\Gamma_\ell(k'/2)}{\Gamma_\ell(k/2)}I_+^{(n-k)/2}[|s|^{(k-\ell-1)/2}f_0(s)]. \tag{4.25}$$

(ii) *There exist functions $\varphi_0(R)$ and $\Phi_0(r)$ on $(0,I_\ell)$ such that $\varphi(\xi)\overset{\text{a.e.}}{=}\varphi_0(R)$, $(\mathcal{R}^*\varphi)(x)\overset{\text{a.e.}}{=}\Phi_0(r)$, and*

$$I_+^{k/2}[|R|^{(n-k-\ell-1)/2}\Phi_0(r)]=\frac{\Gamma_\ell((n-k)/2)}{\Gamma_\ell((n-k')/2)}I_+^{k'/2}[|R|^{(n-k'-\ell-1)/2}\varphi_0(R)]. \tag{4.26}$$

Proof. (i) By Lemma 2.7(b), $f(x)$ can be wrtitten as $f_0(s)$. Owing to (4.13), for any $v\in V_{k',\ell}$, Lemma 2.1 yields

$$\begin{aligned}(\mathcal{R}f)(\xi) &= \frac{1}{\sigma_{k',k}}\int\limits_{V_{k',k}} f_0(S^{1/2}v'uu'vS^{1/2})du\\ &\overset{(2.18)}{=} \frac{1}{\sigma_{k',\ell}}\int\limits_{V_{k',\ell}} f_0(S^{1/2}v'u_0u_0'vS^{1/2})dv=F_0(s),\quad u_0=\left[\begin{array}{c} I_k\\ 0\end{array}\right]\in V_{k',k}.\end{aligned}$$

Hence by (2.22) (with k replaced by k'), the left hand side of (4.25) can be written as

$$\frac{1}{\sigma_{k',\ell}\,\Gamma_\ell((n-k')/2)}\int\limits_0^t |t-S|^{(n-k'-\ell-1)/2}|S|^{(k'-\ell-1)/2}dS\int\limits_{V_{k',\ell}} f_0(S^{1/2}v'u_0u_0'vS^{1/2})dv$$

$$(4.27)\qquad = \frac{2^\ell\,\sigma_{n,\ell}\,|t|^{(n-\ell-1)/2}}{\sigma_{k',\ell}\,\sigma_{n,k'}\,\sigma_{n-k',\ell}\,\Gamma_\ell((n-k')/2)}\int\limits_{V_{n,k'}} f_0(A'yu_0u_0'y'A)dy$$

provided $k'+\ell\le n,\ A\in\mathfrak{M}_{n,\ell},\ A'A=t\in\mathcal{P}_\ell$. As in (4.24), the integral in (4.27) can be replaced by $(\sigma_{n,k'}/\sigma_{n,k})\int_{V_{n,k}} f_0(A'zz'A)dz$ and transformed by (2.22). Proceeding as in the proof of Lemma 4.6, we get (4.25).

(ii) Existence of φ_0 satisfying $\varphi(\xi)=\varphi_0(R)$ follows from Lemma 2.7(c) provided $\ell\le\min(k',n-k')$. By (4.15), $\mathcal{R}^*\varphi$ has the form $\Phi_0(r)$, and by (2.22) (with k replaced by $n-k$), the left hand side of (4.26) is represented as

$$\frac{1}{\sigma_{n-k,\ell}\,\Gamma_\ell(k/2)}\int\limits_0^t |t-r|^{(k-\ell-1)/2}|r|^{(n-k-\ell-1)/2}dr\int\limits_{V_{n-k,\ell}}\varphi_0(r^{1/2}v_1'\tilde{x}_0\tilde{x}_0'v_1r^{1/2})dv_1$$

$$(4.28)\qquad = \frac{2^\ell\,\sigma_{n,\ell}\,|t|^{(n-\ell-1)/2}}{\sigma_{n-k,\ell}\,\sigma_{n,n-k}\,\sigma_{k,\ell}\,\Gamma_\ell(k/2)}\int\limits_{V_{n,n-k}}\varphi_0(A'y\tilde{x}_0\tilde{x}_0'y'A)dy$$

where $A\in\mathfrak{M}_{n,\ell},\ A'A=t$ provided $\ell\le\min(k,n-k)$. The integral $\int_{V_{n,n-k}}$ can be written as

$$(4.29)\qquad \frac{\sigma_{n,n-k}}{\sigma_{n,n-k'}}\int\limits_{V_{n,n-k'}}\varphi_0(A'zz'A)dz.$$

We plug (4.29) in (4.28) and apply (2.22) (with k replaced by $n-k',\ 1\le\ell\le\min(k',n-k')$). This gives (4.26). □

Theorem 4.7 can be easily reformulated for $f(x)$ and $(\mathcal{R}^*\varphi)(x)$ replaced by the corresponding functions on $G_{n,k}$. This will give an extension of Theorem 4.3.

Having (4.22), (4.25) and (4.26), we can play with fractional integrals in both sides by making use of their semigroup property. Note that it was not poosible in (4.19), (4.11) and (4.12), because these equalities were derived with inevitable additional restrictions.

Corollary 4.8. *Let $x\in V_{n,k},\ \xi\in G_{n,k'},\ 1\le k<k'\le n-1,\ \alpha=(k'-k)/2$ and $m\in\mathbb{N}$.*

(i) *If $f(x)\in L^1(V_{n,k}),\ \varphi(\xi)=(\mathcal{R}f)(\xi),\ k+k'\le n$, then for $m>(k'-1)/2$,*

$$(4.30)\qquad I_+^{m-\alpha}[|s|^{\alpha-1/2}M_s^*\varphi]=\frac{\Gamma_k(k'/2)}{\Gamma_k(k/2)}I_+^m\psi,$$

ψ being the function (4.18).

(ii) *If $f\in L^1(V_{n,k})$ and $\varphi\in L^1(G_{n,k'})$ are ℓ-zonal, $1\le\ell\le\min(k,n-k')$, and f is $O(k)$ right-invariant (i.e. $f=f_0(s),\ \varphi=\varphi_0(R),\ \mathcal{R}f=F_0(S),\ \mathcal{R}^*\varphi=\Phi_0(r)$;*

see Theorem 4.7), then for $m > \alpha + (\ell - 1)/2$,

$$I_+^{m-\alpha}[|S|^{(k'-\ell-1)/2} F_0(s)] = \frac{\Gamma_\ell(k'/2)}{\Gamma_\ell(k/2)} I_+^m[|s|^{(k-\ell-1)/2} f_0(s)], \tag{4.31}$$

$$I_+^{m-\alpha}[|r|^{(n-k-\ell-1)/2} \Phi_0(r)] = \frac{\Gamma_\ell((n-k)/2)}{\Gamma_\ell((n-k')/2)} I_+^m[|R|^{(n-k'-\ell-1)/2} \varphi_0(R)]. \tag{4.32}$$

Proof. Equalities (4.30)-(4.32) follow from (4.22), (4.25) and (4.26) if we apply $I_+^{m-(n-k)/2}$ (on $\mathcal{P}_k$) to (4.22), $I_+^{m-(n-k)/2}$ (on $\mathcal{P}_\ell$) to (4.25), and $I_+^{m-k'/2}$ (on $\mathcal{P}_\ell$) to (4.26). If m is not big enough, the action of these operators is treated in the sense of distributions (see Sec. 2.2). The resulting equalities (4.30)-(4.32) still hold pointwise almost everywhere (on $(0, I_k)$ for (4.30), and on $(0, I_\ell)$ for (4.31), (4.32)), because fractional integrals in these equalities are well defined and represent integrable functions. □

Proof of Theorem 1.2. By (4.33) and (4.18),

$$\psi(r) \equiv |r|^{-1/2} \mathcal{M}_{r^{1/2}} f = \frac{\Gamma_k(k/2)}{\Gamma_k(k'/2)} D_+^m I_+^{m-\alpha}[|s|^{\alpha-1/2} M_s^* \varphi](r). \tag{4.33}$$

Note that owing to Remark 3.6, $\psi(r)$ behaves like $|r|^{-1/2}$ as $|r| \to 0$, and therefore (unlike the case $k = 1$) we cannot differentiate (4.30) pointwise, even if f is smooth. Thus we have to invoke distributions. The equalities (1.13), (1.14) follow by Lemma 3.4. □

In a similar way we get the following inversion formulae for the Radon transform and its dual in the ℓ-zonal case:

$$f_0(s) = \frac{\Gamma_\ell(k/2)}{\Gamma_\ell(k'/2)} |s|^{-(k-\ell-1)/2} D_+^m I_+^{m-\alpha}[|S|^{(k'-\ell-1)/2} F_0(s)](s), \tag{4.34}$$

(4.35)
$$\varphi_0(R) = \frac{\Gamma_\ell((n-k')/2)}{\Gamma_\ell((n-k)/2)} |R|^{-(n-k'-\ell-1)/2} D_+^m I_+^{m-\alpha}[|r|^{(n-k-\ell-1)/2} \Phi_0(r)](R).$$

These equalities hold under assumptions of Corollary 4.8(ii) and follow from (4.31), (4.32). The formula (4.36) was exhibited in Theorem 1.1(ii).

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