RADON, COSINE AND SINE TRANSFORMS ON REAL HYPERBOLIC SPACE

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ABSTRACT. New pointwise inversion formulae are obtained for the d-dimensional totally geodesic Radon transform on the n-dimensional real hyperbolic space $1 \le d \le n-1$, in terms of polynomials of the Laplace-Beltrami operator and intertwining fractional integrals. Similar results are established for hyperbolic cosine and sine transforms.

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1. Introduction

The present article is motivated by the following problems.

Problem 1. Let $X = \mathbb{H}^n$, $n \geq 2$, be the n-dimensional real hyperbolic space, and let Ξ be the set of all d-dimensional totally geodesic submanifolds of X, $1 \leq d \leq n-1$. For the Radon transform $(Rf)(\xi)$, $\xi \in \Xi$, $f \in C_c^\infty(X)$, which was introduced by Helgason [15], the following inversion formula holds [18, p. 92]:

$$cf = Q_d(\Delta) R^* Rf, \qquad c = \frac{(-4\pi)^{d/2}\,\Gamma(n/2)}{\Gamma\big((n-d)/2\big)}, \quad d \ \text{even} \, . \tag{1.1}$$

Here R^* is the dual Radon transform,

$$Q_d(\Delta) = \big[\Delta + (d-1)(n-d)\big]\big[\Delta + (d-3)(n-d+2)\big] \ldots \big[\Delta + 1 \cdot (n-2)\big],$$

and Δ designates the Laplace-Beltrami operator on X. We wonder how to extend (1.1) to d odd.

Problem 2. Denote by $d(\cdot,\cdot)$ the geodesic distance on X and consider the following fractional integrals

$$(R^\alpha f)(\xi) = \gamma_{n,d}(\alpha) \int_X f(x) \big(\sinh d(x,\xi)\big)^{\alpha+d-n} dx, \qquad \xi \in \Xi; \tag{1.2}$$

$$(K^\alpha f)(x) = c_{n,\alpha} \int_X f(y) \big(\sinh d(x,y)\big)^{\alpha-n} dy, \qquad x \in X. \tag{1.3}$$

Suppose that $f \in C_c^\infty(X)$, $Re\alpha > 0$; $\alpha + d - n \neq 0, 2, 4, \ldots$ in (1.2), and $\alpha - n \neq 0, 2, 4, \ldots$ in (1.3). The normalizing coefficients $\gamma_{n,d}(\alpha)$ and $c_{n,\alpha}$ will be defined below. How to invert operators (1.2) and (1.3)? The case $\alpha = n - d + 1$ in (1.2), leading to the integral

$$(\mathfrak{C}f)(\xi) = \int_X f(x) \sinh d(x,\xi) dx, \tag{1.4}$$

is of particular interest in view of possible geometrical applications.

Berenstein and Casadio Tarabusi [2, 3] suggested to solve Problem 1 as follows: $f = Q(\Delta) S R^* Rf$, where S is a certain convolution operator on X and the polynomial $Q(\Delta)$ is understood in the sense of distributions. Helgason [18, pp. 99–102] reproduced the argument from [2] for $n = 2$ and presented the inversion formula in a pointwise form. This approach relies heavily on the Fourier analysis and injectivity of the Fourier transform on radial functions. Justification of the injectivity was not given in [18] (see Remark 4.9), so that in fact the result was established only in the distribution sense.

We give an alternative pointwise solution to Problem 1 without using the Fourier analysis as follows. Let

$$(\overset{*}{R}{}^{\alpha}\varphi)(x) = \gamma_{n,d}(\alpha) \int_{\Xi} \varphi(\xi) \big(\sinh d(x,\xi)\big)^{\alpha+d-n} d\xi, \qquad \alpha + d - n \neq 0, 2, 4, \ldots,$$

be the dual of (1.2). In the limit case $\alpha = 0$ it coincides with the dual Radon transform R^* up to a constant multiple (see Lemma 4.1). Denote

$$P_m(\Delta) = \prod_{i=1}^{m} \big[-\Delta + (2i-n)(2i-1)\big], \qquad m \in \mathbb{N}. \tag{1.5}$$

Theorem A. *Let* $\varphi = Rf$, $f \in C_c^\infty(X)$, $1 \le d \le n-1$.

(i) *If* n *is odd, then*

$$f = P_m(\Delta) \overset{*}{R}{}^{2m-d} \varphi \quad \forall m \ge d/2. \tag{1.6}$$

(ii) *If* n *is even, then (1.6) is applicable only for* $d/2 \le m \le n/2 - 1$, *and another inversion formula also holds:*

$$\begin{aligned} f = &-P_{n/2}(\Delta)\Big[\frac{2^{1-n}}{\pi^{n/2}\Gamma(n/2)} \int_{\Xi} \varphi(\xi) \log\big(\sinh d(x,\xi)\big) d\xi\Big] \\ &+ \frac{(-1)^{n/2}\Gamma\big((n+1)/2\big)}{\pi^{(n+1)/2}} \int_{\Xi} \varphi(\xi) d\xi. \end{aligned} \tag{1.7}$$

This theorem contains a series of inversion formulae. The case $m = d/2$ gives (1.1).

Example 1.1. For $n = 2$, $\varphi = Rf$ is inverted by

$$f(x) = \Delta\Big[\frac{1}{2\pi} \int_{\Xi} \varphi(\xi) \log\big(\sinh d(x,\xi)\big) d\xi\Big] + \frac{1}{2\pi} \int_{\Xi} \varphi(\xi) d\xi. \tag{1.8}$$

In the case $d = n-1$, the set Ξ can be identified with the one-sheeted hyperboloid $H^* = \{\eta \in \mathbb{R}^{n+1} \colon [\eta,\eta] = -1\}$, so that $\sinh d(x,\xi) = [x,\eta] = -x_1\eta_1 - \ldots - x_n\eta_n + x_{n+1}\eta_{n+1}$ (the Lorentzian inner product, [29]).

For the sake of completeness we mention some papers by Berenstein and Casadio Tarabusi [4], Gindikin [12, 13], Helgason [17], Ishikawa [19], Kurusa [20–22] related to inversion and characterization of geodesic transforms Rf for smooth f. Our approach differs from these papers. The case of rough functions $f \in L^p(X)$ was studied by Strichartz [29]. Inversion formulas for Rf, $f \in L^p(X)$, were obtained in [5, 6]. An analog of Theorem A for the compact case $X = S^n$ was obtained in [27]. Further references can be found in cited papers and books.

Let us proceed to Problem 2. It is known [18, pp. 92, 101] that

$$R^* Rf = K^d f. \tag{1.9}$$

More general operators (1.3) were introduced in [3] (up to normalization). They serve as analogs of Riesz potentials [18] (don't confuse with "Riesz transforms" and "Riesz potentials $\overset{\text{def}}{=} (-\Delta)^{-\alpha/2} f$" [1]). Spherical counterparts of (1.3)

$$(K_s^\alpha f)(x) \,=\, c_{n,\alpha} \int_{S^n} f(y)\big(\sin d(x,y)\big)^{\alpha-n} dy, \qquad x \in S^n,$$

were inverted explicitly in [26, 27] for all $\alpha > 0$ in the smooth and L^p settings (if $\alpha - n = 0, 2, 4, \dots$, the integrand must be multiplied by $\log[\sin d(x,y)]$). Much less has been done in the hyperbolic case, and the inversion problems for $K^\alpha f$ in the case of nonintegral α is still open. We hope this problem can be tackled using the results of Semyanistyi [28] as in [27] for $X = S^n$. If α is an even integer, $K^\alpha f$ can be inverted by a polynomial of Δ as in [18, p. 92]. For α odd, $K^\alpha f$ was inverted in [2, 3] by making use of the Fourier analysis and distribution theory. This argument was reproduced with slight changes in [18, pp. 99–102] with lack of justification of injectivity of the Fourier transform. We complete this gap by using results of Flensted-Jensen and Koornwinder [10].

We have already seen that Rf can be inverted without using inversion formula for $K^\alpha f$ with α odd, but we could not avoid this formula when treating analytic family (1.2). The latter contains Rf. Namely, $\lim\limits_{\alpha \to 0} R^\alpha f = cRf$, $c = \text{const}$. In the spherical set-up, $X = S^n$, (1.2) contains also the generalized cosine transforms (for $\alpha = n-d+1$) arising in integral geometry and stochastic theory of flat processes [11, 14]. In [27] we obtained inversion formulas for such operators with arbitrary α. In the present paper we show that inversion problem for (1.2) reduces to that for (1.3) and obtain pointwise inversion formulas for $\mathfrak{C}f$. Hopefully this transformation will find geometrical applications similar to those for $X = S^n$, but in the non-euclidean set-up.

Theorem B. *Let* $\psi = \mathfrak{C}f,\ f \in C_c^\infty(X),\ 1 \le d \le n-1,$

$$c = -\frac{2^{-n}\pi^{(1-n)/2}\,\Gamma\big((n-d)/2\big)}{\Gamma(n/2)\,\Gamma\big((n-d+1)/2\big)}.$$

(i) *If* n *is odd, then* $f = cP_{(n+1)/2}(\Delta)R^*\psi$ *(see (1.5)).*

(ii) *If* n *is even, then* $f = c(-\Delta+2-n)TQ(\Delta)R^*\psi$*, where*

$$Q(\Delta) = \prod_{i=1}^{n/2}[-\Delta + 2i(2i+1-n)],$$

$$(Tu)(x) = \begin{cases} \dfrac{1}{\pi\sigma_{n-2}}\displaystyle\int_X u(y)\,\frac{\coth d(x,y)}{\big(\sinh d(x,y)\big)^{n-2}}dy & \text{if } n>2,\\[2ex] \dfrac{1}{2\pi}\displaystyle\int_X u(y)[\coth d(x,y)-1]dy & \text{if } n=2.\end{cases}$$

Sections 2 and 3 contain preliminaries and auxiliary facts. In Section 4 we study fractional integrals (1.2), (1.3), and obtain main results.

2. Preliminaries

Let $E^{n,1}$, $n \ge 2$, be the pseudo-Euclidean space of points $x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1}$ with the inner product

$$[x,y] = -x_1y_1 - \ldots - x_ny_n + x_{n+1}y_{n+1}. \tag{2.1}$$

The n-dimensional hyperbolic space is realized as the upper sheet of the two-sheeted hyperboloid

$$X = \mathbb{H}^n = \{x \in \mathbb{E}^{n,1} : [x,x] = 1, x_{n+1} > 0\}. \tag{2.2}$$

Let Ξ be the set of all d-dimensional totally geodesic submanifolds $\xi \subset \mathbb{H}^n$, $1 \le d \le n-1$. We write $\mathbb{E}^{n,1} = \mathbb{R}^{n-d} \times \mathbb{E}^{d,1}$ where

$$\mathbb{R}^{n-d} = \mathbb{R}e_1 \times \ldots \times \mathbb{R}e_{n-d}, \qquad \mathbb{E}^{d,1} \sim \mathbb{R}^{d+1} = \mathbb{R}e_{n-d+1} \times \ldots \times \mathbb{R}e_{n+1},$$

$e_1, \dots, e_{n+1}$ being coordinate unit vectors. In the following $x_0 = (0, \dots, 0, 1)$ and $\xi_0 = \mathbb{H}^n \cap \mathbb{E}^{d,1} = \mathbb{H}^d$ denote the origins in X and Ξ respectively; $G = SO_0(n,1)$ is the identity component of the pseudo-orthogonal group $O(n,1)$ preserving the bilinear form (2.1); $K = SO(n)$ and $H = SO(n-d) \times SO_0(d,1)$ are the isotropy subgroups of x_0 and ξ_0, so that $X = G/K$, $\Xi = G/H$. One can write $f(x) \equiv$

$f(gK),\ \varphi(\xi) \equiv \varphi(gH),\ g \in G$. The geodesic distance between points x and y in X is defined by $d(x,y) = \cosh^{-1}[x,y]$.

We must take care of consistent normalization of various invariant measures to be used below. For each rotation group $SO(k)$ an invariant measure $d\gamma$ will be normalized by $\int_{SO(k)} d\gamma = 1$, so that for the relevant unit sphere $S^{k-1} \subset \mathbb{R}^k$ we have $\int_{S^{k-1}} f(\omega)d\omega = \sigma_{k-1}\int_{SO(k)} f(\gamma\omega)d\gamma,\ \sigma_{k-1} = |S^{k-1}| = 2\pi^{k/2}/\Gamma(k/2)$. An invariant measure dg on G will be normalized by

$$\int_G f(ge_{n+1})dg = \int_X f(x)dx \tag{2.3}$$

where dx stands for the usual Riemannian measure on $X = G/K$ [25, 30]. Given $x \in X, \xi \in \Xi$, we denote by $r_x, r_\xi (\in G)$ arbitrary pseudo-rotations such that $r_x x_0 = x,\ r_\xi \xi_0 = \xi$, and write $f_\xi(x) = f(r_\xi x), \varphi_x(\xi) = \varphi(r_x \xi)$. As is [6], let

$$g_\theta = \begin{pmatrix} \cosh\theta & 0 & \sinh\theta \\ 0 & I_{n-1} & 0 \\ \sinh\theta & 0 & \cosh\theta \end{pmatrix}, \qquad d\nu(\theta) = (\sinh\theta)^{n-d-1}(\cosh\theta)^d d\theta, \tag{2.4}$$

$$(R_\theta f)(\xi) = \int_H f_\xi(hg_\theta x_0)dh = \int_{d(x,\xi)=\theta} f(x)dm(x), \tag{2.5}$$

$$(R^*_\theta \varphi)(x) = \int_K \varphi_x(kg_\theta^{-1}\xi_0)dk = \int_{d(x,\xi)=\theta} \varphi(\xi)d\mu(\xi), \tag{2.6}$$

where $dm(x)$ and $d\mu(\xi)$ are the relevant normalized measures. By setting $\theta = 0$ in (2.5), (2.6), we obtain the following representations of the Radon transform and its dual:

$$(Rf)(\xi) = \int_{SO_0(d,1)} f_\xi(\gamma x_0)d\gamma, \qquad (R^*\varphi)(x) = \int_K \varphi_x(k\xi_0)dk. \tag{2.7}$$

Lemma 2.1 [6, 19]. *If $f \in L^1(X),\ \varphi \in L^1(\Xi)$, then*

$$\int_X f(x)dx = \sigma_{n-d-1}\int_0^\infty \Big[\int_H f(hg_\theta x_0)dh\Big]d\nu(\theta), \tag{2.8}$$

$$\int_\Xi \varphi(\xi)d\xi = \sigma_{n-d-1}\int_0^\infty \Big[\int_K \varphi(kg_\theta^{-1}\xi_0)dk\Big]d\nu(\theta). \tag{2.9}$$

Lemma 2.2. *Let*

$$(Af)(\xi) = \int_X a\big(\sinh d(x,\xi)\big)f(x)dx, \ \ (A^*\varphi)(x) = \int_\Xi a\big(\sinh d(x,\xi)\big)\varphi(\xi)d\xi.$$

Then $R^*Af = A^*Rf$ *provided that either side of this equality is finite for* a *and* f *replaced by* $|a|$ *and* $|f|$ *respectively.*

Proof. By (2.7), (2.8),

$$(R^*Af(x) = \int_K (Af)(r_x k\xi_0)dk$$
$$= \sigma_{n-d-1}\int_0^\infty a(\sinh\theta)d\nu(\theta)\int_K \Big[\int_H f(r_x khg_\theta e_{n+1})dh\Big]dk. \quad (2.10)$$

In order to transform $\int_K[\dots]$ we introduce the averaging operator

$$(M_t f)(x) = \frac{(t^2-1)^{(1-n)/2}}{\sigma_{n-1}}\int_{\{y\in X:\,[x,y]=t\}} f(y)d\sigma(y), \qquad t>0. \quad (2.11)$$

One can write

$$(M_{[z,e_{n+1}]}f)(x) = \int_K f(r_x kz)dk \qquad \forall z\in X, \quad (2.12)$$

and therefore

$$\int_{\mathbb{H}^d}(M_{[g_\theta e_{n+1},\eta]}f)(x)d\eta = (R^*_\theta Rf)(x). \quad (2.13)$$

Indeed, by (2.12) the left hand side of (2.13) reads

$$\int_{\mathbb{H}^d} d\eta\int_K f(r_x kg_\theta^{-a}\eta)d\eta = \int_K (Rf)(r_x kg_\theta^{-1}\xi_0)dk = (R^*_\theta Rf)(x).$$

Now let us return to (2.10). By (2.12),

$$\int_K[\dots] = \int_H dh\int_K f(r_x khg_\theta x_0)dk = \int_H (M_{[hg_\theta x_0,x_0]}f)(x)dh$$
$$= \int_{SO_0(d,1)}(M_{[g_\theta x_0,\gamma x_0]}f)(x)d\gamma = \int_{\mathbb{H}^d}(M_{[g_\theta x_0,\eta]}f)(x)d\eta.$$

Hence (2.13) and (2.9) yield

$$(R^*Af)(x) = \sigma_{n-d-1}\int_0^\infty a(\sinh\theta)(R^*_\theta Rf)(x)d\nu(\theta) = (A^*Rf)(x).$$

□

The following statement, which is a consequence of the duality

$$\int_\Xi \varphi(\xi)(Rf)(\xi)d\xi = \int_X f(x)(R^*\varphi)(x)dx \quad (2.14)$$

[16], plays a crucial role in our approach.

Lemma 2.3. *Let $\alpha > 0$, $c_\alpha = \pi^{d/2}\sigma_{n-d-1}\Gamma(\alpha/2)/\sigma_{n-1}\Gamma\big((d+\alpha)/2\big)$. Then*

$$\int_\Xi (Rf)(\xi)\big(\sinh d(x_0,\xi)\big)^{\alpha+d-n} d\xi = c_\alpha \int_X f(y)(\sinh d(x_0,y)\big)^{\alpha+d-n} dy \tag{2.15}$$

provided the integral in either side converges for f replaced by $|f|$.

Proof. One can derive (2.15) from (2.14) as we did in [6, Corollary 3.3]. A simpler proof is as follows. Denote by $I_\ell f$ ($I_r f$) the left hand side (the right hand side) of (2.15). Let $f^0(x) = \int_K f(kx)dk$ be the "radialization" of f. Since $I_\ell f = I_\ell f^0$ and $I_r f = I_r f^0$ (change variables $y \to ky$, $\xi \to k\xi$ and integrate in $k \in K$), then it suffices to prove (2.15) for radial functions $f \equiv f(\cosh\omega)$, $\omega = d(x, x_0)$. For such f,

$$(Rf)(\xi) = \sigma_{d-1}(\cosh\theta)^{1-d} \int_{\cosh\theta}^\infty f(s)(s^2-\cosh^2\theta)^{\frac{d}{2}-1} ds, \qquad \theta = d(x_0,\xi) \tag{2.16}$$

([18, p. 87], [6, Lemma 3.1]). Now we plug (2.16) in (2.15) and write $I_\ell f$ and $I_r f$ as one-dimensional integrals using the formulae

$$\int_\Xi \varphi\big(\cosh d(x_0,\xi)\big)d\xi = \sigma_{n-d-1} \int_0^\infty \varphi(\cosh\theta)d\nu(\theta), \tag{2.17}$$

$$\int_X f\big(\cosh d(x_0,y)\big)dy = \sigma_{n-1} \int_0^\infty f(\cosh\omega)(\sinh\omega)^{n-1}d\omega \tag{2.18}$$

((2.17) is a special case of (2.9)). The coincidence of these one-dimensional integrals for $\varphi(\cosh d(x_0,\xi)) = (Rf)(\xi)$ can be checked in an obvious way. □

3. Some aspects of the Fourier analysis on $\mathbb{H}^n$

This area is very large. It was treated by many authors in various general contexts (see, e.g., [7, 10, 16] for references). For our purposes we select only a few facts which are exhibited on an elementary level. The Laplace-Beltrami operator Δ on $X = \mathbb{H}^n$ is the tangential part of the de'Alembertian

$$\Box = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} - \frac{\partial^2}{\partial x_{n+1}^2}$$

on $\Gamma = \{x \in \mathbb{E}^{n,1}\colon [x,x] > 0,\ \ x_{n+1} > 0\}$. Namely, if

$$x = (\sinh r\,\omega, \cosh r) \in \Gamma, \quad r \ge 0, \quad \omega \in S^{n-1},$$

then

$$\Box = -\Big(\frac{\partial^2}{\partial t^2} + \frac{n}{t}\frac{\partial}{\partial t}\Big) + \frac{1}{t^2}\Delta, \qquad \Delta = \frac{\partial^2}{\partial r^2} + 2\rho\coth r\frac{\partial}{\partial r} + \frac{1}{\sinh^2 r}\Delta_s,$$

Δ_s being the Laplace-Beltrami operator on S^{n-1}, $t > 0$, $\rho = \frac{n-1}{2}$. Let $C = \{x\colon [x,x] = 0,\ \ x_{n+1} > 0\}$ be the boundary cone of Γ; $B = \{x \in C\colon x_{n+1} = 1\}$. For $\omega \in S^{n-1} \subset \mathbb{R}^n$ we set $b(\omega) = (\omega, 1) \in B$. Clearly, $[x, b(\omega)] > 0$ for $x \in X$, $\omega \in S^{n-1}$.

Lemma 3.1 [7, p. 82]. *Let $\mu \in \mathbb{C}$, $\omega \in S^{n-1}$. Then $x \to [x, b(\omega)]^{\mu}$ is an eigenfunction of Δ with the eigenvalue $\mu(\mu - 1 + n)$. In particular, for $\mu = i\lambda - \rho$, $\rho = \frac{n-1}{2}$, we have*

$$\Delta[x, b(\omega)]^{i\lambda-\rho} = -(\lambda^2 + \rho^2)[x, b(\omega)]^{i\lambda-\rho}.$$

Corollary 3.2. *The spherical function*

$$\Phi_\lambda(x) \overset{\text{def}}{=} \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} [x, b(\omega)]^{i\lambda-\rho} d\omega, \tag{3.1}$$

is an eigenfunction of Δ with the eigenvalue $-(\lambda^2 + \rho^2)$.

In polar coordinates $x = \text{cosh}r\, e_{n+1} + \text{sinh}r\, \omega'$, $r \geq 0$, $\omega' \in S^{n-1}$, we have [9, 3.7(7)]

$$\begin{aligned} \Phi_\lambda(x) &= \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_0^\pi (\text{cosh}r \ - \text{sinh}r \cos\theta)^{i\lambda-\rho} (\sin\theta)^{2\rho-1} d\theta = \\ &= 2^{\rho-1/2} \Gamma(\rho + 1/2)(\text{sinh}r)^{1/2-\rho} P^{1/2-\rho}_{i\lambda-1/2}(\text{cosh}r), \end{aligned} \tag{3.2}$$

$P^\mu_\nu(z)$ being the associated Legendre function. We redenote $\Phi_\lambda(x) \equiv \Phi_\lambda(r)$ up to abuse of notation.The Fourier transform of $f \in C^\infty_c(X)$ is defined by

$$\tilde{f}(\lambda, \omega) = \int_X f(x)[x, b(\omega)]^{i\lambda-\rho} dx, \tag{3.3}$$

$\lambda \in \mathbb{R}$, $\omega \in S^{n-1}$. If f is radial, i.e.$f(x) = f_0(x_{n+1})$, then $\tilde{f}(\lambda, \omega) \equiv \tilde{f}(\lambda)$ is independent of ω, and

$$\tilde{f}(\lambda) = \sigma_{n-1} \int_0^\infty f_0(\text{cosh}r)\Phi_\lambda(r)(\text{sinh}r)^{n-1} dr = \int_X f(x)\Phi_\lambda(x) dx. \tag{3.4}$$

This expression is called the spherical transform of the radial function f. More general Fourier-Jacobi transforms and related convolution operators were studied in [10].

Lemma 3.3. *Let $\Phi_0(x) = \Phi_\lambda(x)|_{\lambda=0}$,*

$$(Kf)(x) = \int_X f(y)k([x, y]) dy \overset{\text{def}}{=} (f * k)(x), \qquad x \in X. \tag{3.5}$$

Suppose that

$$\text{(a)} \int_X |f(x)|\Phi_0(x) dx < \infty, \quad \textit{and} \quad \text{(b)} \int_X |k(x_{n+1})|\Phi_0(x) dx < \infty.$$

Then for almost all $\omega \in S^{n-1}$ *and all* $\lambda \in \mathbb{R}$*, the Fourier transforms* $\tilde{k}(\lambda)$, $\tilde{f}(\lambda,\omega)$, $(Kf)^{\sim}(\lambda,\omega)$ *are finite and*

$$(Kf)^{\sim}(\lambda,\omega) = \tilde{k}(\lambda)\tilde{f}(\lambda,\omega). \tag{3.6}$$

Proof. This statement is well known, although it is not so easy to find a simple self-contained proof of it in the literature (cf. [18, p. 100] for $n = 2$). The proof is as follows. By changing the order of integration (this step will be justified below), we get

$$(Kf)^{\sim}(\lambda,\omega) \overset{(!)}{=} \int_X f(y)dy \int_X [x, b(\omega)]^{i\lambda-\rho} k([x,y])dx. \tag{3.7}$$

Let $x = r_y z$ (as above, $r_y \in G$, $r_y x_0 = y$), $r_y^{-1} b(\omega) = \xi \in \Gamma$. Then the inner integral can be written as

$$\int_X k(z_{n+1})[z,\xi]^{i\lambda-\rho}dz = \xi_{n+1}^{i\lambda-\rho} \int_X k(z_{n+1})[z, b(\tilde{\omega})]^{i\lambda-\rho}dz = \xi_{n+1}^{i\lambda-\rho}\tilde{k}(\lambda)$$

(by homogeneity, $\xi = \xi_{n+1} b(\tilde{\omega})$ for some $\tilde{\omega} \in S^{n-1}$). Since $\xi_{n+1} = [\xi, e_{n+1}] = [y, b(\omega)]$, the result follows. In order to justify (!) in (3.7) we note that for non-negative f and k, as above we have $(Kf)^{\sim}(0,\omega) = \tilde{f}(0,\omega)\tilde{k}(0)$. This expression is finite for almost all ω, because $\hat{k}(0) < \infty$ (by (b)) and $\int_{S^n} \tilde{f}(0,\omega)d\omega = \sigma_{n-1}\int_X f(y)\Phi_0(y)dy < \infty$ (by (a)). Thus the repeated integral in (3.7) is absolutely convergent, and the Fubini theorem is applicable. □

Example 3.4. Let $r = d(x_0, x)$,

$$k_+^\alpha(x) = c_{n,\alpha}(\mathrm{sinh}r)^{\alpha-n}, \qquad k_-^\alpha(x) = c_{n,\alpha}(\mathrm{sinh}r)^{\alpha-n}\mathrm{cosh}r,$$

$$c_{n,\alpha} = \frac{2^{-\alpha}\Gamma\big((n-\alpha)/2\big)}{\pi^{n/2}\Gamma(\alpha/2)};$$

$$h^\alpha(x) = d_{n,\alpha}(\mathrm{cosh}r - 1)^{\alpha/2-1}(\mathrm{sinh}r)^{2-n}, \qquad d_{n,\alpha} = \frac{2^{-1-\alpha/2}\Gamma\big((n-\alpha)/2\big)}{\pi^{n/2}\Gamma(\alpha/2)}.$$

Then for $\lambda \in \mathbb{R}$,

$$\tilde{k}_\pm^\alpha(\lambda) = \frac{\Gamma\Big(\frac{n\pm1}{4} - \frac{\alpha}{2} + \frac{i\lambda}{2}\Big)\Gamma\Big(\frac{n\pm1}{4} - \frac{\alpha}{2} - \frac{i\lambda}{2}\Big)}{2^\alpha\,\Gamma\Big(\frac{n\pm1}{4} + \frac{i\lambda}{2}\Big)\Gamma\Big(\frac{n\pm1}{4} - \frac{i\lambda}{2}\Big)}, \qquad 0 < Re\,\alpha < \frac{n\pm1}{2}, \tag{3.8}$$

$$\tilde{h}^\alpha(\lambda) = \frac{\Gamma\Big(\frac{n-1}{2} - \frac{\alpha}{2} + i\lambda\Big)\Gamma\Big(\frac{n-1}{2} - \frac{\alpha}{2} - i\lambda\Big)}{\Gamma\Big(\frac{n-1}{2} + i\lambda\Big)\Gamma\Big(\frac{n-1}{2} - i\lambda\Big)}, \qquad 0 < Re\,\alpha < n-1, \tag{3.9}$$

so that

$$\tilde{k}_+^\alpha(\lambda)\tilde{k}_-^\alpha(\lambda) = 2^{2\alpha}\tilde{h}^{2\alpha}(\lambda), \qquad 0 < Re\,\alpha < \frac{n-1}{2}. \tag{3.10}$$

The equality (3.8) follows from (3.4) and (3.2) by [24, 2.17.1(15)] (for $\tilde{k}_-(\lambda)$ one should also use [9, 3.8(2)]); $\tilde{h}(\lambda)$ can be calculated with the aid of [24, 2.17.3(6)].

We recall Young's inequality

$$\|f * k\|_q \le \|f\|_p \|k\|_r, \quad 1 \le p \le q \le \infty, \quad 1 - p^{-1} + q^{-1} = r^{-1}, \tag{3.11}$$
$$\|k\|_r^r = \sigma_{n-1} \int_1^\infty |k(t)|^r (t^2-1)^{n/2-1} dt.$$

The following statement is a particular case of Theorem 3.5 from [10].

Theorem 3.5. *The spherical transform (3.4) is injective on the space $L^p_{rad}(X)$ of radial functions from $L^p(X)$ provided $1 \le p \le 2$.*

4. Intertwining fractional integrals. Main results

Owing to G-invariance, x_0 in (2.15) can be replaced by arbitrary $x \in X$. The resulting equality leads to the following fractional integrals. For $Re\,\alpha > 0$ let

$$(K^\alpha f)(x) = \begin{cases} c_{n,\alpha} \int_X f(y) \big(\sinh d(x,y)\big)^{\alpha-n} dy, & \alpha - n \ne 0, 2, 4, \dots, \\ c'_{n,\alpha} \int_X f(y) \big(\sinh d(x,y)\big)^{\alpha-n} \log\big(\sinh d(x,y)\big) dy & \text{otherwise}, \end{cases} \tag{4.1}$$

$$c_{n,\alpha} = \frac{2^{-\alpha}\Gamma\big((n-\alpha)/2\big)}{\pi^{n/2}\,\Gamma(\alpha/2)}, \qquad c'_{n,\alpha} = \frac{2^{1-\alpha}(-1)^{(\alpha-n)/2+1}}{\big((\alpha-n)/2\big)!\,\pi^{n/2}\Gamma(\alpha/2)}.$$

The normalizing coefficient $c_{n,\alpha}$ has been chosen according to (3.8), so that the symbol $\tilde{k}_+^\alpha(\lambda)$ of K^α behaves like $(i\lambda)^{-\alpha}$ at infinity. The second line in (4.1) then follows by continuity $\lim\limits_{\alpha \to n+2m} K^\alpha f = K^{n+2m} f$ provided

$$\int_X f(x) x^j dx = 0 \quad \forall\, |j| \le m. \tag{4.2}$$

By keeping in mind the left hand side of (2.15), for $Re\,\alpha > 0$ we set

$$(R^\alpha f)(\xi) = \int_X f(x) r_\alpha(x,\xi) dx, \qquad (\overset{*}{R}{}^\alpha \varphi)(x) = \int_\Xi \varphi(\xi) r_\alpha(x,\xi) d\xi. \tag{4.3}$$

The kernel $r_\alpha(x,\xi)$ is defined as follows. If $\alpha + d - n \ne 0, 2, 4, \dots$, then

$$r_\alpha(x,\xi) = \gamma_{n,d}(\alpha)\big(\sinh d(x,\xi)\big)^{\alpha+d-n},$$

$$\gamma_{n,d}(\alpha)=\frac{2^{-\alpha-d}\,\Gamma((n-\alpha-d)/2)\,\Gamma((n-d)/2)}{\pi^{n/2}\Gamma(n/2)\,\Gamma(\alpha/2)}.$$

If $\alpha+d-n=0,2,4,\dots$, then

$$r_\alpha(x,\xi)=\gamma'_{n,d}(\alpha)\big(\sinh d(x,\xi)\big)^{\alpha+d-n}\log\big(\sinh d(x,\xi)\big),$$

$$\gamma'_{n,d}(\alpha)=\frac{2^{1-\alpha-d}(-1)^{(\alpha+d-n)/2+1}\,\Gamma((n-d)/2)}{\pi^{n/2}\,\Gamma(n/2)\,\Gamma(\alpha/2)\,((\alpha+d-n)/2)!}.$$

Analytic families $\{R^\alpha\}$ and $\{\overset{*}{R}{}^\alpha\}$ include the Radon transform and its dual in the following sense.

Lemma 4.1. *If $f(x)$ and $\varphi(\xi)$ are compactly supported continuous functions, then*

$$\lim_{\alpha\to 0}R^\alpha f=cRf,\qquad \lim_{\alpha\to 0}\overset{*}{R}{}^\alpha\varphi=cR^*\varphi,\qquad c=\frac{2^{-d}\Gamma\big((n-d)/2\big)}{\pi^{d/2}\Gamma(n/2)}. \tag{4.4}$$

Proof. By (2.8),

$$\begin{aligned}(R^\alpha f)(\xi)\;&=\;\gamma_{n,d}(\alpha)\int_X f_\xi(x)\big(\sinh d(x,\xi_0)\big)^{\alpha+d-n}dx\\ &=\gamma_{n,d}(\alpha)\sigma_{n-d-1}\int_0^\infty(\sinh\theta)^{\alpha-1}(\cosh\theta)^d d\theta\\ &\times\int_H f_\xi(h[e_1\sinh\theta+e_{n+1}\cosh\theta])dh.\end{aligned}$$

By setting $t=\sinh\theta$ and passing to the limit as $\alpha\to 0$, we obtain the first equality in (4.4). The second one follows similarly by (2.9). □

Theorem 4.2. *For $\alpha>0$, $\alpha+d-n\neq 0,2,4,\dots,$*

$$\overset{*}{R}{}^\alpha Rf=K^{\alpha+d}f,\qquad R^*R^\alpha f=K^{\alpha+d}f \tag{4.5}$$

provided that either part of the corresponding equality is finite for f replaced by $|f|$. For $\alpha=n-d$,

$$\overset{*}{R}{}^{n-d}Rf=\overset{*}{R}R^{n-d}f=K^nf+c_f,\;\; c_f=\frac{\psi\big((n-d)/2\big)-\psi(n/2)}{2^n\pi^{n/2}\,\Gamma(n/2)}\int_X f(x)dx, \tag{4.6}$$

with the psi function $\psi(z)=\Gamma'(z)/\Gamma(z)$.

Proof. Let $\alpha+d-n\neq 0,2,4,\dots$. The first equality in (4.5) follows from (2.15) due to normalization. The second one holds by Lemma 2.2, according to which

$$\overset{*}{R}{}^\alpha Rf=R^*R^\alpha f\quad\forall\alpha>0. \tag{4.7}$$

Let us prove (4.6). We set $\varphi = Rf$ so that $\int_X = \int_\Xi \varphi$ (put $\alpha = n-d$ in (2.15)). For $\alpha + d \neq n$ but close to n one can write $\overset{*}{R}{}^\alpha \varphi = K^{\alpha+d} f$ as

$$\gamma_{n,d}(\alpha) \int_\Xi \varphi(\xi) \Big[\big(\sinh d(x,\xi)\big)^{\alpha+d-n} - 1 \Big] d\xi =$$
$$= c_{n,\alpha+d} \int_X f(y) \Big[\big(\sinh d(x,y)\big)^{\alpha+d-n} - 1 \Big] dy + [c_{n,\alpha+d} - \gamma_{n,d}(\alpha)] \int_X f(y) dy.$$

It remains to pass to the limit as $\alpha \to n-d$ and make use of (4.7). □

Remark 4.3. For $\alpha + d - n = 2m, \; m = 0,1,2,\ldots,$ (4.5) holds without additional terms provided that f obeys (4.2).

Theorem 4.4. *Let $f \in C_c^\infty(X), \; \alpha \geq 2, \; \Delta_\alpha = -\Delta + (\alpha - n)(\alpha - 1)$. If $\alpha - n \neq 0, 2, 4, \ldots,$ then*

$$\Delta_\alpha K^\alpha f \;=\; K^{\alpha-2} f \qquad (K^0 f = f). \tag{4.8}$$

For $\alpha = n + 2m, \; m = 0,1,2,\ldots,$ this equality holds provided

$$\int_X f(y) y^j dy = 0 \quad \forall |j| = 0, 2, \ldots, 2m. \tag{4.9}$$

For $\alpha = n$,

$$-\Delta K^n f = a_f + \begin{cases} K^{n-2} f, & n > 2 \\ f, & n = 2, \end{cases} \qquad a_f \;=\; \frac{2^{1-n}(n-1)}{\pi^{n/2}\,\Gamma(n/2)} \int_X f(x) dx. \tag{4.10}$$

Proof. The proof mimics that of Theorem 1.5 from [18, p. 91] (with slight changes in the case $\alpha = n + 2m$). The key point is the "Darboux equation"

$$\Delta_x (M_{\cosh r} f)(x) = \Delta_{rad} (M_{\cosh r} f)(x) \tag{4.11}$$

where Δ_x designates the Laplace-Beltrami operator Δ in the x-variable, and $\Delta_{rad} = \frac{\partial^2}{\partial r^2} + (n-1)\coth r \frac{\partial}{\partial r}$ is the radial part of Δ. The equality (4.11) is transparent in Fourier terms by taking into account that

$$(\Delta f)^\sim(\lambda, \omega) = -(\lambda^2 + \rho^2) \tilde{f}(\lambda, \omega), \qquad \rho = (n-1)/2, \tag{4.12}$$

$$(M_{\cosh r} f)^\sim(\lambda, \omega) = \Phi_\lambda(f) \tilde{f}(\lambda, \omega), \qquad \Delta_{rad} \Phi_\lambda(r) = -(\lambda^2 + \rho^2) \Phi_\lambda(r); \tag{4.13}$$

see (3.2) and [23, Section 4]. Alternative proofs of (4.11) are indicated in [18, p. 91]. Let us sketch the proof of (4.10). By (4.1) and (4.11), integration by parts yields

$$\Delta K^n f = c'_{n,n}\sigma_{n-1}\int_0^\infty (\text{sinh}r)^{n-1}\log(\text{sinh}r)\Delta_{rad}M_{\text{cosh}r}f\,dr =$$
$$= -\frac{2^{1-n}\sigma_{n-1}}{\pi^{n/2}\,\Gamma(n/2)}\Big[(n-2)\int_0^\infty (\text{sinh}r)^{n-3}M_{\text{cosh}r}f\,dr$$
$$+ (n-1)\int_0^\infty (\text{sinh}r)^{n-1}M_{\text{cosh}r}f\,dr\Big]$$

(for $n = 2$ the first term in brackets should be replaced by f).This gives (4.10). □

Corollary 4.5. *Let* $f \in C_c^\infty(X)$, $P_\ell(\Delta) = \Delta_{2\ell}\Delta_{2\ell-2}\ldots\Delta_2$, $\ell \in \mathbb{N}$. *Then* $P_\ell(\Delta)K^{2\ell}f = f$. *If* $2\ell = n + 2m$, $m = 0, 1, 2, \ldots$, *this pointwise inversion formula holds provided (4.9).*

For $2\ell \le n - 1$, this statement was proved by Helgason [18 pp. 91–93].

Our next goal is to obtain a pointwise inversion formula for $K^\alpha f$, $f \in C_c^\infty(X)$, if α is odd. By Theorem 4.4 the problem reduces to $\alpha = 1$. We start with the case $n > 2$ and consider the kernels in Example 3.4. Denote $K_-^\alpha f = f * k_-^\alpha$, $H^\alpha f = f * h^\alpha$.

Lemma 4.6. *Let* $f \in C_c^\infty(X)$, $n > 2$, $0 < Re\,\alpha < n/2$. *Then*

$$K_-^\alpha K^\alpha f = H^{2\alpha} f. \tag{4.14}$$

Proof. It suffices to prove that

$$k_+^\alpha * k_-^\alpha = h^{2\alpha}. \tag{4.15}$$

The convolution $(k_+^\alpha * k_-^\alpha)(x)$ can be transformed to

$$I(\alpha) = c\int_0^\infty (\text{sinh}r)^{\alpha-1}\text{cosh}r\,dr\int_0^\pi \left[(\text{cosh}r\,x_{n+1} - \text{sinh}r\cos\theta)^2 - 1\right]^{\frac{\alpha-n}{2}}(\sin\theta)^{n-2}d\theta,$$

$c = const.$ We could not compute this integral directly and had to take a longer way. One can show that $I(\alpha)$ is well-defined for $0 < Re\,\alpha < n/2$ and represents an analytic function of α in this strip (we omit these technicalities). Thus, it suffices to prove (4.15) for $0 < Re\,\alpha < \delta$ and some $\delta > 0$. By Lemma 3.3 and Theorem 3.5, the latter will follow from (3.10) if both sides of (4.15) belong to $L^q(X)$ for some $q \in [1, 2]$. Assume for a while that α is real and less than 1. A direct calculation shows that $h^{2\alpha} \in L^q(X)$ provided

$$1 - \frac{2\alpha}{n} < \frac{1}{q} < 1 - \frac{\alpha}{n-1}. \tag{4.16}$$

The left hand side of (4.15) will be treated by Young's inequality (3.11). The function k_-^α does not belong to $L^p(X)\ \forall p \in [1,\infty]$. Hence we set $k_-^\alpha(x) \equiv k_-^\alpha(r) = \varphi_1(r)+\varphi_2(r)$, where $\varphi_1(r) = k_-^\alpha(r)$ for $r \le 2$ and 0 for $r > 2$, $r = d(x_0, x)$. Clearly, $\varphi_1 \in L^{p_1}(X)$, $p_1^{-1} > 1-\alpha/n$, and $\varphi_2 \in L^{p_2}(X)$, $p_2^{-1} < 1-\alpha/(n-1)$. By (3.11), for $1 \le p \le q \le \infty$,

$$K^\alpha\colon L^p(X) \to L^q(X), \quad 0<\alpha<1+\frac{n-1}{p}, \quad \frac{1}{p}-\frac{\alpha}{n}<\frac{1}{q}<\frac{1}{p}-\frac{\alpha-1}{n-1}. \tag{4.17}$$

Hence $K^\alpha\varphi_i \in L^{q_i}(X), \quad i=1,2$, if

$$1-\frac{2\alpha}{n}<\frac{1}{q_1}<1-\frac{\alpha}{n}-\frac{\alpha-1}{n-1}, \qquad 1-\frac{\alpha}{n-1}-\frac{\alpha}{n}<\frac{1}{q_2}<1-\frac{2\alpha-1}{n-1}.$$

These intervals contain (4.16) for $\alpha \le 1$. Furthermore, the interval (4.16) lies in $[\frac{1}{2},1)$ for $\alpha \le n/4$. It means that for $0 < Re\,\alpha < \min(1, n/4)$ there exists $q \in (1,2)$ such that both sides of (4.15) belong to $L^q(X)$. This completes the proof. □

Corollary 4.7. *For $f \in C_c^\infty(X)$, $n > 2$,*

$$(-\Delta+2-n)K_-^1K^1 f = f. \tag{4.18}$$

Proof. By (4.14), $K_-^1K^1 = H^2 = K^2$ (see Example 3.4). It remains to apply Theorem 4.4. □

In the case $n = 2$ one can proceed as in [18, pp. 101, 102]. We reproduce briefly this argument with slight corrections related to justification. Since $k_-^1(r) = \pi^{-1}\coth r \notin L^1(X)$ we "correct" this kernel (as in [2]) by setting $\sigma = (2\pi)^{-1}(\coth r - 1) \in L^p(X), \quad 1 \le p < 2$. By (4.17), $\omega = k_+^1 * \sigma = K^1\sigma \in L^q(X) \quad \forall q \in (1,\infty)$, and therefore by Lemma 3.3, as in [18, p. 102] we have $\tilde\omega(\lambda) = \tilde k_+^1(\lambda)\tilde\sigma(\lambda) = (\lambda^2+1/4)^{-1}$. Now let $f \in C_c^\infty(X)$,

$$(Sf)(x) = (f*\sigma)(x) = \frac{1}{2\pi}\int_X f(y)\big[\coth d(x,y) - 1\big]\,dy. \tag{4.19}$$

Theorem 4.8. *If $f \in C_c^\infty(X)$, $n = 2$, then*

$$-\Delta SK^1 f = f. \tag{4.20}$$

Proof. As in [18, p. 102], it suffices to prove (4.20) for f radial. By (4.19), $-\Delta SK^1 f = -\Delta(f*\omega) = (-\Delta f)*\omega$. By Lemma 3.3 and (4.12), $(-\Delta f)*\omega)^\sim(\lambda) =$

$\tilde{f}(\lambda)$. Since f and $(-\Delta f) * \omega$ belong to $L^q(X)$ $\forall q \in (1, \infty)$, by Theorem 3.5 we get $-\Delta f * \omega = f$. This gives (4.20). □

Remark 4.9. In the proof presented in [18, p. 102] injectivity of the spherical transform was not justified. The argument on p. 100 of [18], cannot be applied directly, because the functions σ and k_+^1 are not compactly supported. The formula (62) in [18, p. 102] can be justified by making use of Young's inequality and Lemma 3.3 (the assertion $\tau \equiv (\sinh r)^{-1} \in L^1(X)$ before this formula is wrong).

Proof of Theorem A. The validity of (1.6) follows from (4.5) and Corollary 4.5. For n even, by (4.6) and Theorem 4.4 we have

$$\begin{aligned} P_{n/2}(\Delta) \overset{*}{R}{}^{n-d} \varphi &= P_{n/2}(\Delta)\big[K^n f + c_f\big] \overset{(4.10)}{=} \Delta_2 \dots \Delta_{n-2}\big[K^{n-2} f + a_f\big] \\ &= f - \frac{(-1)^{n/2}\,\Gamma\big((n+1)/2\big)}{\pi^{(n+1)/2}} \int_X f(x)dx. \end{aligned}$$

This gives (1.7). □

Proof of Theorem B. (i) By the second equality in (4.5),

$$K^{n+1} f = \gamma_{n,d}(n-d+1) R^* \psi = c R^* \psi, \tag{4.21}$$

and the result follows from (4.8).

(ii) In this case $n+1$ is odd. Hence by (4.21) and (4.8),

$$K^1 f = c \Delta_3 \Delta_5 \dots \Delta_{n+1} R^* \psi = c Q(\Delta) R^* \psi.$$

It remains to apply inversion formulae (4.18) and (4.20). □

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