# A POSITIVE SOLUTION TO THE GENERALIZED BUSEMANN-PETTY PROBLEM FOR TWO AND THREE-DIMENSIONAL SECTIONS

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ABSTRACT. The generalized Busemann-Petty problem asks: If $K$ and $L$ are origin-symmetric convex bodies in $\mathbb{R}^n$, and the volume of $K \cap \xi$ is smaller than the volume of $L \cap \xi$ for every $i$-dimensional subspace $\xi$, $1 \le i \le n-1$, does it follow that the volume of $K$ is also smaller than the volume of $L$ ? For $i = 1$, the affirmative answer is obvious. Bourgain and Zhang gave a negative answer for all $3 < i \le n-1$. We show that in the cases (a) $i = 2$, $n \ge 3$, and (b) $i = 3$, $n \ge 4$, the answer is positive. This completes the problem.

## 1. INTRODUCTION

Let $G_{n,i}$ be the Grassmann manifold of $i$-dimensional subspaces $\xi$ of $\mathbb{R}^n$, and $vol_i(\cdot)$ denote the $i$-dimensional volume function, $1 \le i \le n$. Is it true that for origin-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$, the inequality

$$vol_i(K \cap \xi) \le vol_i(L \cap \xi) \quad \forall \xi \in G_{n,i} \tag{1.1}$$

implies

$$vol_n(K) \le vol_n(L) \quad ? \tag{1.2}$$

This question is known as *the generalized Busemann-Petty problem* (GBP). For $i = n-1$, the problem was posed by Busemann and Petty [BP] in 1956 and has a long history (see [G2]). In the past decade a good deal of studying various aspects of this problem was made by Bourgain, Gardner, Koldobsky, Zhang and others. It was proven [GKS], [BFM], [P], [Z2] that for $i = n-1$, the answer is YES if and only if $n \le 4$, and in the general case the answer is NO for $3 < i \le n-1$ [BZ]. In [Z1]

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a negative answer was also claimed for $i = 3$. The case $i = 2$, $n \geq 5$ remained open. In the present article we show that GBP has an affirmative answer if $i = 2$ and $i = 3$. This disproves the aforementioned statement from [Z1]. One should note that there is a series of accompanying results which are of independent interest but do not solve GBP. We shall not discuss these results here.

The present article is almost self-contained, and the argument is simpler than that in [GKS], [Z2] and [BFM]. The Busemann-Petty case $i = n - 1$ is also included. The key idea is a remarkable interrelation between euclidean $i$-plane transforms on $\mathbb{R}^n$, the dual geodesic Radon transforms on the unit sphere $S^{n-1}$ in $\mathbb{R}^n$, and the classical fractional calculus [R1], [SKM]. This interrelation was used implicitly in [BFM] for $i = n - 1$. We extend it to all $i \leq n - 1$ and simplify technicalities.

*Remark* 1.1. In contrast to [GKS] (and some other papers by A. Koldobsky), our approach does not employ the Fourier analysis on $\mathbb{R}^n$. In this connection one should mention earlier papers by V.I. Semyanistyi [Se1], [Se2] containing a systematic study of interrelation between homogeneous distributions on $\mathbb{R}^n$ and analytic families of convolution operators on the sphere (including the Minkowski-Funk and cosine transforms). According to the Gel'fand-Šapiro theory [GŠ], the Fourier transform serves as a bridge between these two classes of objects. Similar results were obtained by Semyanistyi and his collaborators for hyperbolic spaces and more general settings.

Section 2 contains auxiliary statements about Radon transforms of different kinds. In Section 3 we prove the following

**Theorem A.** *If origin-symmetric convex bodies $K$ and $L$ in $\mathbb{R}^n$ satisfy $vol_i(K \cap \xi) \leq vol_i(L \cap \xi) \quad \forall \xi \in G_{n,i}$, where $i = 2$, $n \geq 3$ (or $i = 3$, $n \geq 4$), then $vol_n(K) \leq vol_n(L)$.*

Since for $3 < i \leq n - 1$, GBP has a negative answer [BZ], Theorem A completes the problem.

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## 2. Passage to Radon transforms and preliminaries

We shall use the following notation: $\sigma_{n-1} = |S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbb{R}^n$; $e_1, e_2, \ldots, e_n$ denote the coordinate unit vectors, $p_0 = \mathbb{R}e_{n-i+1} + \ldots + \mathbb{R}e_n$ is the coordinate $i$-plane; $du$ and $d\xi$ are invariant measures so that $\int_{S^{n-1}} du = \sigma_{n-1}$ and $\int_{G_{n,i}} d\xi = 1$. The totally geodesic Radon transform of an integrable function $f$ on

$S^{n-1}$ is defined by

$$(R_i f)(\xi) = \int\limits_{S^{n-1}\cap\xi} f(u) d_\xi u, \qquad \xi \in G_{n,i}, \tag{2.1}$$

where $d_\xi u$ is the induced Lebesgue measure on the section $S^{n-1}\cap\xi$. We identify the $i$-plane $\xi$ with the $(i-1)$-dimensional circle $S^{n-1}\cap\xi$. The case $i = n-1$ in (2.1) corresponds to the Minkowski-Funk transform $(Rf)(u) = (R_{n-1}f)(u^\perp)$ . The dual Radon transform of a function $\varphi \in L^1(G_{n,i})$ is defined by

$$(R_i^* \varphi)(u) = \int_{SO(n-1)} \varphi(r_u \gamma p_0) d\gamma, \qquad u \in S^{n-1}, \tag{2.2}$$

where $r_u \in SO(n)$ is a rotation so that $r_u e_n = u$, and $SO(n-1)$ is the isotropy subgroup of $e_n$ [He], [R2]. The relevant duality reads

$$\frac{1}{\sigma_{i-1}} \int\limits_{G_{n,i}} (R_i f)(\xi)\varphi(\xi) d\xi = \frac{1}{\sigma_{n-1}} \int\limits_{S^{n-1}} f(u)(R_i^*\varphi)(u) du. \tag{2.3}$$

Each origin-symmetric star body $K$ can be identified with its radial function

$$\rho_K(u) = \sup\{\lambda \geq 0 : \lambda u \in K, \, u \in S^{n-1}\} \qquad (\in C_{even}(S^{n-1})).$$

By passing to polar coordinates, we have

$$vol_n(K) = \frac{1}{n} \int\limits_{S^{n-1}} \rho_K^n(u) du = V(K), \tag{2.4}$$

$$vol_i(K \cap \xi) = \frac{1}{i} \int\limits_{S^{n-1}\cap\xi} \rho_K^i(u) \, d_\xi u = \frac{1}{i}(R_i \rho_K^i)(\xi). \tag{2.5}$$

It is worth mentioning the paper by Hadwiger [H] who studied more general "radialpotenzintegrale" like (2.4) and (2.5) for zonal functions on $S^2$ with $\rho_K^n$ and $\rho_K^i$ replaced by $\rho_K^\alpha$ and $\rho_K^\beta$ $\forall \alpha, \beta \geq 0$.

We shall deal with the following classes of star bodies:

$$\begin{aligned} \mathcal{K}_e &= \text{the set of all origin-symmetric star bodies;} \\ \mathcal{K}_c &= \{K : K \in \mathcal{K}_e, \quad K \quad \text{is convex}\}; \\ \mathcal{I}_{n,i} &= \{K : K \in \mathcal{K}_e, \quad \rho_K^{n-i} \in R_i^*(L_+^1(G_{n,i}))\} \end{aligned} \tag{2.6}$$

where $R_i^*(L_+^1(G_{n,i}))$ is the range of the dual transform (2.2) on non-negative functions $\varphi \in L^1(G_{n,i})$.

The definition (2.6) is motivated by the following

**Lemma 2.1.** *If $K \in \mathcal{I}_{n,i}$ then (1.1) implies (1.2) $\forall L \in \mathcal{K}_e$.*

*Proof.* Let $\rho_K^{n-i} = R_i^*\varphi$, $\varphi \in L^1_+(G_{n,i})$. By (2.3)-(2.5),

$$\begin{aligned} V(K) &= \frac{1}{n}\int\limits_{S^{n-1}} \rho_K^i(u)\rho_K^{n-i}(u)du = \frac{1}{n}\int\limits_{S^{n-1}} \rho_K^i(u)(R_i^*\varphi)(u)du \\ &= \frac{\sigma_{n-1}}{n\sigma_{i-1}}\int\limits_{G_{n,i}} (R_i\rho_K^i)(\xi)\varphi(\xi)d\xi \overset{(1.1)}{\leq} \frac{\sigma_{n-1}}{n\sigma_{i-1}}\int\limits_{G_{n,i}} (R_i\rho_L^i)(\xi)\varphi(\xi)d\xi \\ &= \frac{1}{n}\int\limits_{S^{n-1}} \rho_L^i(u)\rho_K^{n-i}(u)du. \end{aligned}$$

By Hölder's inequality, $V(K) \leq V(L)^{i/n}V(K)^{1-i/n}$, and (1.2) follows. $\square$

The above lemma is well known. Such a simple argument was used, e.g., by Hadwiger [H] and Lutwak [L].

**Lemma 2.2.** *If $\varphi \in L^1(G_{n,i})$ then $R_i^*\varphi = \sigma_{n-2}^{-1}R\psi$ where $R\psi$ is the Minkowski-Funk transform of a function*

$$\psi(\lambda e_{n-1}) = \int\limits_{SO(n-2)} \varphi(\lambda\gamma p_0)d\gamma, \qquad \lambda \in SO(n), \tag{2.7}$$

*satisfying $\int_{S^{n-1}} \psi(u)du = \sigma_{n-1}\int_{G_{n,i}} \varphi(\xi)d\xi$.*

*Proof.* By (2.1) and (2.2),

$$\frac{1}{\sigma_{n-2}}(R\psi)(u) = \int\limits_{SO(n-1)} \psi(r_u\rho e_{n-1})d\rho$$

$$= \int\limits_{SO(n-1)} d\rho \int\limits_{SO(n-2)} \varphi(r_u\rho\gamma p_0)d\gamma = \int\limits_{SO(n-2)} d\gamma \int\limits_{SO(n-1)} \varphi(r_u\rho\gamma p_0)d\rho.$$

Replacing $\rho\gamma$ by $\rho$, we get $(R_i^*\varphi)(u)$. Furthermore,

$$\int\limits_{S^{n-1}} \psi(u)du = \sigma_{n-1}\int\limits_{SO(n)} \psi(\lambda e_{n-1})d\lambda = \sigma_{n-1}\int\limits_{SO(n)} d\lambda \int\limits_{SO(n-2)} \varphi(\lambda\gamma p_0)d\gamma$$

$$= \sigma_{n-1}\int\limits_{SO(n)} \varphi(\lambda p_0)d\lambda = \sigma_{n-1}\int\limits_{G_{n,i}} \varphi(\xi)d\xi.$$

$\square$

We shall need some facts from [Mu], [Herz] about functions of matrix argument. Let $\mathfrak{M}_{n,k}$, $n \geq k$, be the space of real matrices $x = (x_{i,j})$ having $n$ rows and $k$ columns with the volume element $dx = \prod_{i=1}^n \prod_{j=1}^k dx_{i,j}$. In the following $x'$ denotes the transpose of $x$, $|x| = \det(x)$ (for square matrices), and $I_k$ is the identity $k \times k$ matrix. Let $V_{n,k} = \{v \in \mathfrak{M}_{n,k} : v'v = I_k\}$ be the Stiefel manifold of orthonormal $k$-frames in $\mathbb{R}^n$. For $n = k$, $V_{n,n} = O(n)$ represents the orthogonal group in $\mathbb{R}^n$. We fix a measure $dv$ on $V_{n,k}$ which is $O(n)$ left-invariant, $O(k)$ right-invariant, and normalized by $\int_{V_{n,k}} dv = \sigma_{n,k}$, where $\sigma_{n,k} = 2^k \pi^{nk/2}/\Gamma_k(n/2)$, and

$$\Gamma_k(\alpha) = \pi^{k(k-1)/4}\Gamma(\alpha)\Gamma(\alpha - \frac{1}{2}) \dots \Gamma(\alpha - \frac{k-1}{2})$$

is the Siegel Gamma function. We denote by $\mathcal{P}_k$ the cone of positive definite symmetric $k \times k$ matrices $r = (r_{i,j})$ with the volume element $dr = \prod_{i \leq j} dr_{i,j}$. The following polar decomposition can be found in ([Herz], p. 482]; [Mu], p. 66]; [GR]).

**Lemma 2.3.** *Almost all $x \in \mathfrak{M}_{n,k}$, $n \geq k$, can be decomposed as*

$$x = vr^{1/2}, \quad v \in V_{n,k}, \quad r = x'x \in \mathcal{P}_k \quad \textit{with} \quad dx = 2^{-k}|r|^{(n-k-1)/2} dr dv.$$

The next statement mimics Lemma 6.1 from [GZ]. We present it in a more complete form, and give an alternative proof.

**Lemma 2.4.** *Let $f = Rg$, $g \in C_+(S^{n-1})$, and let $K$ be a star body so that $\rho_K^{n-1}(u) = (n-1)g(u)$. Then for $1 < i < n$,*

$$(2.8) \quad f^{n-i}(u) = (R_i^* \varphi)(u), \quad \varphi(\eta) = \int\limits_{\eta^\perp \cap K} dx_1 \dots \int\limits_{\eta^\perp \cap K} |x'x|^{(i-1)/2} dx_{n-i},$$

*$x = [x_1, \dots, x_{n-i}] \in \mathfrak{M}_{n,n-i}$, and*

$$(2.9) \qquad \int_{G_{n,i}} \varphi(\eta) d\eta = c(vol_n(K))^{n-i}, \qquad c = \frac{\sigma_{n-i,n-i}}{\sigma_{n,n-i}}.$$

*Proof.* By (2.5), $f(u) = vol_{n-1}(K \cap u^\perp)$, and therefore

$$f^{n-i}(u) = \prod_{j=1}^{n-i} \int\limits_{K \cap u^\perp} dx_j = \int\limits_{\mathfrak{M}_{n-1,n-i}} F(y) dy$$

where $F(y)$ is the product of characteristic functions $\chi_{r_u^{-1}K}(y_j)$, $j = 1, \ldots, n-i$, of the rotated body $r_u^{-1}K$. Now Lemma 2.3 yields

$$
\begin{aligned}
f^{n-i}(u) &= 2^{i-n} \int\limits_{V_{n-1,n-i}} dv \int\limits_{\mathcal{P}_{n-i}} F(vr^{1/2})|r|^{(i-2)/2} dr \\
&= \frac{2^{i-n}}{\sigma_{n-i,n-i}} \int\limits_{V_{n-1,n-i}} dv \int\limits_{O(n-i)} d\gamma \int\limits_{\mathcal{P}_{n-i}} F(v\gamma r^{1/2})|r|^{(i-2)/2} dr \\
&= \frac{1}{\sigma_{n-i,n-i}} \int\limits_{V_{n-1,n-i}} dv \int\limits_{\mathfrak{M}_{n-i,n-i}} F(vz)|z'z|^{(i-1)/2} dz \\
&= \int\limits_{SO(n-1)} d\alpha \int\limits_{\mathfrak{M}_{n-i,n-i}} F\left(\alpha \left[\begin{array}{c} I_{n-i} \\ 0 \end{array}\right] z\right) |z'z|^{(i-1)/2} dz \\
&= \int\limits_{G_{n-1,n-i}} d\xi \Big(\int\limits_{\xi} \ldots \int\limits_{\xi}\Big) F(y)|y'y|^{(i-1)/2} dy_1 \ldots dy_{n-i}.
\end{aligned}
$$

After changing variables this reads

$$
\int\limits_{\eta \supset u} \Big(\int\limits_{\eta^\perp \cap K} dx_1 \ldots \int\limits_{\eta^\perp \cap K} |x'x|^{(i-1)/2} dx_{n-i}\Big) d\eta, \qquad \eta \in G_{n,i},
$$

and (2.8) follows. In order to prove (2.9) we proceed in the opposite direction with slight changes:

$$
\begin{aligned}
\int\limits_{G_{n,i}} \varphi(\eta) d\eta &= \int\limits_{SO(n)} d\beta \int\limits_{\mathfrak{M}_{n-i,n-i}} F\left(\beta \left[\begin{array}{c} I_{n-i} \\ 0 \end{array}\right] z\right) |z'z|^{(i-1)/2} dz \\
&= \frac{1}{\sigma_{n,n-i}} \int\limits_{V_{n,n-i}} dw \int\limits_{\mathfrak{M}_{n-i,n-i}} F(wz)|z'z|^{(i-1)/2} dz \\
&= \frac{2^{i-n}}{\sigma_{n,n-i}} \int\limits_{V_{n,n-i}} dw \int\limits_{O(n-i)} d\gamma \int\limits_{\mathcal{P}_{n-i}} F(w\gamma r^{1/2})|r|^{(i-2)/2} dr \\
&= \frac{2^{i-n}\sigma_{n-i,n-i}}{\sigma_{n,n-i}} \int\limits_{V_{n,n-i}} dw \int\limits_{\mathcal{P}_{n-i}} F(wr^{1/2})|r|^{(i-2)/2} dr \\
&= c \int\limits_{\mathfrak{M}_{n,n-i}} F(y) dy = c \int\limits_{r_u^{-1}K} dy_1 \ldots \int\limits_{r_u^{-1}K} dy_{n-i} = c(vol_n(K))^{n-i}.
\end{aligned}
$$

$\square$

The next statement is especially important. It concerns interrelation between spherical and euclidean Radon transforms. We consider the euclidean $i$-plane transform [He]

$$(P_i f)(\xi, x'') = \int_{\xi} f(x' + x'') dx', \qquad \xi \in G_{n,i}, \quad x'' \in \xi^{\perp}. \tag{2.10}$$

Since $\int_{\xi^\perp} (P_i f)(\xi, x'') dx'' = \int_{\mathbb{R}^n} f(x) dx \, \forall \xi$, $P_i f$ is well defined for all $f \in L^1(\mathbb{R}^n)$. The case $i = 1$ in (2.10) corresponds to the $X$-ray transform of $f$.

**Lemma 2.5.** *Let $f \in L^1(\mathbb{R}^n)$, $\xi \in G_{n,i}$. Then for all $\omega \in S^{n-1} \cap \xi^{\perp}$ and $t \in \mathbb{R}$,*

$$R_i^* : (P_i f)(\xi, \omega t) \to \sigma_{i-2} \int_{|t|}^{\infty} \psi_u(r)(r^2 - t^2)^{(i-3)/2} r dr \tag{2.11}$$

*where*

$$\psi_u(r) = \frac{1}{\sigma_{n-2}} \int_{S^{n-1} \cap u^{\perp}} (P_1 f)(\theta, r\theta) d\theta \tag{2.12}$$

*is the average of the $X$-ray transform of $f$ over all lines parallel to $u$ at distance $r$ from the origin.*

*Proof.* Let $p_0 = \mathbb{R}e_{n-i+1} + \ldots + \mathbb{R}e_n$, $y = y' + y''$, $y' \in p_0$, $y'' \in p_0^{\perp}$. Given $u \in S^{n-1}$, we fix a rotation $r_u \in SO(n)$ so that $r_u : e_n \to u$, and denote $f_u(x) = f(r_u x)$. By (2.2),

$$\begin{aligned} (R_i^*[(P_i f)(\xi, x'')])(u) &= \int_{SO(n-1)} (P_i f_u)(\rho p_0, \rho y) d\rho \\ &= \int_{SO(n-1)} d\rho \int_{p_0} f_u(\rho(y' + y'')) dy'. \end{aligned}$$

For $y = \sum_{k=1}^{n} y_k e_k$, we write

$$y = z + y_n e_n, \quad z = z' + z'', \quad z' = \sum_{k=n-i+1}^{n-1} y_k e_k, \quad z'' = \sum_{k=1}^{n-i} y_k e_k = y''.$$

Then the last integral reads

$$\int\limits_{\mathbb{R}^{i-1}} dz' \int\limits_{-\infty}^{\infty} dy_n \int\limits_{SO(n-1)} f_u(\rho z + y_n e_n) d\rho$$

$$= \int\limits_{\mathbb{R}^{i-1}} dz' \int\limits_{-\infty}^{\infty} dy_n \int\limits_{SO(n-1)} f_u(\rho\sqrt{|z'|^2+|z''|^2} e_1 + y_n e_n) d\rho$$

$$= \sigma_{i-2} \int\limits_0^{\infty} s^{i-2} \psi_u(\sqrt{s^2+t^2}) ds = \sigma_{i-2} \int\limits_t^{\infty} \psi_u(r)(r^2-t^2)^{(i-3)/2} r dr,$$

where $t = |z''| = |y''| = |x''|$, and

$$\psi_u(r) = \int\limits_{SO(n-1)} d\rho \int\limits_{-\infty}^{\infty} f_u(r\rho e_1 + y_n e_n) dy_n = \frac{1}{\sigma_{n-2}} \int\limits_{S^{n-1}\cap u^\perp} (P_1 f)(u, r\theta) d\theta.$$

□

## 3. Proof of the main result

Denote

$$M_\xi(\omega, t) = (P_i f)(\xi, \omega t), \qquad \omega \in S^{n-1} \cap \xi^\perp, \quad t \in \mathbb{R}. \tag{3.1}$$

If $f = \chi_K$ is the characteristic function of $K$, then (3.1) is a volume of the $i$-dimensional plane section of $K$ in the direction $\omega$ at distance $|t|$ from the origin. Our nearest goal is to apply Lemma 2.5 to $f = \chi_K$, $K \in \mathcal{K}_e$, and to express $\psi_u(0) = (P_1 f)(u, 0) = 2\rho_K(u)$ through $M_\xi(\omega, t)$. Let

$$F_u(t) = (R_i^*[M_\xi(\omega, t)])(u), \qquad t \geq 0, \tag{3.2}$$

and suppose that $K$ has a $C^\infty$-boundary. If $i$ is odd we follow [BFM] and write (2.11) as

$$\frac{F_u(t)}{\sigma_{i-2}} = \int\limits_0^{\infty} \psi_u(r)(r^2-t^2)^{(i-3)/2} r dr - t^{i-1} \int\limits_0^1 \psi_u(ts)(s^2-1)^{(i-3)/2} s ds.$$

The first term in the right hand side is a polynomial of degree $i-3$, and simple calculation yields

$$\begin{aligned} F_u^{(i-1)}(0) &= -\sigma_{i-2}(i-1)!\psi_u(0) \int\limits_0^1 (s^2-1)^{(i-3)/2} s ds \\ &= \pi^{i/2-1} 2^i (-1)^{(i-1)/2} \Gamma(i/2) \rho_K(u). \end{aligned} \tag{3.3}$$

For $i$ even, the required result can be easily obtained by making use of Marchaud's fractional derivative which was introduced in 1927 [M], and is well known in fractional calculus. Numerous generalizations and applications of this notion can be found in [R1], [SKM]. We recall some basic facts from the book [R1] containing a detailed account and further development of Marchaud's method. For a large class of functions $\varphi$ on a half-line, the Weyl fractional integral

$$\Phi(t) = (I^\alpha_- \varphi)(t) = \frac{1}{\Gamma(\alpha)} \int_t^\infty \varphi(r)(r-t)^{\alpha-1} dr, \qquad Re\,\alpha > 0, \tag{3.4}$$

can be inverted as $\varphi = \mathbb{D}^\alpha_- \Phi$ where

$$(\mathbb{D}^\alpha_- \Phi)(r) = \frac{1}{\kappa_\ell(\alpha)} \int_0^\infty \Big[ \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j \Phi(r+jt) \Big] \frac{dt}{t^{1+\alpha}}, \tag{3.5}$$

$$\kappa_\ell(\alpha) = \int_0^\infty \frac{(1-e^{-t})^\ell}{t^{1+\alpha}} dt, \qquad \ell > Re\,\alpha, \quad \ell \in \mathbb{N}.$$

The expression (3.5) containing a finite difference of $\Phi$ of order $\ell$, is called *the Marchaud fractional derivative* of $\Phi$. The integer $\ell > Re\,\alpha$ can be arbitrary due to normalization. If $\Phi$ has $\ell+1$ continuous derivatives in a neighborhood of $r$ then $\mathbb{D}^\alpha_- \Phi$ exists as an improper integral. Otherwise it represents a hypersingular integral which is understood in a certain special sense (see Theorem 10.21 in [R1]). For $r = 0$, (3.5) yields

$$(\mathbb{D}^\alpha_- \Phi)(0) = \frac{1}{\kappa_\ell(\alpha)} \int_0^\infty \Big[ \sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j \Phi(jt) \Big] \frac{dt}{t^{1+\alpha}}. \tag{3.6}$$

This expression represents analytic continuation ($a.c.$) of the integral $I(\beta) = \frac{1}{\Gamma(\beta)} \int_0^\infty \Phi(r) r^{\beta-1} dr$ in the strip $\mathcal{L} = \{\beta : -\ell < Re\,\beta < 0\}$ at the point $-\alpha \in \mathcal{L}$. If $\Phi^{(2m+1)}(0) = 0$, $m = 0, 1, \dots,$ then $\ell$ can be reduced so that (3.6) holds for all $\ell > 2[Re\,\alpha/2]$, $[a]$ being the integral part of $a$ (see Corollary 10.15 in [R1]).

We note that in applications of fractional calculus, a difference regularization is sometimes more effective then the classical one [GS]

$$a.c. \int_0^\infty \Phi(r) r^{\beta-1} dr = \int_0^\infty \Big[ \Phi(r) - \sum_{j=0}^{\ell-1} \frac{r^j}{j!} \Phi^{(j)}(0) \Big] r^{\beta-1} dr, \tag{3.7}$$

$$-\ell < Re\,\beta < -\ell + 1.$$

We shall use both regularizations of $I(\beta)$. By (2.11) and (3.2), $F_u(\sqrt{t}) = \pi^{(i-1)/2}(I_-^{(i-1)/2}\psi_u(\sqrt{\cdot}))(t)$. Hence

$$\begin{aligned}
\psi_u(0) &= \frac{\pi^{(1-i)/2}}{\kappa_\ell((i-1)/2)} \int\limits_0^\infty \Big[\sum_{j=0}^{\ell} \binom{\ell}{j} (-1)^j F_u(\sqrt{jt})\Big] \frac{dt}{t^{(i+1)/2}} \\
&= a.c. \frac{\pi^{(1-i)/2}}{\Gamma(\alpha)} \int\limits_0^\infty t^{\alpha-1} F_u(\sqrt{t})dt\Big|_{\alpha=(1-i)/2} \\
&= a.c. \frac{2\pi^{(1-i)/2}}{\Gamma(\beta/2)} \int\limits_0^\infty \tau^{\beta-1} F_u(\tau)d\tau\Big|_{\beta=1-i}.
\end{aligned}$$

Since odd derivatives of $F_u(t)$ are zeros at $t=0$, and $\psi_u(0) = 2\rho_K(u)$, according to (3.7) we get

$$\rho_K(u) = c_i \int\limits_0^\infty \Big[F_u(\tau) - \sum_{m=0}^{(i-2)/2} \frac{\tau^{2m}}{(2m)!} F_u^{(2m)}(0)\Big] \frac{d\tau}{\tau^i}, \tag{3.8}$$

$$c_i = \frac{\pi^{(1-i)/2}}{\Gamma((1-i)/2)} = \frac{(-1)^{i/2}(i-1)!!}{(2\pi)^{i/2}}.$$

This implies the following

**Theorem 3.1.** *Let* $\rho_K \in C^\infty(S^{n-1})$, $\quad 2 \le i \le n-1$. *For* $\xi \in G_{n,i}$, $\quad \omega \in S^{n-1} \cap \xi^\perp$, *and* $t \ge 0$, *we set*

$$M_\xi(\omega, t) = vol_i(K \cap \{\xi + \omega t\}), \tag{3.9}$$

*and denote by* $M_\xi^{(j)}(\omega, 0)$, $j = 1, 2, \dots$, *directional derivatives of* $M_\xi(\omega, \cdot)$ *in the direction* $\omega$ *at* $t=0$. *Then* $\rho_K = R_i^* h$, *where* $h \equiv h(\xi, \omega) \in C^\infty(G_{n,i})$. *The function* $h$ *has the form*

$$h(\xi, \omega) = \frac{(-1)^{(i-1)/2}}{\pi^{i/2-1} 2^i \Gamma(i/2)} M_\xi^{(i-1)}(\omega, 0) \tag{3.10}$$

*if* $i$ *is odd, and*

$$h(\xi, \omega) = \frac{(-1)^{i/2}(i-1)!!}{(2\pi)^{i/2}} \int\limits_0^\infty \Big[M_\xi(\omega, \tau) - \sum_{m=0}^{(i-2)/2} \frac{\tau^{2m}}{(2m)!} M_\xi^{(2m)}(\omega, 0)\Big] \frac{d\tau}{\tau^i} \tag{3.11}$$

*if* $i$ *is even.*

For $i = n-1$, Theorem 3.1 was proven in [GKS] and [BFM]. Our proof is essentially simpler and based on different ideas. We recall that the argument in [GKS] relies heavily on the Fourier analysis of homogeneous distributions in $\mathbb{R}^n$. Derivation of (3.11) in [BFM] (for $i = n-1$) is more complicated than ours. It is worth noting that for $i < n-1$, one deals with the *family* of functions $h$ parameterized by the vector $\omega$ belonging to the $(n-i-1)$-dimensional unit sphere. In the case $i = n-1$ there exist only one such function.

Now we can return to the generalized Busemann-Petty problem. We denote by $\mathcal{A}(GBP)$ the set of all pairs $(i,n)$, $2 \le i \le n-1$, so that (1.1) $\Rightarrow$ (1.2) for all $K, L \in \mathcal{K}_c$, and let $\kappa_K(x)$ be the Gaussian curvature of the boundary $\mathrm{bd}(K)$ at the point $x \in \mathrm{bd}(K)$ [Sch]. Consider the following class of "test bodies":

$$\mathcal{K}_{c+}^\infty = \{K : K \in \mathcal{K}_c; \quad \rho_K \in C^\infty(S^{n-1}); \quad \kappa_K(x) > 0 \quad \forall x \in \mathrm{bd}(K)\}.$$

**Lemma 3.2.** *If $\mathcal{K}_{c+}^\infty \subset \mathcal{I}_{n,i}$ (see (2.6)) then $(i,n) \in \mathcal{A}(GBP)$.*

*Proof.* Following Gardner's scheme ([G1], Theorem 3.1), we suppose that for some $K, L \in \mathcal{K}_c$ satisfying (1.1), $vol_n(K) > vol_n(L)$. There exists an approximating body $K' \in \mathcal{K}_{c+}^\infty$ so that $K' \subset K$ and $\mathrm{vol}_n(K') > \mathrm{vol}_n(L)$ (see, e.g., [G2], p. 297, and [G1], p. 438). By Lemma 2.1, $K' \notin \mathcal{I}_{n,i}$, and we arrive at contradiction. □

**Proof of Theorem A.** Let $K \in \mathcal{K}_{c+}^\infty$ be a "test body". By Lemma 3.2 it suffices to show that

$$\rho_K^{n-i}(u) \in R_i^*(L_+^1(G_{n,i})). \tag{3.12}$$

By Theorem 3.1,

$$h(\xi,\omega) = \frac{1}{2\pi} \int_0^\infty \frac{M_\xi(\omega,0) - M_\xi(\omega,\tau)}{\tau^2} d\tau \tag{3.13}$$

if $i = 2$, and

$$h(\xi,\omega) = -\frac{1}{4\pi} M_\xi''(\omega,0) \tag{3.14}$$

if $i = 3$. Fix $\omega \in S^{n-1} \cap \xi^\perp$ and consider the $(i+1)$-dimensional plane $\pi_{i+1}$ spanned by $\xi$ and $\omega$. The $(i+1)$-dimensional body $K \cap \xi_i$ belongs to the class $\mathcal{K}_{c+}^\infty$ (on $\pi_{i+1}$), and by the Brunn-Minkowski theory [Sch] the function $h$ in (3.13) and (3.14) is positive. Moreover, it is continuous in $\xi \in G_{n,i}$. Let $\tilde{h}(\xi)$ be the average of $h$ over all $\omega \in S^{n-1} \cap \xi^\perp$. By Theorem 3.1, $\rho_K(u) = (R_i^* \tilde{h})(u)$, and owing to Lemma 2.2, $\rho_K(u)$ is represented by the Minkowski-Funk transform of a function $\psi$ which is defined by (2.7) with $\varphi$ replaced by $\tilde{h}$. Since $\tilde{h} \in C_+(G_{n,i})$, then $\psi \in C_+(S^{n-1})$. Next we use Lemma 2.4, which implies (3.12). This completes the proof.

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