SPECIAL SETS OF POINTS ON COMPACT RIEMANN SURFACES

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1. Introduction

The first part of this note is an expanded version of the talk given by the first author at the 1998 Ahlfors-Bers Colloquium. The second part contains a general discussion of Weierstrass points associated to arbitrary finite dimensional vector spaces of differentials on Riemann surfaces of finite conformal type with applications to the study of $\mathbb{H}^2/\Gamma(k)$, where $\mathbb{H}^2$ is the upper half plane and $\Gamma(k)$ is the level k principal congruence subgroup of the modular group.

One of the topics central to investigations of functions on compact Riemann surfaces is the Riemann theta function. In their published works and lectures, Ahlfors and Bers "neglected" this function and hence many of their students were never exposed to this fascinating aspect of the theory of Riemann surfaces. For the most part, this theory became the domain of the algebraic geometers; this note continues the efforts to bring this theory back to function theory practitioners.

We begin by recalling a number of statements concerning this subject made by Ahlfors and Bers. The first one was in writing when Ahlfors ended his review of Lewittes' Acta paper [**?**], with the statement "theta functions is not a spectator sport." The second is a statement made privately to the first author by Ahlfors after a lecture given in La Jolla explaining how the theta function can be used to generalize the notion of the cross ratio of four points on a sphere to four points on an arbitrary compact surface of genus $g \geq 0$. Ahlfors said that if he were younger he would begin to study theta functions. Lipman Bers once told the first author that in his opinion we really do not work on the same problems.

The purpose of this note is an attempt to make theta functions into a "spectator sport," and to show that Bers and we really were working on the "same" problems[1]. The first part of this note should be viewed of as a continuation of [**?**] and [**?**], where relations between Weierstrass points and the theta divisor are discussed.

PART I. Some of the concepts discussed in Part II, particularly §**??**, are relevant for the material presented here. The reader might want to review that section before proceeding to this part.

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[1] There may be little doubt about this statement applied to the second author; a recent "convert" to theta.

2. Weierstrass points on compact hyperbolic surfaces

It is well known that on every compact Riemann surface S of genus $g \geq 2$, there is a finite nonempty collection W of classical Weierstrass points with

$$2g+2 \leq |W| \leq g^3 - g;$$

the lower bound occurs if and only if the surface is hyperelliptic, the upper bound being the generic situation. In the classical theory one takes a basis for the space of holomorphic one forms on the surface[2] $\theta_1, \ ..., \ \theta_g$, and forms the Wronskian

$$W(\theta_1, ..., \theta_g) = \ \det \left[\begin{array}{ccc} \theta_1 & \theta_2, ..., & \theta_g \\ \theta_1' & \theta_2', ..., & \theta_g' \\ ... & ... & ... \\ \theta_1^{(g-1)} & \theta_2^{(g-1)}, ..., & \theta_g^{(g-1)} \end{array} \right].$$

One then shows that the Wronskian is a holomorphic $\frac{g(g+1)}{2}$-differential and therefore that $W(\theta_1, ..., \theta_g)$ has $g^3 - g$ zeros counting multiplicity and that there are at least $2g+2$ distinct zeros each with multiplicity at most $\frac{g(g-1)}{2}$. If the maximum multiplicity is achieved at one Weierstrass point, then the surface is hyperelliptic and it has $2g+2$ Weierstrass points each with multiplicity $\frac{g(g-1)}{2}$.

In a similar way, if one starts with a basis for the space of holomorphic q-differentials, $q \geq 2$, one can define and obtain the Weierstrass points for this vector space generalizing the classical situation. This is treated in [**?**, Ch. III] and the reader can consult this book for further information on this subject. We point out that we show in our book that a set of Weierstrass points can be defined for any finite dimensional vector space of holomorphic differentials (see also §**??** and the less known fact that, the set of classical Weierstrass points W can be defined also by Riemann's theta function. For the sake of completeness we recall the statement of this last result ([**?**, Th. VII.1.10]).

Let θ denote the Riemann theta function[3]. Let φ_{P_0} denote the Abel Jacobi map of the Riemann surface S into its Jacobi variety and let $\mathcal{K}_{P_0}$ denote the vector of Riemann constants, both with base point P_0. Then the nontrivial zeros of $\theta(g\varphi_{P_0}(P) + \mathcal{K}_{P_0})$ are classical Weierstrass points and conversely, each classical Weierstrass point is a zero of this multivalued function. Here, nontrivial zero means that $P \neq P_0$, the base point of the map φ_{P_0}. If we choose $P_0 \notin W$, then the function $P \to \theta(g\varphi_{P_0}(P) + \mathcal{K}_{P_0})$, has a g-th order zero at P_0 and that the remaining $g^3 - g$ zeros are classical Weierstrass points. A similar statement holds for the zeros of

$$\theta((2q-1)(g-1)\varphi_{P_0}(P) + (2q-1)\mathcal{K}_{P_0});$$

namely that they are the Weierstrass points for the space of holomorphic q-differentials, $q \geq 2$.

It is a well known consequence of the Riemann vanishing theorem that if α is a positive integer with $0 < \alpha < g$, then $\theta(\alpha\varphi_{P_0}(P) + \mathcal{K}_{P_0}) = 0$ for all $P \in S$. Thus $\alpha = g$ is the first nontrivial case and it gives rise to the classical Weierstrass points. It thus seems reasonable to ask what happens when $g \leq \alpha$, $\alpha \in \mathbb{Z}$.

[2]Unless otherwise noted, we are following the notation and conventions of [**?**]; in particular, we are identifying, when there is little room for confusion, a differential θ with its expression in local coordinates $\varphi(z)dz$ as well the function φ in this representation.

[3]Also defined for $g = 1$; although in this section, $g \geq 2$.

3. $g = 1$

When $g = 1$ and $\alpha = g$ we are considering the holomorphic function $\theta(\varphi_{P_0}(P) + \mathcal{K}_{P_0})$. This function has a simple zero at the point $P = P_0$ and no other zeros; a reflection the fact that there are no classical Weierstrass points when $g = 1$. If for $g = 1$ we take $\alpha \in \mathbb{Z}^+$, $\alpha \geq 2$, we find easily that $\theta(\alpha\varphi_{P_0}(P) + \mathcal{K}_{P_0})$ has α^2 zeros on the surface and that only one of these zeros is at the point P_0. The remaining zeros are at points P which satisfy $\left(\frac{P}{P_0}\right)^\alpha$ is a principal divisor on the surface. Alternatively, we can say that P is a nontrivial zero of this function provided there is a meromorphic one form on the surface whose only singularity is a pole of order at most α at P_0 and a zero of order α at P. The dimension of the space of meromorphic one forms with singularity a pole of order at most $\alpha \geq 2$ at P_0 is α; so that in this sense, P is a Weierstrass point for this space. On the other hand if we think of the point P_0 as the origin we are also constructing the points of order α on the torus. The above material for higher genus surfaces will lead to a generalization of points of finite order on tori.

4. The general case

We now consider a compact Riemann surface S of genus $g \geq 2$ and the multivalued function on this surface

$$f : P \mapsto \theta((g + k)\varphi_{P_0}(P) + \mathcal{K}_{P_0}),$$

with $k \in \mathbb{Z}^+ \cup \{0\}$. The case $k = 0$ has already been studied; it yields the classical Weierstrass points. There are now two ways to proceed and it is advantageous to study each of them.

Theorem 1. *The zeros of $\theta((g + k)\varphi_{P_0}(P) + \mathcal{K}_{P_0})$ consist of the point P_0 and those points $P \in S$ for which there exists an integral divisor ζ of degree $g - 1$ such that $\frac{P_0^{k+1}\zeta}{P^{g+k}}$ is principal. Alternatively, the zeros of $\theta((g + k)\varphi_{P_0}(P) + \mathcal{K}_{P_0})$ consist of the point P_0 and the points $P \in S$ such that P is a Weierstrass point for the vector space of meromorphic differentials on the surface whose only singularity is a pole of order at most $k + 1$ at P_0.*

Proof. The Riemann vanishing theorem asserts that the zeros of the Riemann theta function are those points in the Jacobi variety which are the images of integral divisors of degree $g - 1$ under the Abel Jacobi map φ_{P_0} translated by the vector of Riemann constants $\mathcal{K}_{P_0}$. It follows that $\mathcal{K}_{P_0}$ is always a zero. This explains why the point P_0 is necessarily a zero of the function f. If $P \in S$ and P is a zero then there exists an integral divisor of ζ of degree $g - 1$ such that

$$\varphi_{P_0}(P^{g+k}) + \mathcal{K}_{P_0} = \varphi_{P_0}(\zeta) + \mathcal{K}_{P_0}.$$

It follows from Abel's theorem that $D = \frac{P_0^{k+1}\zeta}{P^{g+k}}$ is a principal divisor. The argument also reverses so that P is a zero if and only if D is principal. Note that we see already here why the case $k = 0$ gives rise to the classical Weierstrass points.

It is however useful to look at this condition in another way. We know that $\frac{P^{g+k}}{P_0^{k+1}\zeta}$ is principal. Since the degree of ζ is $g - 1$ there is always a holomorphic differential η whose divisor is a multiple of ζ. If the divisor of the function f, (f), is $\frac{P^{g+k}}{P_0^{k+1}\zeta}$, then the divisor of the meromorphic differential $f\eta$ is $\frac{P^{g+k}\zeta'}{P_0^{k+1}}$ for some integral divisor ζ' of degree $g - 1$. Notice this meromorphic differential can only have a pole at P_0, and the order of the pole is at most $k + 1$. Since the dimension of the space of meromorphic differentials with this type of singularity is precisely $g+k$ and the order of vanishing of $f\eta$ at the point $P \neq P_0$ is at least $g+k$, we conclude that every zero $P \neq P_0$ of f is a Weierstrass point for this space V.

The general theory of theta functions (see, for example, [**?**, Ch. VI]) tells us that the function $\theta((g+k)\varphi_{P_0}(P)+\mathcal{K}_{P_0})$ has $(g+k)^2 g$ zeros on the surface, counting multiplicity, and that if we denote the divisor of zeros by $Q_1 \ldots Q_{(g+k)^2 g}$, then

$$\varphi_{P_0}(Q_1 \ldots Q_{(g+k)^2 g}) + (g+k)\mathcal{K}_{P_0} = -(g+k)^2 \mathcal{K}_{P_0};$$

so

$$\varphi_{P_0}(Q_1 \ldots Q_{(g+k)^2 g}) = \frac{(g+k)(g+k+1)}{2}(-2\mathcal{K}_{P_0}).$$

Since $-2\mathcal{K}_{P_0}$ is the image under φ_{P_0} of a canonical divisor, $\frac{Q_1 \ldots Q_{(g+k)^2 g}}{P_0^{(g+k)(k+1)}}$ is $q = \frac{(g+k)(g+k+1)}{2}$-canonical (the divisor of a meromorphic q-differential ω). $\square$

The q-differential ω should be closely related to another q-differential Ω, the Wronskian of a basis for the $(g+k)$-dimensional vector space of meromorphic q-differentials V with at most a pole of order $k+1$ at P_0 that we introduced in the proof of the last theorem. Both divisors (ω) and (Ω) have degree $(g-1)(g+k)(g+k+1)$. Since the space V cannot contain a differential with a simple pole at P_0, the lowest and highest possible orders at P_0 of elements of V are respectively

$$-(k+1),\ -k,\ \ldots,\ -2,\ 0,\ 1,\ \ldots,\ g-1$$

and

$$-(k+1),\ -k,\ \ldots,\ -2,\ 0,\ 2,\ \ldots,\ 2g-2;$$

(note that V certainly contains differentials φ with $\mathrm{ord}_{P_0} = l$, $-(k+1) \leq l \leq 0$, $l \neq -1$) hence

$$\mathrm{ord}_{P_0}\Omega \geq -\frac{k(k+3)}{2} + \frac{g(g-1)}{2} - \frac{(g+k-1)(g+k)}{2} = -k(g+k+1)$$

and

$$\mathrm{ord}_{P_0}\Omega \leq -\frac{k(k+3)}{2} + g(g-1) - \frac{(g+k-1)(g+k)}{2} = \frac{g^2 - g(1+2k) - 2k(1+k)}{2}.$$

We conclude that the number, N, of zeros of Ω on $S - \{P_0\}$ is bounded by

$$\frac{g\left(2g^2 + g(4k-1) + (2k^2-1)\right)}{2} \leq N \leq g\left((g+k)^2 - 1\right).$$

A point $P \in S - \{P_0\}$ is a Weierstrass point for V if and only if $i\left(\frac{P^{g+k}}{P_0^{k+1}}\right) > 0$. This condition does not apply to the special point P_0 because a compact Riemann surface does not carry an abelian differential of the third kind whose only singularity is one simple pole. Note that P_0 is a Weierstrass point for V if and only if it is a classical Weierstrass point (if and only if $i(P_0^g) > 0$). Since each nontrivial zero of f is a Weierstrass point for the space V, it is clear that f has at least g trivial zeros at P_0. It may also occur that P_0 itself is a Weierstrass point for the space V. We have proven

Theorem 2. *The (multivalued holomorphic) function* $P \mapsto \theta((g+k)\varphi_{P_0}(P)+\mathcal{K}_{P_0})$ *has at least a g-th order zero at the point P_0 and its remaining, at most $g\left((g+k)^2-1\right)$, zeros are the zeros of the (meromorphic) Wronskian of a basis for the space of meromorphic differentials with at most a pole of order $k+1$ at P_0.*

Remark 1. We do not claim that ω is a nonzero constant multiple of Ω. This equality would follow if we could establish that

$$\mathrm{ord}_P\omega = \mathrm{ord}_P\Omega, \text{ for all } P \in S.$$

In this connection see the Problem at the end of [**?**, §6].

The above is capable of some generalization. In place of considering only one point P_0 we can choose more than one point, say for argument's sake two distinct points P_1 and P_2. We can now study the function

$$f(P) = \theta((g+k_1)\varphi_{P_1}(P) + (k_2+1)\varphi_{P_2}(P) + \mathcal{K}_{P_0}),$$

for arbitrary nonnegative integers k_1 and k_2. It is easy to see that the above can be rewritten as

$$\theta((g+k_1)\varphi_{P_1}(P) + (k_2+1)(\varphi_{P_2}(P_1) + \varphi_{P_1}(P) + \mathcal{K}_{P_1})$$

which is the same as

$$\theta((g+k_1+k_2+1)\varphi_{P_1}(P) - (k_2+1)\varphi_{P_1}(P_2) + \mathcal{K}_{P_1}).$$

It is thus clear that the point P is a zero of f if and only if there exists an integral divisors ζ of degree $g-1$ such that

$$\varphi_{P_1}\left(\frac{P^{g+k_1+k_2+1}\zeta}{P_2^{k_2+1}}\right) = -2\mathcal{K}_{P_1} = \varphi_{P_1}\left(\frac{P^{g+k_1+k_2+1}\zeta}{P_2^{k_2+1}P_1^{k_1+1}}\right);$$

in other words, if and only if P is a Weierstrass point for the space of meromorphic differentials with poles of orders at most k_1+1 at P_1 and k_2+1 at P_2. The dimension of this space is $g+k_1+k_2+1$ and a basis for the space is the union of a bases for the spaces of holomorphic differentials, the meromorphic differentials with a poles of order l, $2 \le l \le k_1+1$, at the point P_1, the meromorphic differentials with poles of l, $2 \le l \le k_2+1$, at the point P_2 and one meromorphic differential with simple poles at P_1 and P_2. A point $P \in S$ is a Weierstrass point for this space if and only if $i\left(\frac{P^{g+k_1+k_2+1}}{P_1^{k_1+1}P_2^{k_2+1}}\right) > 0$.

The general θ-function theory in this case yields that f has $(g+k_1+k_2+1)^2g$ zeros $Q_1, \ldots, Q_{(g+k_1+k_2+1)^2g}$ which satisfy the relation

$$(g+k_1+k_2+1)((k_1+1)\varphi_{P_1}(P_2) - \mathcal{K}_{P_1}) = \varphi_{P_1}(Q_1 \ldots Q_{(g+k_1+k_2+1)^2g}) + (g+k_1+k_2+1)^2\mathcal{K}_{P_1}$$

or that

$$\varphi_{P_1}\left(\frac{Q_1 \ldots Q_{(g+k_1+k_2+1)^2g}}{P_2^{(g+k_1+k_2+1)(k_2+1)}}\right) = \frac{(g+k_1+k_2+1) + (g+k_1+k_2+1)^2}{2}(-2\mathcal{K}_{P_1}).$$

Hence

$$\frac{Q_1 \ldots Q_{(g+k_1+k_2+1)^2g}}{P_1^{(g+k_1+k_2+1)(k_1+1)}P_2^{(g+k_1+k_2+1)(k_2+1)}}$$

is the divisor of a meromorphic $\frac{(g+k_1+k_2+1)(g+k_1+k_2+2)}{2}$-differential with poles of orders at most

$$(g+k_1+k_2+1)(k_1+1) \text{ and } (g+k_1+k_2+1)(k_2+1)$$

at the points P_1 and P_2, respectively. This differential is closely associated with the Wronskian of a basis for the space of meromorphic differentials which are permitted to have poles only at P_1 and P_2 of orders at k_1+1 and k_2+1, respectively.

The Weierstrass points for the spaces we have been discussing are in fact objects that we should be familiar with. Let us return to the space V where we started with the pair (P_0, k), with arbitrary $P_0 \in S$ and $k \in \mathbb{Z}^+ \cup \{0\}$. Let $P \in S$. Consider the sequence of divisors

$$D_0 = P_0^{k+1},\ D_1 = \frac{P_0^{k+1}}{P},\ ...,\ D_k = \frac{P_0^{k+1}}{P^k},\ ...,\ D_{2g+k} = \frac{P_0^{k+1}}{P^{2g+k}}.$$

For each divisor D_k consider the vector space $L(D_k)$, the space of meromorphic functions on S whose divisors are multiples of D_k and its dimension $r(D_k)$. Call the integer $r \geq 1$ a *gap* provided $r(D_r) - r(D_{r-1}) = 0$, and otherwise call it a *nongap*. It is an immediate consequence of the Riemann-Roch theorem that $0 = r(D_0) = r(D_1) = ...\ r(D_k)$ and that $r(D_{2g+k}) = g$. We therefore obtain

$$g = r(D_{2g+k}) - r(D_1) = (r(D_{2g+k}) - r(D_{2g+k-1})) + ...\ + (r(D_2) - r(D_1)).$$

It is thus clear that there are precisely g nongaps and therefore $g+k$ gaps among the integers $1,\ ...,\ 2g+k$ (and that $1,\ ...,\ k$ are gaps). The Riemann-Roch theorem also gives us the connection with the space of differentials since

$$r\left(\frac{P_0^{k+1}}{P^r}\right) = r - (k+1) - g + i\left(\frac{P^r}{P_0^{k+1}}\right)$$

and $i\left(P_0^{-(k+1)}\right) = \dim V$. As in the classical case, r is a gap if and only if $i\left(\frac{P^{r-1}}{P_0^{k+1}}\right) - i\left(\frac{P^r}{P_0^{k+1}}\right) = 1$. If the point $P \neq P_0$ is not a classical Weierstrass point[4], then the first nongap is necessarily $\geq g+1$. Moreover, as remarked earlier, P_0 is a Weierstrass point for V if and only if it is a classical Weierstrass point.

5. An example

The example we discuss, where we compute Weierstrass points, is not the most general but is designed to indicate the possibilities and the advantages of the theta function approach when used together with the Wronskian approach. It is clear that in the study of the space of meromorphic differentials V with poles at P_0 the question of the order of pole of the Wronskian of a basis for V at P_0 is not totally trivial and this is in fact reflected in the ambiguity of the order of the zero of the theta function at this point.

Let S be a hyperelliptic surface of genus $g \geq 2$, $P_0 \in S$ and $k = 1$. We consider the Weierstrass points for the vector space V of meromorphic differentials with at most a second order pole at P_0. There are two cases depending on whether P_0 is a classical Weierstrass point or not.

Assume that P_0 is a classical Weierstrass point on S and let $P \neq P_0$ be any other classical Weierstrass point. In this case the relevant sequence of dimensions is:

$$r(P_0^2) = 0 = r\left(\frac{P_0^2}{P}\right),\ r\left(\frac{P_0^2}{P^2}\right) = 1 = \left(\frac{P_0^2}{P^3}\right),\ ...,\ r\left(\frac{P_0^2}{P^{2i}}\right) = i = r\left(\frac{P_0^2}{P^{2i+1}}\right),$$
$$...,\ r\left(\frac{P_0^2}{P^{2g}}\right) = g = r\left(\frac{P_0^2}{P^{2g+1}}\right);$$

so that the sequence of gaps at P is the sequence of odd integers $1,\ 3,\ ...,\ 2g+1$ and the weight of the Weierstrass point P for the space is $0+1+2+...+g = \frac{g(g+1)}{2}$. There are $2g+1$

[4]If the point $P \neq P_0$ is not a Weierstrass point for V, then the first nongap is necessarily $\geq g+k+1$.

such points; so that the number of Weierstrass points accounted for by the $2g+1$ classical Weierstrass points P with $P \neq P_0$ is $\frac{g(g+1)(2g+1)}{2}$.

As remarked earlier, the classical Weierstrass point P_0 is a Weierstrass point for the space V. In this case, the gap sequence is identical to the ones for $P \neq P_0$, but the weight of the Weierstrass point is different since we allow poles of order at most 2 at P_0. The weight of this point is $(-2)+(-1)+0+\ldots+(g-2)=\frac{(g-4)(g+1)}{2}$. The total weight for V of the $2g+2$ classical Weierstrass points is hence $(g+1)(g^2+g-2)$. Since the total weight of V is $(g-1)(g+1)(g+2)$, we have accounted for all the Weierstrass points of this space. The set of Weierstrass points for the space V is the same as the set of classical Weierstrass points but the weights of the points are different.

If P_0 is not a classical Weierstrass point then the appropriate sequence for P a classical Weierstrass point is

$$r(P_0^2)=0=r\left(\frac{P_0^2}{P}\right)=r\left(\frac{P_0^2}{P^2}\right)=r\left(\frac{P_0^2}{P^3}\right),\ r\left(\frac{P_0^2}{P^4}\right)=1=r\left(\frac{P_0^2}{P^5}\right),\ \ldots,$$
$$r\left(\frac{P_0^2}{P^{2i}}\right)=i-1=r\left(\frac{P_0^2}{P^{2i+1}}\right),\ \ldots,\ r\left(\frac{P_0^2}{P^{2(g-1)}}\right)=g-2=r\left(\frac{P_0^2}{P^{2g-1}}\right),$$
$$r\left(\frac{P_0^2}{P^{2g}}\right)=g-1,\ r\left(\frac{P_0^2}{P^{2g+1}}\right)=g.$$

Hence the gaps at the classical Weierstrass point P for the space V are the positive integers $1,\ 2,\ 3,\ 5,\ 7,\ \ldots,\ 2g-1$; its weight is therefore $\frac{(g-2)(g-1)}{2}$, a Weierstrass point for V if and only if $g \geq 3$. There are $2g+2$ such points so that their total weight for the space V is $(g-2)(g-1)(g+1)$. Before estimating the number of additional Weierstrass points, we must consider the point P_0 which may have negative weight. The sequence of dimensions to be studied is now

$$r(P_0^2)=0=r(P_0),\ r(1)=1=r(P_0^{-1})=r(P_0^{-2})=\ldots r(P_0^{-g}),$$
$$r(P_0^{-(g+1)})=1,\ r(P_0^{-2g})=g;$$

so the gaps at P_0 are $1,\ 3,\ 4,\ \ldots,\ g+2$ and its weight is $-(g+2)$ (since this is the minimum possible weight at P_0, this point is not a Weierstrass point for V). There are therefore an additional $4g^2+g-2$ Weierstrass points for the space V.

The approach used above to calculate the weights of the Weierstrass points is of course the Wronskian approach. The remark after Theorem 2 pointed out that we make no claim that ω is a constant multiple of Ω even though we believe this to be so since we believe that the generic Riemann surface has only simple Weierstrass points. We of course have not proven this here. For the remainder of this section though we make the implicit assumption that this is so.

Let us see what this means for the simplest cases, $g=2$ and 3.

We begin with $g=2$. Here the function $f(P)$ is $\theta(3\varphi_{P_0}(P)+\mathcal{K}_{P_0})$. Let us assume that P_0 is a classical Weierstrass point. The function f vanishes to order three at the six classical Weierstrass points (also the Weierstrass points for V) on S. The Wronskian for V has a third order zero at each Weierstrass point $P \neq P_0$ and a third order pole at P_0. If P_0 is not a classical Weierstrass point then the function f vanishes to order two at the point P_0 and at 16 other points. One of the reasons we get this situation is the following. In the case of $g=2$ if we wanted to describe the Weierstrass points for the holomorphic quadratic differentials, we would be looking at the zeros of $\theta(3\varphi_{P_0}(P)+3\mathcal{K}_{P_0}$ and would indeed find that there are 18 such points; the classical Weierstrass points each of weight 3. If P_0 is a classical Weierstrass point, it is a point of order 2 in the Jacobi variety $J(S)$; so that $3\mathcal{K}_{P_0} = \mathcal{K}_{P_0}$ and the function $\theta(3\varphi_{P_0}(P) + \mathcal{K}_{P_0})$ whose zeros are the Weierstrass points we seek are the same as the zeros of the function f. This is why we get 6 zeros each of weight 3.

Let us see whether we can deduce anything from this. In the case that P_0 is not a classical Weierstrass point, there are 16 Weierstrass points for V and none of these are classical Weierstrass points on S. The function f has 16 non trivial zeros (P_0 is a trivial double zero). These 16 zeros decompose into two sets, simple zeros and double zeros. Let us now suppose that we have an automorphism T of our surface of prime order, and that P_0 is a fixed point of this automorphism.Thee sets of simple and double zeros are each invariant under the automorphism T. This is fairly obvious from the Wronskian approach. It is however also not too difficult to see that when $g = 2$ and P_0 is not a classical Weierstrass point, the function $\theta(3\phi_{P_0}(P) + K_{P_0})$ has a double zero at P_0 and the remaining 16 zeros are either simple or of order 2. Consider

$$\theta(\phi_{P_0}(P) + K_{P_0}), \qquad \theta(\phi_{P_0}(P) - K_{P_0}).$$

The first vanishes identically on the Riemann surface and the second has a double zero at the point P_0. This implies that

$$\theta(K_{P_0}) = \sum_{i=1}^{2} \theta_i(K_{P_0})\mu_i(P_0) = \sum_{i=1}^{2} \theta_i(K_{P_0})\mu_i'(P_0) + \sum_{i,j=1}^{2} \theta_{ji}(K_{P_0})\mu_i(P_0)\mu_j(P_0) = 0$$

and that

$$\sum_{i=1}^{2} \theta_i(-K_{P_0})\mu_i'(P_0) + \sum_{i,j=1}^{2} \theta_{ji}(-K_{P_0})\mu_i(P_0)\mu_j(P_0) \neq 0$$

where the subcripts on the theta function mean partial differentiation and μ_1, μ_2 are a basis for the holomorphic differentials.

Suppose now that $\theta(3\phi_{P_0}(P) + K_{P_0})$ had a third order zero at P_0 it would follow that

$$3\sum_{i=1}^{2} \theta_i(K_{P_0})\mu_i'(P_0) + 9\sum_{i,j=1}^{2} \theta_{ji}(-K_{P_0})\mu_i(P_0)\mu_j(P_0) = 0$$

This together with the preceding would imply that

$$\sum_{i=1}^{2} \theta_i(K_{P_0})\mu_i'(P_0)$$

which is a contradiction since in order that this differential have a double zero at P_0, P_0 need be a classical Weierstrass point which it isn't.

We have already seen that the Weierstrass points for the space V in this case are not classical Weierstrass points and that all nontrivial zeros are Weierstrass points for V, it thus follows by a similar argument that all zeros can be at most of order 2. In fact a characterization of the points of weight 2 can be given which of course agrees with the one in the Wronskian approach. We outline the argument.

Let $Q \neq P_0$ be a zero of $\theta(3\phi_{P_0}(P) + K_{P_0})$. We then have that

$$3\phi_{P_0}(Q) + K_{P_0} = \phi_{P_0}(R) + K_{P_0}$$

or that $\frac{Q^3}{P_0^2 R}$ is principal and $\frac{Q^3 h(R)}{P_0^2}$ is canonical. Let us write

$$e = \phi_{P_0}(R) + K_{P_0}.$$

Then as is well known from the theory of theta functions on Riemann surfaces we have

$$\sum_{i=1}^{2} \theta_i(e)\mu_i(P)$$

is a holomorphic diferential which vanishes at the two points R and $h(R)$ on the surface where h represents the hyperelliptic involution. The condition for second order vanishing is thus that

$$\sum_{i=1}^{2} \theta_i(e)\mu_i(Q) = 0$$

and this occurs only when $Q = R$ which is ruled out since this would imply that Q is a classical Weierstrass point, or $Q = h(R)$. This of course implies that $\frac{Q^4}{P_0^2}$ is canonical which gives weight 2 in the Wronskian approach.

Let the cardinality of the sets of simple and double zeros be x and y, respectively. Since $x+2y = 16$, x must be an even integer and the possible solutions to the Diophantine equation are the pairs (x, y)

$$(0, 16)(2, 7), (4, 6), (6, 5), (8, 4), (10, 3), (12, 2), (14, 1), (16, 0).$$

The only possible orders of T are 2, 3 and 5.

If the order of T is 2 then either we have 6 fixed points and the automorphism is the hyperelliptic involution (which we have excluded by the hypothesis that P_0 a nonclassical Weierstrass point is fixed by T), or there are 2 fixed points in which case we actually know the other fixed point. It is just $h(P_0)$, where h is the hyperelliptic involution. If the solution to our Diophantine equation is either (2,7), (6,5), (10,3) or (14,1) then the fixed point ***WHICH ONE*** is in fact one of the Weierstrass points above ***MORE SPECIFIC***. If the solution is one of the remaining cases we have no further information.

NOT CHECKED REMAINDER OF THIS SECTIONIf the order of T is 3 then it is clear that some additional fixed points are always in the set of Weierstrass points above. None of the possible solutions are of the form (x, y) with both x and y congruent to zero mod 3 so there must be fixed points. In fact a case by case description is easily done.

If the order of T is 5, then there are 3 fixed points, 2 in addition to the one at P_0 and the reader can check that some of the above possibilities are in fact not possible and again will show that at least one additional fixed point is in the set of Weierstrass points for our space.

Let us now consider the case of $g = 3$ and P_0 not a classical Weierstrass point and T an automorphism with $T(P_0) = P_0$. In this case we have seen that the classical Weierstrass points are all Weierstrass points for our space as well but are all simple Weierstrass points. There therefore remains an additional 37 Weierstrass points for the space and no matter how they distribute themselves into points of weight 1,2 or 3 each of those sets will be invariant under T and an additional fixed point of T will be in this set of Weierstrass points. The argument is simple. Since 37 is prime there is no way that each of the sets can be congruent to zero mod the order of T. If they were the order of T would have to divide 37 which it can't since it is less than 37 and 37 is prime.

We therefore see that the sets of Weierstrass points we have here constructed can be useful for example in locating fixed points of automorphisms.

PART II. More material on the subject discussed here can be found in [?][5].

6. Factors of automorphy

Let G be a subgroup of $\mathrm{SL}(2,\mathbb{R})$ whose projection (denoted by same symbol) to $\mathrm{PSL}(2,\mathbb{R})$ is a finitely generated Fuchsian group of the first kind[6]. We restrict our attention to the action of G on the upper half plane $\mathbb{H}^2 = \{\tau \in \mathbb{C}; \Im\tau > 0\}$. A *factor of automorphy* for G is a function

$$e : G \times \mathbb{H}^2 \to \mathbb{C}^*$$

($\mathbb{C}^* = \mathbb{C} - \{0\}$) with

$$e(g,\cdot) : \mathbb{H}^2 \to \mathbb{C}^*$$

holomorphic for all $g \in G$ and

$$e(g_1 g_2, \tau) = e(g_1, g_2(\tau))\ e(g_2,\tau)$$

for all $g_1,\ g_2 \in G$ and all $\tau \in \mathbb{H}^2$. The most important example of a factor of automorphy is the *canonical* factor $\mathcal{K}$ defined by

$$\mathcal{K}(g,\tau) = g'(\tau),\ g \in G,\ \tau \in \mathbb{H}^2.$$

We are interested only in a restricted class of factors of automorphy. A factor of automorphy e is called *parabolic* if there exists a real constant q, which we will call the *weight* of e, and for each $g \in G$ there exists a complex number of absolute value 1, $c(g)$, such that

$$e(g,\tau) = c(g)\ g'(\tau)^q,\ \text{all } \tau \in \mathbb{H}^2.$$

This requirement involves only finitely many conditions (one for each generator of G). If $q \in \mathbb{Z}$, then c is a *(normalized) character* for G; that is, c is a homomorphism of G into the unit circle, the complex numbers of absolute value 1. It is convenient to write

$$c(g) = \exp\{2\pi\imath\alpha(g)\},\ \alpha = \alpha(g) \in \mathbb{R},\ 0 \le \alpha < 1.$$

A meromorphic function φ on $\mathbb{H}^2$ is *e-automorphic* if it satisfies

$$\varphi(\gamma(\tau))\ e(\gamma,\tau) = \varphi(\tau),\ \text{all } \tau \in \mathbb{H}^2,\ \text{all } \gamma \in G, \tag{1}$$

and has a limit, $\varphi(x)$ (as a point of $\mathbb{C} \cup \{\infty\}$), as $\tau \in \mathbb{H}^2$ approaches a parabolic fixed point[7] $x \in \mathbb{R} \cup \{\infty\}$ of G through a cusped region determined by x. The e-automorphic function φ for the parabolic factor of automorphy of weight q is *holomorphic* if it is holomorphic function on $\mathbb{H}^2$ and $\varphi(x)$ is finite for all parabolic fixed points x of G. The condition imposed at the cusps on a function to be e-automorphic needs further explanation.

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[5]We are following the notational conventions of that manuscript. In contrast to Part I, the genus of a surface is denoted by p rather that g; the latter symbol being reserved for elements of a group G.

[6]We also use the same symbol for an element of $\mathrm{SL}(2,\mathbb{R})$ and the Möbius transformation it induces.

[7]The set of parabolic fixed points of G will be denoted by $\Lambda_{\mathrm{par}}(G)$; these fixed points will also be called *cusps*.

Let $C \in \mathrm{SL}(2,\mathbb{R})$ and assume that e is a parabolic factor of automorphy for G of weight q. Then

$$\tilde{e}(C^{-1}gC,\tau) = e(g, C(\tau))\frac{(C^{-1}gC)'(\tau))^q}{g'(C(\tau))^q},\ g \in G, \tau \in \mathbb{H}^2,$$

defines a parabolic factor of automorphy $\tilde{e}$ for $\tilde{G} = C^{-1}GC$ of weight q. Let C_*^q be the map that sends a function φ to $\tilde{\varphi} = (\varphi \circ C)(C')^q$. If φ satisfies (**??**), then $\tilde{\varphi}_x = \tilde{\varphi}$ satisfies

$$\tilde{\varphi}(\tilde{\gamma}(\tau))\ \tilde{e}(\tilde{\gamma},\tau) = \tilde{\varphi}(\tau),\ \text{all}\ \tau \in \mathbb{H}^2,\ \text{all}\ \tilde{\gamma} \in \tilde{G}.$$

Let $x \in \mathbb{R} \cup \{\infty\}$ be a parabolic fixed point of G. Let P be a generator for the stabilizer of x in G. Choose a Möbius transformation $C \in\ \mathrm{PSL}(2,\mathbb{R})$ such that

$$C(x) = \infty \text{ and } CPC^{-1} = B^{\pm 1} = \begin{bmatrix} 1 & \pm 1 \\ 0 & 1 \end{bmatrix}.$$

Replacing P by its inverse, we may assume that $CPC^{-1} = B$. Let[8] $\tilde{e}(B,\tau) = \exp\{2\pi\imath\alpha\}$, $\alpha \in \mathbb{R}$, $0 \le \alpha < 1$. The function $\tilde{\varphi}$ satisfies

$$\tilde{\varphi}(\tau+1)\exp(2\pi\imath\alpha) = \tilde{\varphi}(\tau),\ \text{all}\ \tau \in \mathbb{H}^2.$$

Hence

$$f(\tau) = \exp(2\pi\imath\alpha\tau)\tilde{\varphi}(\tau),\ \tau \in \mathbb{H}^2,$$

has a Fourier series expansion

$$f(\tau) = \sum_{n\in\mathbb{Z}} a_n \exp(2\pi\imath n\tau),\ \tau \in \mathbb{H}^2.$$

We say that $\tilde{\varphi}$ is *meromorphic at* $\imath\infty$ if there exists an $N \in \mathbb{Z}$, such that $a_n = 0$ for $n < N$. It involves no loss of generality to assume that $a_N \neq 0$. In this case,

$$\tilde{\varphi}(\tau) = \exp(-2\pi\imath\alpha\tau)\sum_{n=N}^{\infty} a_n \exp(2\pi\imath n\tau),\ a_N \neq 0,\ \tau \in \mathbb{H}^2. \tag{2}$$

We say that $\tilde{\varphi}$ is *holomorphic (satisfies the cusp condition) at* $\imath\infty$ if $N - \alpha \ge 0$ $(N-\alpha > 0)$ and set

$$\tilde{\varphi}(\imath\infty) = \begin{cases} \infty \text{ if } N-\alpha < 0 \\ a_N \text{ if } N-\alpha = 0 \\ 0 \text{ if } N-\alpha > 0 \end{cases}.$$

We claim that our definitions are well defined (independent of the choices made). Our only choice was the transformation C. Had we used C_1 instead, then $C_1C^{-1} \in\ \mathrm{PSL}(2,\mathbb{R})$ and $C_1C^{-1}BCC_1^{-1} = B^{\pm 1}$. The minus sign cannot hold since a real Möbius transformation cannot conjugate B to its inverse. Thus there exists a $b \in \mathbb{R}$, such that $C_1(\tau) = C(\tau) + b$. Hence

$$\tilde{\varphi}_1(\tau) = \left((C_1)_*^q\varphi\right)(\tau) = \left(C_*^q\varphi\right)(\tau+b) = \tilde{\varphi}(\tau+b);$$

showing that

$$\tilde{\varphi}_1(\tau) = \exp(-2\pi\imath\alpha(\tau+b))\sum_{n=N}^{\infty} a_n \exp(2\pi\imath n(\tau+b))$$

$$= \exp(-2\pi\imath\alpha\tau)\sum_{n=N}^{\infty} \exp(2\pi\imath(n-\alpha)b)a_n \exp(2\pi\imath n\tau).$$

[8]This corresponds to $e(P,\tau) = \exp\{2\pi\imath\alpha\}P'(\tau)^q$.

Even though the Fourier series expansion of of a meromorphic $\tilde{e}$-automorphic function depends on the choice of the fractional linear transformation C, its value at the cusp $\imath\infty$ does not. We can now define φ to be *meromorphic, holomorphic* or satisfy *the cusp condition* at x if $\tilde{\varphi}_x$ is meromorphic, holomorphic or satisfies the cusp condition, respectively, at $\imath\infty$.

We let $\mathbb{A}^+(\mathbb{H}^2, G, e)$ and $\mathbb{A}(\mathbb{H}^2, G, e)$ denote the spaces of e-automorphic functions that are holomorphic, respectively satisfy the cusp condition, at each $x \in \Lambda_{\text{par}}(G)$. Note that $\mathbb{A}^+(\mathbb{H}^2, G, e) \supset \mathbb{A}(\mathbb{H}^2, G, e)$. Further

$$C_*^q : \mathbb{A}^+(\mathbb{H}^2, G, e) \to \mathbb{A}^+(\mathbb{H}^2, \tilde{G}, \tilde{e})$$

and

$$C_*^q : \mathbb{A}(\mathbb{H}^2, G, e) \to \mathbb{A}(\mathbb{H}^2, \tilde{G}, \tilde{e})$$

are $\mathbb{C}$-linear isomorphisms. If $q \notin \mathbb{Z}$, then we use the same branch of $(C')^q$ in all the above formulae. The resulting factor of automorphy and its space of holomorphic automorphic functions depend in a mild way on this choice.

7. Multiplicative q-forms

Let G be a finitely generated Fuchsian group of the first kind operating on $\mathbb{H}^2$. Let c be a character on G and $q \in \mathbb{Z}$. For the parabolic factor of automorphy $e = c\,\mathcal{K}^q$ for G defined by

$$e(g, \tau) = g'(\tau)^q \ c(g), \ g \in G, \ \tau \in \mathbb{H}^2,$$

we write

$$\mathbb{A}^+(\mathbb{H}^2, G, e) = \mathbb{A}_q^+(\mathbb{H}^2, G, c) \text{ and } \mathbb{A}(\mathbb{H}^2, G, e) = \mathbb{A}_q(\mathbb{H}^2, G, c).$$

The trivial character $c = 1$ is, as usual, dropped from the notation.

Let e be factor of automorphy for G of weight q. A nonzero e-automorphic meromorphic function φ (in particular, a $\varphi \in \mathbb{A}^+(\mathbb{H}^2, G, e)$) has a well defined order, $\text{ord}_x\varphi \in \mathbb{R} \cup \{0\}$, at each point $x \in \mathbb{H}^2$ and each point $x \in \Lambda_{\text{par}}(G)$, the set of parabolic fixed points $x \in \mathbb{R} \cup \{\infty\}$ of G. For $x \in \mathbb{H}^2$, $\text{ord}_x\varphi$ is defined as the order of vanishing[9] of φ at x <u>divided</u> by the order of, G_x, the stabilizer of x in G. To define the order of φ at a parabolic fixed point x of G we use the material and notation of the previous section (especially (**??**)) and set

$$\text{ord}_x\varphi = N - \alpha.$$

While the order of φ at a parabolic fixed point x is only a nonnegative real number,

$$\text{ord}_x\varphi_1 - \ \text{ord}_x\varphi_2 \in \mathbb{Z},$$

for all φ_1 and $\varphi_2 \in \mathbb{A}^+(\mathbb{H}^2, G, e) - \{0\}$[10]. It is routine to prove that if G has signature $(p, n;\ \mu_1,\ \ldots,\ \mu_n)$, then (see, for example, [**?**, §III.8] for a similar argument)

$$\sum_{x \in \overline{\mathbb{H}^2/G}} \text{ord}_x\varphi = q\left(2p - 2 + \sum_{j=1}^{n}\left(1 - \frac{1}{\mu_j}\right)\right) = D(q) \in \mathbb{Q}. \tag{3}$$

It is useful to observe that for all $\varphi \in \mathbb{A}^+(\mathbb{H}^2, G, e)$ (more generally for all meromorphic e-automorphic functions for G), all $C \in \text{SL}(2, \mathbb{R})$, and all $x \in \mathbb{H}^2 \cup \Lambda_{\text{par}}(C^{-1}GC)$,

$$\text{ord}_{C(x)}\varphi = \ \text{ord}_x C_q^*(\varphi), \tag{4}$$

[9] A rational, not necessarily an integer.

[10] More generally, for all pairs of nontrivial meromorphic e-automorphic functions.

where q is the weight of e. The possible orders of vanishing of a $0 \neq \varphi \in \mathbb{A}^+(\mathbb{H}^2, G, e)$ at an ordinary point $x \in \mathbb{H}^2$ are the positive integers

$$0,\ 1,\ \ldots,\ \lfloor D(q) \rfloor.$$

At an elliptic fixed point x, there are restrictions for the possible orders of vanishing of functions $0 \neq \varphi \in \mathbb{A}^+(\mathbb{H}^2, G, e)$ that can be readily determined. It suffices to observe that these rational numbers satisfy as a consequence of (**??**),

$$0 \leq \ \mathrm{ord}_x \varphi \leq D(q);$$

also at parabolic fixed points.

Let $q \in \mathbb{Z}$ and let c be a character on G. If $\varphi \in \mathbb{A}_1^+(\mathbb{H}^2, G, c) - \{0\}$, then $\varphi^q \in \mathbb{A}_q^+(\mathbb{H}^2, G, c^q) - \{0\}$, and for all $x \in \mathbb{H}^2 \cup \Lambda_{\mathrm{par}}(G)$,

$$\mathrm{ord}_x \varphi^q = q \ \mathrm{ord}_x \varphi;$$

hence for *rational* characters (characters c with c^N the trivial character for some $N \in \mathbb{Z}^+$), (**??**) is a consequence of the fact that the degree of a canonical divisor on a compact surface of genus $p \geq 0$ is $2p - 2$.

Proposition 1. *Let e be a factor of automorphy for the finitely generated Fuchsian group of the first kind G operating on $\mathbb{H}^2$. Then*

$$\dim \mathbb{A}^+(\mathbb{H}^2, G, e) < \infty.$$

Proof. If $\dim \mathbb{A}^+(\mathbb{H}^2, G, e) > 0$, choose a nonzero element $\varphi_o \in \mathbb{A}^+(\mathbb{H}^2, G, e)$. For any $\varphi \in \mathbb{A}^+(\mathbb{H}^2, G, e)$, the ratio $\frac{\varphi}{\varphi_o}$ projects to a meromorphic function on $\overline{\mathbb{H}^2/G}$, the compactification of $\mathbb{H}^2/G$ obtained by filling in the punctures) with poles only at the zeros of φ_o. These poles have orders at most $D(q)$. □

Remark 2. Let c be a character on G. A form $\varphi \in \mathbb{A}_1(\mathbb{H}^2, G, c) - \{0\}$ defines a Prym differential on the compact Riemann surface $\overline{\mathbb{H}^2/G}$ if and only if $c(\gamma) = 1$ for all elliptic and parabolic $\gamma \in G$.

8. Weierstrass points for subspaces of $\mathbb{A}^+(\mathbb{H}^2, G, e)$

We fix a finitely generated Fuchsian group of the first kind G operating on $\mathbb{H}^2$. Let e be a parabolic factor of automorphy of weight q for G. Let V be a nontrivial d-dimensional vector space of meromorphic e-automorphic functions (in particular, V could be a subspace of $\mathbb{A}^+(\mathbb{H}^2, G, e)$). We proceed to define an invariant for this space. For $x \in \mathbb{H}^2$ or x a parabolic fixed point of G. Let

$$r_0 < r_1 < \ldots < r_{d-1}$$

be the possible orders of vanishing[11] at x for elements of $V-\{0\}$. We define the *weight* of the point x with respect to the vector space V as

$$\tau_V(x)=\sum_{i=0}^{d-1}\left(r_i-\frac{i}{\mu}\right)$$

if $x\in\mathbb{H}^2$ and $|G_x|=\mu=\mu_x$, and

$$\tau_V(x)=\sum_{i=0}^{d-1}r_i$$

if x is a parabolic fixed point for G. Note that $r_i\in\mathbb{R}$ for $x\in\Lambda_{\text{par}}(G)$, $r_i\in\mathbb{Q}$ whenever $x\in\mathbb{H}^2$, and $r_i\in\mathbb{Z}$ if in addition $\mu_x=1$. We call x a *Weierstrass* point with respect to V if and only if every element of V is holomorphic[12] at x and its weight is above the minimum that it can be; that is, if and only if $r_0\geq 0$ and

$$\tau_V(x)>\left\{\begin{array}{r}0 \text{ if } x\in\mathbb{H}^2\\ \frac{d(d-1)}{2} \text{ if } x\in\Lambda_{\text{par}}(G)\end{array}\right. .$$

The *weight* of the vector space V is

$$\tau(V)=\sum_{x\in\overline{\mathbb{H}^2/G}}\tau_V(x).$$

Choose a basis $\varphi_0,\ ...,\ \varphi_{d-1}$ for V, and as in [?, §III.5] form their *Wronskian*

$$W=\ \det\left[\begin{array}{ccc}\varphi_0 & \cdots & \varphi_{d-1}\\ \varphi_0' & \cdots & \varphi_{d-1}'\\ & \cdots & \\ \varphi_0^{(d-1)} & \cdots & \varphi_{d-1}^{(d-1)}\end{array}\right]=\ \det\,[\varphi_0,\ ...,\ \varphi_{d-1}].$$

It is easily seen that a change of basis for V results in a nonzero constant multiple of W and that W is a $e^d\kappa^{\frac{d(d-1)}{2}}$-automorphic function ($W\in\mathbb{A}^+(\mathbb{H}^2,G,e^d\kappa^{\frac{d(d-1)}{2}})$ whenever $V\subset\mathbb{A}^+(\mathbb{H}^2,G,e)$), where κ is the canonical factor of automorphy for G. Hence deg $(W)=d\ \chi(G)\left(q+\frac{d-1}{2}\right)$, where $\chi(G)$ is the negative Euler characteristic of G.

Proposition 2. *For all $x\in\mathbb{H}^2\cup\Lambda_{par}(G)$,*

$$\text{ord}_x W=\tau_V(x).$$

Hence

$$\tau(V)=\text{deg }(W)=d\ \chi(G)\left(q+\frac{d-1}{2}\right).$$

[11]If $x\in\mathbb{H}^2$ is stabilized by an elliptic subgroup of order μ, then the ordinary order of vanishing of meromorphic functions is divided by μ; at a parabolic fixed point, we use the order of vanishing of the Fourier series in terms of an appropriate local coordinate vanishing at the corresponding puncture. In this context, the weight of a parabolic fixed point can be interpreted as the limiting case of the weight of an elliptic fixed point as $\mu\to\infty$. Thus the second formula below should be interpreted as a special case of the first one with $\mu=\infty$. We will use this convention, usually without further remark, throughout the remainder of the manuscript.

[12]It is often desirable to have a way of labeling points at which functions in V have singularities as Weierstrass points. This extension is very sensitive to the definition of the space V. For our example in §**??** the point P_0 is naturally considered a Weierstrass point for V if and only if it is a classical Weierstrass point.

Proof. The arguments of [**?**, §III.5], whose notation we employ in this section, easily establish the equality for $x \in \mathbb{H}^2$. For $x \in \mathbb{R} \cup \{\infty\}$ a parabolic fixed point, we use the fact that

$$\det\,[\varphi_0,\ ...,\ \varphi_{d-1}] = \varphi_0^d \det\left[1,\ \frac{\varphi_1}{\varphi_0},\ ...,\ \frac{\varphi_{d-1}}{\varphi_0}\right] = \varphi_0^d \det\left[\frac{\varphi_1}{\varphi_0},\ ...,\ \frac{\varphi_{d-1}}{\varphi_0}\right],$$

and hence by induction on d that

$$\mathrm{ord}_x \det\,[\varphi_0,\ ...,\ \varphi_{d-1}] = d\ r_0 + \sum_{i=1}^{d-1} r_i - (d-1) r_0.$$

□

9. Holomorphic maps of $\overline{\mathbb{H}^2/\Gamma(k)}$ into low dimensional projective spaces

The purpose of this section is to give an application of the material on Weierstrass points for finite dimensional spaces of holomorphic functions developed in §**??**.

Let $k \in \mathbb{Z}^+$. Let $V(k)$ be the finite dimensional vector space of holomorphic functions on $\mathbb{H}^2$ spanned by the modified theta constants

$$\tau \mapsto \varphi_l(\tau) = \theta[\chi_l](0, k\tau),$$

where the characteristic $\chi_l = \begin{bmatrix} \frac{2l+1}{k} \\ 1 \end{bmatrix}$, $l = 0,\ ...,\ \frac{k-3}{2}$, for odd $k \geq 3$ and $\chi_l = \begin{bmatrix} \frac{2l}{k} \\ 0 \end{bmatrix}$, $l = 0,\ ...,\ \frac{k}{2}$, for even k.

For odd k, let $V'(k)$ be the finite dimensional vector space of holomorphic functions on $\mathbb{H}^2$ spanned by the modified theta constant derivatives ($'$ denotes differentiation with respect to z, the first variable of $\theta[\chi](z,\tau)$)

$$\tau \mapsto \varphi_l'(\tau) = \theta' \begin{bmatrix} \frac{2l+1}{k} \\ 1 \end{bmatrix} (0, k\tau),\ l = 0,\ ...,\ \frac{k-1}{2}.$$

For even k, let $V'(k)$ be the finite dimensional vector space of holomorphic functions on $\mathbb{H}^2$ spanned by the modified theta constant derivatives

$$\tau \mapsto \varphi_l'(\tau) = \theta' \begin{bmatrix} \frac{2l}{k} \\ 0 \end{bmatrix} (0, k\tau),\ l = 1,\ ...,\ \frac{k-2}{2}.$$

The four spaces introduced have respective dimensions $\frac{k-1}{2}$, $\frac{k}{2}+1$, $\frac{k+1}{2}$ and $\frac{k}{2}-1$. We will denote this number by $d+1$. The easiest way to obtain the linear independence of the functions introduced is through the observation that in terms of the local coordinate $x = \exp\left(\frac{2\pi\imath\tau}{k}\right)$, for odd k, and $x = \exp\left(\frac{\pi\imath\tau}{k}\right)$, for even k,

$$\mathrm{ord}_\infty \varphi_l = \frac{(1+2l)^2}{8} = \mathrm{ord}_\infty \varphi_l' \text{ for odd } k \text{ and } \mathrm{ord}_\infty \varphi_l = l^2 = \mathrm{ord}_\infty \varphi_l' \text{ for even } k.$$

We need to look at several families of subgroups of the modular group $\mathrm{PSL}(2,\mathbb{Z}) = \Gamma$:

(a) $\Gamma(k)$ is the level k principal congruence subgroup of elements represented by matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ with $a \equiv 1 \mod k \equiv d$ and $b \equiv 0 \mod k \equiv c$,

(b) $\Gamma_o(k)$ consisting of elements represented by matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma$ with $c \equiv 0 \mod k$ and

(c) $H(k)$ defined only for even k and consisting of elements represented by matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma(k)$ with $b \equiv 0 \mod 2k \equiv c$.

The needed facts concerning the first two families of groups are well known (see, for example, [?, Ch. 1]). The groups $H(k)$ appear less often in the literature. We need to know that $H(2) = \Gamma(4)$ and that for all even $k \geq 4$,

$$\Gamma(2k) \subset H(k) \subset \Gamma(k);$$

the two inclusions being of index 2 and 4, respectively. The group $H(k)$ is torsion free of type $(4p(k) - 3 + n(k), 2n(k))$, where $(p(k), n(k))$ is the type of $\Gamma(k)$; its (negative) Euler characteristic is

$$\chi(H(k)) = \begin{cases} 4 \text{ for } k = 2 \\ \frac{k^3}{3} \prod_{\mathfrak{P} \text{ prime}, \, \mathfrak{P}|k} \left(1 - \frac{1}{\mathfrak{P}^2}\right) \text{ for } k \geq 4 \end{cases}.$$

The basic facts about the spaces $V(k)$ and $V'(k)$ are summarized in

Lemma 1. *Let*

$$s = \begin{cases} \frac{1}{4} \textit{ when considering } V(k) \\ \frac{3}{4} \textit{ when considering } V'(k) \end{cases} \textit{ and } G = \begin{cases} \Gamma(k) \textit{ for odd } k \\ H(k) \textit{ for even } k \end{cases}.$$

(a) *The Petersson inner product*

$$< \varphi, \psi >= \int \int_{\mathbb{H}^2/G} (\Im z)^{2-2s} \varphi(z) \overline{\psi(z)} \left| \frac{dz\overline{dz}}{2} \right|$$

endows $V(k)$ and $V'k)$ with a Hilbert space structure; this space is invariant under the s-action of Γ: an element $\gamma \in \Gamma$ acts on these spaces as the operator γ_s^* that sends φ to $(\varphi \circ \gamma)(\gamma')^s$.

(b) *The spaces $V(k)$ and $V'(k)$ consist of automorphic functions for a factor of automorphy of weight s for the group G.*

(c) *Assume that k is even. For each element $\gamma = \begin{bmatrix} * & * \\ c & * \end{bmatrix} \in \Gamma(k)$, $\gamma_{\frac{1}{4}}^*$ sends φ_l, $l = 0, \ \ldots, \ \frac{k}{2}$, to a nonzero constant multiple of itself if $2k|c$ and to a nonzero constant multiple of $\varphi_{\frac{k}{2}-l}$ otherwise, and $\gamma_{\frac{3}{4}}^*$ sends φ_l', $l = 1, \ \ldots, \ \frac{k}{2} - 1$, to a nonzero constant multiple of itself if $2k|c$ and to a nonzero constant multiple of $\varphi'_{\frac{k}{2}-l}$ otherwise.*

(d) *For all $\gamma \in \Gamma$, γ_s^* is a unitary operator on $V(k)$, $V'(k)$.*

(e) *Assume that k is odd. For each motion $\gamma \in \Gamma_o(k)/\Gamma(k)$ there is a permutation σ_γ of the first $\frac{k-1}{2}$ nonnegative integers such that*

$$\gamma_{\frac{1}{4}}^* \varphi_l = \kappa(l, \gamma) \varphi_{\sigma_\gamma(l)}, \ \gamma_{\frac{3}{4}}^* \varphi_l' = \kappa(l, \gamma) \varphi'_{\sigma_\gamma(l)}, \ l = 0, \ 1, \ \ldots, \ \frac{k-3}{2}, \textit{ and } \gamma_{\frac{3}{4}}^* \varphi'_{\frac{k-1}{2}} = \kappa \varphi'_{\frac{k-1}{2}},$$

where $\kappa(l, \gamma)$ and κ are constants of absolute value 1.

(f) *Assume that k is an odd prime. The $\frac{k-1}{2}$ functions $\{\varphi_0, \ \ldots, \ \varphi_{\frac{k-3}{2}}\}$ form an orthogonal basis for the Hilbert space $V(k)$; these functions have the same norm. The $\frac{k+1}{2}$ functions $\{\varphi_0', \ \ldots, \ \varphi'_{\frac{k-1}{2}}\}$ form an orthogonal basis for the Hilbert space $V'(k)$; the first $\frac{k-1}{2}$ of these functions have the same norm.*

Proof. For odd k, most of the lemma has been established in [**?**]. Consult [**?**] for a complete treatment. □

Proposition 3. *For each odd integer $k \geq 5$, the maps*

$$\Phi : \tau \mapsto (\varphi_0(\tau),\ \varphi_1(\tau),\ ...,\ \varphi_{\frac{k-5}{2}}(\tau),\ \varphi_{\frac{k-3}{2}}(\tau))$$

and

$$\Phi' : \tau \mapsto (\varphi_0'(\tau),\ \varphi_1'(\tau),\ ...,\ \varphi_{\frac{k-3}{2}}(\tau),\ \varphi_{\frac{k-1}{2}}(\tau))$$

from $\mathbb{H}^2 \cup \mathbb{Q} \cup \{\infty\}$ to $\mathbb{C}^{\frac{k-1}{2}}$ and $\mathbb{C}^{\frac{k+1}{2}}$, define holomorphic mappings (also to be called Φ and Φ') from $\overline{\mathbb{H}^2/\Gamma(k)}$ into $\mathbf{P}\mathbb{C}^{\frac{k-3}{2}}$ and $\mathbf{P}\mathbb{C}^{\frac{k-1}{2}}$. If k is prime, the distinguished punctures on the surface $\mathbb{H}^2/\Gamma(k)$ are sent (injectively) onto the coordinate vectors in projective space. Further, in this case, the maps Φ and Φ' restricted to the punctures are injective, and of maximal rank at each distinguished puncture (punctures on $\mathbb{H}^2/\Gamma(k)$ that are $\Gamma_o(k)/\Gamma(k)$-equivalent to P_∞, the puncture on $\mathbb{H}^2/\Gamma(k)$ obtained from the cusp $\imath\infty$.

Proof. The case of $V(k)$ has been treated in [**?**]. The arguments for $V'(k)$ are similar. □

We assume for the remainder of this note that k is odd. The following corollary and proposition were also established in [**?**].

Corollary 1. *For every prime $k \geq 7$, the map Φ is also defined by $\frac{k-1}{2}$ linearly independent abelian differentials of the first kind on $\overline{\mathbb{H}^2/\Gamma(k)}$.*

Remark 3. $\Phi(\overline{\mathbb{H}^2/\Gamma(k)})$ is a curve of degree $\frac{(k^2-1)(k-3)}{48}$ in $\mathbf{P}\mathbb{C}^{\frac{k-3}{2}}$. For $k = 7$ it is a curve of genus 3 and degree 4 in 2-dimensional complex projective space; as a consequence of the last corollary, the canonical curve.

Problem 1. Are the maps Φ and Φ' injective?

Proposition 4. *For each prime $k > 3$, the homomorphism*

$$\Theta : \Gamma/\Gamma(k) \to \ \text{Aut}\ \mathbf{P}V(k)$$

induced by Φ is injective.

Remark 4. It is probably true that Φ is injective. The injectivity of Θ would follow trivially from this conjecture.

Although we have few tools for the investigation of the maps Φ and Φ' at points $x \in \mathbb{H}^2/\Gamma(k)$, at times enough information at the punctures can be translated to results about interior points. Among the results so obtained are the following two propositions.

Proposition 5. (a) *For odd $k > 1$, each puncture on $\mathbb{H}^2/\Gamma(k)$ is a Weierstrass point for $V(k)$ and $V'(k)$, and only the punctures are Weierstrass points.*
(b) *For $k = 2$, the 6 punctures on $\mathbb{H}^2/H(2)$ are the Weierstrass points for $V(2)$.*
(c) *For $k \geq 4$, the punctures on $\mathbb{H}^2/H(k)$ are the Weierstrass points for $V(k)$ and $V'(k)$.*

Proof. Assume that k is odd. It is obvious that ∞ is a Weierstrass point for $V(k)$ since

$$r_l = \ \text{ord}_\infty \varphi_l = \frac{(2l+1)^2}{8} > l \text{ for } l = 0,\ ...,\ \frac{k-3}{2}.$$

The same argument shows that ∞ is a Weierstrass point for $V'(k)$. Since Γ preserves $V(k)$ and $V'(k)$ and acts transitively on the punctures of $\mathbb{H}^2/\Gamma(k)$, we conclude that all the cusps are Weierstrass points of the same weight. Now straight forward calculations (using formulae for the sums of squares of odd integers and the fact that the negative Euler characteristic of $\Gamma(k)$ is $\frac{k^3}{12}\prod_{\mathfrak{P}\text{ prime, }\mathfrak{P}|k}\left(1-\frac{1}{\mathfrak{P}^2}\right)$) show that

$$\tau_{V(k)}(\infty)=\frac{k(k-1)(k-2)}{48},\ \tau_{V'(k)}(\infty)=\frac{k(k+1)(k+2)}{48},$$
$$\tau(V(k))=\frac{k^3(k-1)(k-2)}{96}\prod_{\mathfrak{P}\text{ prime, }\mathfrak{P}|k}\left(1-\frac{1}{\mathfrak{P}^2}\right)$$

and

$$\tau(V'(k))=\frac{k^3(k+1)(k+2)}{96}\prod_{\mathfrak{P}\text{ prime, }\mathfrak{P}|k}\left(1-\frac{1}{\mathfrak{P}^2}\right).$$

Since the $\frac{k^2}{2}\prod_{\mathfrak{P}\text{ prime, }\mathfrak{P}|k}\left(1-\frac{1}{\mathfrak{P}^2}\right)$ punctures account for the total weight of $V(k)$ and $V'(k)$, the proof is complete in this case. For the odd prime k, we have the simplification

$$\tau(V(k))=\frac{(k-2)(k-1)^2k(k+1)}{96}\text{ and }\tau(V'(k))=\frac{(k-1)k(k+1)^2(k+2)}{96}.$$

For even k,

$$\tau_{V(k)}(\infty)=\frac{k(k+1)(k+2)}{24},\ \tau_{V'(k)}(\infty)=\frac{k(k-1)(k-2)}{24},$$
$$\tau(V(k))=\begin{cases}6 & \text{for } k=2\\ \frac{k^3(k+1)(k+2)}{24}\prod_{\mathfrak{P}\text{ prime, }\mathfrak{P}|k}\left(1-\frac{1}{\mathfrak{P}^2}\right) & \text{for } k\geq 4\end{cases}$$

and

$$\tau(V'(k))=\frac{k^3(k-1)(k-2)}{24}\prod_{\mathfrak{P}\text{ prime, }\mathfrak{P}|k}\left(1-\frac{1}{\mathfrak{P}^2}\right).$$

Once again, for $k=2$ (for $k\geq 4$), the 6 $\left(k^2\prod_{\mathfrak{P}\text{ prime, }\mathfrak{P}|k}\left(1-\frac{1}{\mathfrak{P}^2}\right)\right)$ punctures account for all the Weierstrass points. $\square$

Proposition 6. *For all odd $k\in\mathbb{Z}^+$, $k\geq 3$, the maps Φ and Φ' have maximal rank everywhere.*

Proof. We need to show that $d\Phi$ and $d\Phi'$ are nonsingular everywhere; equivalently that for all $x\in\mathbb{H}^2\cup\mathbb{Q}\cup\{\infty\}$ we can find functions f and g in $V(k)$ ($V'(k)$) that are regular at x with $\frac{g}{f}$ having a simple zero at x. We have seen that we can choose f and g in both $V(k)$ and $V'(K)$ so that

$$\mathrm{ord}_\infty f=\frac{1}{8}\text{ and }\ \mathrm{ord}_\infty g=\frac{9}{8}.$$

Hence our maps have maximal rank at the cusps. The previous proposition showed that the weight of each ordinary point $x\in\mathbb{H}^2$ with respect to either $V(k)$ or $V'(k)$ is zero. Hence there certainly are functions f and g in these spaces with

$$\mathrm{ord}_x f=0\text{ and }\ \mathrm{ord}_x g=1.$$

Thus the two maps have maximal rank globally. $\square$

Remark 5. The maps Φ and Φ' can be defined for even k (using the groups $H(k)$ rather than $\Gamma(k)$). Results similar to the last proposition follow. The details are left to the reader.

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