# PARTITIONS AND THETA CONSTANT IDENTITIES

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## 1. Introduction

In the elementary combinatorial theory of numbers one finds many results whose proofs are consequences of solutions to function theoretic problems. Perhaps a good example of this is what is generally regarded as the first problem in partition theory, the problem of Euler [?]: Given a positive integer $N$, partition[1]it into parts without allowing repetitions. Call a partition *even* if it contains an even number of parts and otherwise call it *odd*. Euler proved that the number of even partitions of $N$ is equal to the number of odd partitions of the integer provided $N$ is not of the form $\frac{3k^2+k}{2}$. If $N$ is of the form $\frac{3k^2+k}{2}$, the difference between the number of even and odd partitions is $(-1)^k$.

Euler's proof is based on an identity he derived that now carries his name:

$$\prod_{n=1}^{\infty}(1-x^n) = \sum_{n=-\infty}^{\infty}(-1)^n x^{\frac{3n^2+n}{2}}.$$

Subsequently, a purely combinatorial proof was obtained of Euler's partition result. A reasonable reference for the above and much of what we will be saying about combinatorial number theory without proof or without further reference is the beautiful book of Hardy and Wright [?]. The basic facts about theta functions that we use can be found in [?] and [?]; our notation is consistent with that used in [?] and our manuscript under preparation [?].

In this note we are interested in three different related partition problems. Their solutions, as with Euler's problem, will follow from identities we derive. The identities we use are in fact not new, though the application to this sort of combinatorial problem seems to us to be. The problems we present are variants of Euler's problem in the sense that we also consider partitions of an integer; but the parts are taken from a specific given set $S \subset \mathbb{Z}^+$. The set, for example, could be as in the case of Euler, the set of positive integers with each positive integer appearing only once (that is, we allow no repetitions of parts). Alternatively, the set

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[1]A partition of $N$ is a set of positive integers, called *parts*, whose sum is $N$.