PARABOLIC WAVELET TRANSFORMS AND LEBESGUE SPACES OF PARABOLIC POTENTIALS

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1. Introduction and main results

Continuous wavelet transforms

$$f(x) \to Wf(x,a) = \frac{1}{a^n} \int_{\mathbb{R}^n} f(y) w\left(\frac{|x-y|}{a}\right) dy, \tag{1.1}$$

$$x \in \mathbb{R}^n, \qquad a > 0, \qquad \int_{\mathbb{R}^n} w(|y|)dy = 0,$$

play an important role in harmonic analysis, function theory and have many applications (see, e.g., [6–8, 12, 14, 15, 21, 22] and references therein). Owing to the formula

$$\int_0^\infty Wf(x,a)\frac{da}{a^{1+\alpha}} = c(-\Delta)^{\alpha/2} f(x), \quad c = c(\alpha, w), \tag{1.2}$$

$\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2$, which can be given a precise sense in the framework of the L^p-theory [21], the wavelet transforms (1.1) can be used for characterization of spaces of functions of fractional smoothness and in a variety of problems dealing with powers of the laplacian (e.g., in integral geometry and in fractional calculus [12, 21, 23]. Our general aim is to find natural anisotropic analogs of (1.1), to develop the relevant L^p-theory and to give applications.

In this paper we introduce continuous wavelet transforms associated with the heat operators $-\Delta_x + \partial/\partial t$, $\quad I - \Delta_x + \partial/\partial t$, $\quad x \in \mathbb{R}^n$, $\quad t \in \mathbb{R}^1$. By making use of

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these transforms we obtain new explicit inversion formulas for parabolic potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$ defined in the Fourier terms by

$$F[H^\alpha f](\xi,\tau) = (|\xi|^2 + i\tau)^{-\alpha/2} F[f](\xi,\tau), \tag{1.3}$$

$$F[\mathcal{H}^\alpha f](\xi,\tau) = (1 + |\xi|^2 + i\tau)^{-\alpha/2} F[f](\xi,\tau). \tag{1.4}$$

These potentials were introduced by B.F. Jones, Jr. [13] and C.H. Sampson [26] and studied in [3–5, 9, 10, 16–19]. We also obtain a parabolic analog of the Calderón reproducing formula and give a new characterization of the relevant anisotropic Sobolev spaces.

One should mention the papers [1, 2] by I.A. Aliev devoted to inversion and characterization of parabolic potentials generated by a generalized translation operator in terms of hypersingular integrals with finite differences. The techniques developed below can be applied to this class of potentials. The corresponding results will be presented in forthcoming publications.

Main results.

Let $\mathbb{R}^{n+1}$ be the $(n+1)$-dimensional euclidean space of points (x,t), $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, $t \in \mathbb{R}^1$. Given a finite Borel measure $m \equiv m(x,t)$ on $\mathbb{R}^{n+1}$, we define an *anisotropic dilation* $m_{\sqrt{a},a}$ of m by

$$\int\limits_{\mathbb{R}^{n+1}} \omega(x,t)\, dm_{\sqrt{a},a}(x,t) = \int\limits_{\mathbb{R}^{n+1}} \omega(\sqrt{a}x, at)\, dm(x,t), \tag{1.5}$$

$a > 0$, $\omega \in C_0 = C_0(\mathbb{R}^{n+1})$ (the space of continuous functions on $\mathbb{R}^{n+1}$ vanishing at infinity). If $m(\mathbb{R}^{n+1}) = 0$, the convolution

$$\begin{aligned} A_m f(x,t;a) = {} & (f * m_{\sqrt{a},a})(x,t) = \\ & = \int\limits_{\mathbb{R}^{n+1}} f(x - \sqrt{a}y, t - a\tau)\, dm(y,\tau) \end{aligned} \tag{1.6}$$

will be called *an anisotropic wavelet transform* of f generated by the wavelet measure m. The choice of m is at our disposal. We choose it as follows. Let

$$W(x,t) = (4\pi t)^{-n/2} \exp(-|x|^2/4t), \quad x \in \mathbb{R}^n, \ t > 0, \tag{1.7}$$

be the Gauss-Weierstrass kernel possessing the following properties [27]:

$$1)\ \int_{\mathbb{R}^n} W(x,t)dx = 1; \qquad 2)\ \int_{\mathbb{R}^n} W(y,t)W(x-y,\tau)dy = W(x,t+\tau);$$
$$3)\ F[W(\cdot,t)](\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} W(x,t)dx = e^{-t|\xi|^2}. \tag{1.8}$$

Suppose that μ is a finite Borel measure on $\mathbb{R}^1$ such that $\mu(\mathbb{R}^1) = 0$ and $\operatorname{supp}\mu \subset [0,\infty)$. Denote $\mu' = \mu - \mu(\{0\})\delta$, where $\delta = \delta(t)$ is the unit mass at $t = 0$. We set

$$m = \mu(\{0\})\delta(x,t) + m', \qquad dm'(x,t) = W(x,t)dxd\mu'(t), \tag{1.9}$$

$W(x,t)$ being extended to $t \le 0$ by zero. Clearly, m is finite and $m(\mathbb{R}^{n+1}) = 0$.

Definition 1.1. *The anisotropic wavelet transform (1.6) with m defined by (1.9) will be called a parabolic wavelet transform and denoted by $P_\mu f(x,t;a)$. Thus,*

$$P_\mu f(x,t;a) = \mu(\{0\})f(x,t) + \int_{\mathbb{R}^n\times(0,\infty)} f(x-\sqrt{a}y, t-a\tau)W(y,\tau)dyd\mu(\tau) \tag{1.10}$$

or

$$P_\mu f(x,t;a) = \int_{\mathbb{R}^n\times[0,\infty)} f(x-\sqrt{a\tau}z, t-a\tau)W(z,1)dzd\mu(\tau). \tag{1.11}$$

Given two measures μ and ν on $\mathbb{R}^1$ supported by $[0,\infty)$ we have

$$P_{\mu*\nu} f(x,t;a) = P_\mu[P_\nu f(\cdot,\cdot;a)](x,t;a). \tag{1.12}$$

Definition 1.2. *A finite Borel measure μ on $\mathbb{R}^1$ supported by $[0,\infty)$ is called admissible if*

$$k(t) \overset{\text{def}}{=} \frac{\mu([0,t))}{t} \in L^1(0,\infty). \tag{1.13}$$

The following statement gives a “parabolic” analog of the Calderón reproducing formula for functions $f \in L^p = L^p(\mathbb{R}^{n+1})$ (cf. [8, 22]).

Theorem A. *Let μ be an admissible measure, and let*

$$k_0 = \int_0^\infty k(t)dt \qquad \text{(see (1.13))}. \tag{1.14}$$

(i) *If $f \in L^p$, $1 < p < \infty$, then*

$$\int_0^\infty P_\mu f(x,t;a)\frac{da}{a} \equiv \lim_{\substack{\varepsilon\to 0\\ \rho\to\infty}} \int_\varepsilon^\rho (\cdots) = k_0 f(x,t) \tag{1.15}$$

where $\lim = \overset{(L^p)}{\lim}$.

(ii) *If* $f \in C_0$, *then (1.15) holds with the limit interpreted in the* $\sup\limits_{(x,t)\in\mathbb{R}^{n+1}}$ *-norm.*

(iii) *If* $f \in L^p$, $1 < p < \infty$, *and* $k(t)$ *has a decreasing integrable majorant, then (1.15) holds a.e. on* $\mathbb{R}^{n+1}$.

As in [22, p. 180] and [21, Section 12], a measure μ is admissible if $\mathrm{supp}\mu \subset [0,\infty)$, $\mu(\mathbb{R}^1) = 0$ and one of the following conditions holds:

$$(a) \qquad \int\limits_0^\infty |\log t| d|\mu|(t) < \infty \tag{1.16}$$

or

$$(b) \qquad d\mu(t) = g(t)dt, \quad g \in H^1 \text{ (the real Hardy space on } \mathbb{R}^1). \tag{1.17}$$

Furthermore,

$$k_0 = \begin{cases} \displaystyle\int_0^\infty \log(1/t) d\mu(t) \ \text{ in the case } (a), \\ \displaystyle\frac{\pi}{2}\int_{-\infty}^\infty Hg(t) sgn\, t\, dt \ \text{ in the case (b),} \end{cases} \tag{1.18}$$

$Hg(t) = p.v.\,\pi^{-1}\int_0^\infty g(\tau)(t-\tau)^{-1}d\tau (\in L^1(0,\infty))$ being the Hilbert transform of g.

Here and on the notation like $\int_a^b f(t)d\mu(t)$ designates $\int_{[a,b)} f(t)d\mu(t)$.

Remark. We do not know whether the statement (iii) of Theorem A holds for $p = 1$ because we cannot assert the validity of the weak estimate for the relevant mixed partial maximal functions (see the proof of Theorem A in Section 2).

The potentials $H^\alpha f$ and $\mathcal{H}^\alpha f$ initially defined by (1.3) and (1.4) are representable by the integrals

$$\begin{aligned} H^\alpha f(x,t) &= \frac{1}{\Gamma(\alpha/2)} \int\limits_{\mathbb{R}^n\times(0,\infty)} \tau^{\alpha/2-1} W(y,\tau) f(x-y,t-\tau) dy d\tau = \\ &= (h_\alpha * f)(x,t), \qquad h_\alpha(x,t) = \frac{1}{\Gamma(\alpha/2)} t_+^{\alpha/2-1} W(x,t); \end{aligned} \tag{1.19}$$

$$\begin{aligned} \mathcal{H}^\alpha f(x,t) &= \frac{1}{\Gamma(\alpha/2)} \int\limits_{\mathbb{R}^n\times(0,\infty)} \tau^{\alpha/2-1} e^{-\tau} W(y,\tau) f(x-y,t-\tau) dy d\tau, \\ &= (\tilde{h}_\alpha * f)(x,t), \quad \tilde{h}_\alpha(x,t) = e^{-t} h_\alpha(x,t). \end{aligned} \tag{1.20}$$

The integrals (1.19), (1.20) have the structure of one-sided potentials (cf. [20], where similar potentials involving the Poisson kernel (instead of Gauss-Weierstrass' one) were investigated).

Theorem 1.3 [3, 9]**.**

I. *Let* $f \in L^p$, $1 \le p < \infty$, $0 < \alpha < (n+2)/p$, $\quad q = (n+2-\alpha p)^{-1}(n+2)p$.

(*a*) *The integral* $(H^\alpha f)(x,t)$ *converges absolutely for almost all* $(x,t) \in \mathbb{R}^{n+1}$.

(b) *For* $p > 1$, *the operator* H^α *is bounded from* L^p *into* L^q.

(c) *For* $p = 1$, H^α *is an operator of the weak* $(1,q)$ *type:*

$$|\{(x,t) : |H^\alpha f)(x,t)| > \gamma\}| \le \left(\frac{c\|f\|_1}{\gamma}\right)^q.$$

II. *The operator* $\mathcal{H}^\alpha$ *is bounded in* L^p *for all* $\alpha \ge 0$, $\quad 1 \le p \le \infty$.

Remark 1.4. If $\alpha \ge (n+2)/p$, the integral (1.19) may be divergent for $f \in L^p$. In this case we interpret $H^\alpha f$ as a Φ_0'-distribution defined by duality $\langle H^\alpha f, \omega\rangle = \langle f, H_-^\alpha \omega\rangle$, $\quad H_-^\alpha \omega = U H^\alpha U \omega$, $\quad U\omega(x,t) = \omega(-x,-t)$ (since $F[H_-^\alpha \omega](\xi,\tau) = (|\xi|^2 - i\tau)^{-\alpha/2} F[\omega](\xi,\tau)$, H_-^α is an automorphism of Φ_0 for any α). Here the test function ω belongs to the space $\Phi_0 = \Phi_0(\mathbb{R}^{n+1})$ of Schwartz functions orthogonal to all polynomials [21, p. 19], and the abbreviation like $\langle f, g\rangle$ is used for $\int f\bar{g}$.

A straightforward calculation enables us to represent (1.19), (1.20) via the relevant wavelet transforms. Namely, for $Re\,\alpha > 0$,

$$H^\alpha f(x,t) = c_{\alpha,\mu}^{-1} \int_0^\infty P_\mu f(x,t;a)\,\frac{da}{a^{1-\alpha/2}}, \tag{1.21}$$

$$c_{\alpha,\mu} = \Gamma(\alpha/2) \int_0^\infty \tau^{-\alpha/2} d\mu(\tau),$$

provided that $\int_0^\infty \tau^{-Re\,\alpha/2} d|\mu|(\tau) < \infty$ and $c_{\alpha,\mu} \ne 0$. Similarly,

$$\mathcal{H}^\alpha f(x,t) = c_{\alpha,\mu}^{-1} \int_0^\infty \mathcal{P}_\mu f(x,t;a)\,\frac{da}{a^{1-\alpha/2}} \tag{1.22}$$

where

$$\begin{aligned}\mathcal{P}_\mu f(x,t;a) &= \mu(\{0\}) f(x,t) + \int\limits_{\mathbb{R}^n \times (0,\infty)} f(x - \sqrt{a}y, t - a\tau) W(y,\tau) e^{-a\tau} dy d\mu(\tau) = \\ &= \int\limits_{\mathbb{R}^n \times [0,\infty)} f(x - \sqrt{a\tau}z, t - a\tau) W(z,1) e^{-a\tau} dz d\mu(\tau)\end{aligned} \tag{1.23}$$

will be called a *weighted parabolic wavelet transform.*

In fact, the equalities (1.21), (1.22) were the first and gave rise to definitions of the wavelet transforms (1.10), (1.23). Numerous examples of continuous wavelet transforms generated by different analytic families of fractional integrals can be found in [23]. An idea to introduce weighted wavelet transforms associated with inhomogeneous differential operators (or fractional integrals) seems to be new (but it was clear to the second author a few years ago; as an exercise, we suggest that the reader defines weighted wavelet transforms corresponding to the Bessel potentials $J^\alpha f = (I-\Delta)^{\alpha/2} f$ and proves the relevant reproducing formula).

The formula (1.12) remains true if P is substituted for $\mathcal{P}$, and an analog of Theorem A also holds for the weighted transform (1.23). In view of (1.3) and (1.4), explicit inverses of H^α and $\mathcal{H}^\alpha$ can be obtained from (1.21), (1.22) if one replaces formally α by $-\alpha$. More precisely we have

Theorem B. *Let μ be a finite Borel measure on $[0,\infty)$ satisfying the following conditions:*

$$\int_0^\infty t^j d\mu(t) = 0 \qquad \forall j = 0, 1, \dots [\alpha/2] \quad \text{(the integer part of } \alpha/2); \tag{1.24}$$

$$\int_0^\infty t^\beta d|\mu|(t) < \infty \quad \textit{for some } \beta > \alpha/2. \tag{1.25}$$

Suppose that $\varphi = H^\alpha f$, $f \in L^p$, $1 \le p < \infty$, $0 < \alpha < (n+2)/p$. Then

$$\int_0^\infty P_\mu \varphi(x,t;a)\, \frac{da}{a^{1+\alpha/2}} \equiv \lim_{\varepsilon\to 0} \int_\varepsilon^\infty (\dots) = d_{\alpha,\mu} f(x,t), \tag{1.26}$$

$$d_{\alpha,\mu} = \begin{cases} \Gamma(-\alpha/2) \displaystyle\int_0^\infty t^{\alpha/2} d\mu(t) & \textit{if } \alpha/2 \notin \mathbb{N}, \\ \dfrac{(-1)^{1+\alpha/2}}{(\alpha/2)!} \displaystyle\int_0^\infty t^{\alpha/2} \log t \, d\mu(t) & \textit{if } \alpha/2 \in \mathbb{N}. \end{cases} \tag{1.27}$$

The limit in (1.26) is interpreted in the L^p-norm for $1 \le p < \infty$ and a.e. on $\mathbb{R}^{n+1}$ for $1 < p < \infty$.

The same statement holds for all $\alpha > 0$ and $1 \le p \le \infty$ (L^∞ is identified with C_0) provided that H^α and P_μ are replaced by $\mathcal{H}^\alpha$ and $\mathcal{P}_\mu$ respectively.

Our next result concerns application of parabolic wavelet transforms to characterization of anisotropic spaces $\mathcal{L}^\alpha_{p,r}(\mathbb{R}^{n+1})$ of parabolic potentials which were introduced and studied in [19]. We recall that given $\alpha > 0$, $1 < p < \infty$, $1 < r < \infty$, the space $\mathcal{L}^\alpha_{p,r} = \mathcal{L}^\alpha_{p,r}(\mathbb{R}^{n+1})$ is defined by

$$\mathcal{L}^\alpha_{p,r} = \{f \colon \|f\|_{\mathcal{L}^\alpha_{p,r}} \equiv \|f\|_r + \|F^{-1}(|\xi|^2 + i\tau)^{\alpha/2} F f\|_p < \infty\} \tag{1.28}$$

where the Fourier transform F is understood in the sense of Φ_0'-distributions (see Remark 1.4).

Theorem C. *Let $\alpha > 0$, $1 < p < \infty$, $1 < r < \infty$. Then*

$$\mathcal{L}^\alpha_{p,r} = \{f \in L^r\colon \sup_{\varepsilon>0} \| \int\limits_\varepsilon^\infty \frac{P_\mu f(x,t;a)}{a^{1+\alpha/2}} da \|_p < \infty\} \tag{1.29}$$

where $P_\mu f$ is the parabolic wavelet transform (see Definition 1.1) generated by an arbitrary measure μ satisfying the conditions of Theorem B.

Example 1.5. Let

$$\mu = \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \delta_k, \qquad \ell > \alpha/2, \tag{1.30}$$

$\delta_k = \delta_k(t)$ being the unit mass at the point $t = k$. By (1.11),

$$\begin{aligned} P_\mu f(x,t;a) &= \sum_{k=0}^{\ell} \binom{\ell}{k} (-1)^k \int\limits_{\mathbb{R}^n} f(s - \sqrt{ak} z, t - ka) W(z,1) dz = \\ &= \int\limits_{\mathbb{R}^n} \Delta^\ell_{y,a} f(x,t) W(y,a) dy \end{aligned}$$

where $\Delta^\ell_{y,a} f(x,t) = \sum\limits_{k=0}^{\ell} \binom{\ell}{k} (-1)^k f(x - \sqrt{k} y, t - ka)$ is an anisotropic finite difference of f. Furthermore,

$$\int\limits_0^\infty \frac{P_\mu f(x,t;a)}{a^{1+\alpha/2}} da = \int\limits_{\mathbb{R}^n \times (0,\infty)} \frac{\Delta^\ell_{y,a} f(x,t)}{a^{1+\alpha/2}} W(y,a) dy da.$$

Hypersingular integrals of this form were introduced in [16–19]. As in [21, Section 17.4] it is not difficult to show that the measure (1.30) satisfies (1.24), (1.25) and

$$d_{\alpha,\mu} = \int\limits_0^\infty \frac{(1 - e^{-t})^\ell}{t^{1+\alpha/2}} dt \neq 0.$$

In Sections 2, 3 and 4 we prove Theorems A, B, C respectively. Section 5 contains concluding remarks and an additional Theorem D characterizing the range $H^\alpha(L^p)$.

2. Proof of Theorem A

Denote

$$I_{\varepsilon,\rho}(\mu, f) = \int_{\varepsilon}^{\rho} P_\mu f(x,t;a)\,\frac{da}{a}, \qquad 0<\varepsilon<\rho<\infty. \tag{2.1}$$

We have

$$I_{\varepsilon,\rho}(\mu, f) = f * K_\varepsilon - f * K_\rho \tag{2.2}$$

where

$$K_\varepsilon(x,t) = \varepsilon^{-1} W(x,t) k(t/\varepsilon), \tag{2.3}$$

$k(t)$ being the function (1.13) (extended by 0 to $t<0$). Indeed, by changing the order of integration and passing to new variables we get

$$\begin{aligned} I_{\varepsilon,\rho}(\mu, f) &= \int_0^\infty d\mu(\tau) \int_{\varepsilon\tau}^{\rho\tau} \left(\frac{\tau}{b}\right)^{n/2} \frac{db}{b} \int_{\mathbb{R}^n} f(x-z,t-b) W\left(\frac{z\sqrt{\tau}}{\sqrt{b}},\tau\right) dz = \\ &= \int_0^\infty \frac{db}{b} \int_{\mathbb{R}^n} f(x-z,t-b) W(z,b) dz \int_{b/\rho}^{b/\varepsilon} d\mu(\tau), \end{aligned}$$

and (2.2) follows. If μ is admissible, i.e. $k(t) \in L^1(0,\infty)$, then

$$(f * K_\varepsilon)(x,t) - k_0 f(x,t) = \int_0^\infty k(\tau) d\tau \int_{\mathbb{R}^n} [f(x - z\sqrt{\varepsilon\tau}, t-\varepsilon\tau) - f(x,t)] W(z,1) dz,$$

and therefore $f * K_\varepsilon \to k_0 f$ in the L^p-norm (or in the sup-norm for $f \in C_0$). Furthermore, if $k(t) \in L^1(0,\infty)$, then

$$\lim_{\rho\to\infty} \|f * K_\rho\|_p = 0 \qquad \forall f \in L^p(\mathbb{R}^{n+1}), \quad 1<p\le\infty \tag{2.4}$$

(L^∞ is identified with C_0). Indeed,

$$\begin{aligned} \|f * K_\rho\|_p &= \|\,\|\int_0^\infty k(\tau) d\tau \int_{\mathbb{R}^n} f(x-y,t-\rho\tau) W(y,\rho\tau) dy\|_{L^p_x}\|_{L^p_t} \le \\ &\le \|\int_0^\infty |k(\tau)|\,\|\int_{\mathbb{R}^n} f(x-y,t-\rho\tau) W(y,\rho\tau) dy\|_{L^p_x}\, d\tau\|_{L^p_t} \le \\ &\le \|\int_0^\infty |k(\tau)| \tilde{f}(t-\rho\tau) d\tau\|_{L^p_t}, \end{aligned} \tag{2.5}$$

where $\tilde{f}(t) = \|f(\cdot,t)\|_p \in L^p(\mathbb{R}^1)$. The expression (2.5) tends to 0 as $\rho \to \infty$ (use, e.g., Theorem 1.15 from [21 p. 3]. This completes the proof of (i) and (ii).

The validity of (iii) follows in a standard way [27] from the maximal estimate

$$\| \sup_{\varepsilon>0} |f * K_\varepsilon| \, \|_p \le c\|f\|_p.$$

The latter is a consequence of the L^p-boundedness of the "partial" Hardy-Littlewood maximal functions

$$\tilde{f}(x,t) = \sup_{r>0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} |f(y,t)|dy, \qquad f^*(x,t) = \sup_{s>0} \frac{1}{2s} \int\limits_{|t-\tau|<s} \tilde{f}(x,\tau)d\tau,$$

$B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$, due to the following estimates:

$$|(f * K_\rho)(x,t)| \le \int\limits_0^\infty |k(\tau)| \, |\int\limits_{\mathbb{R}^n} f(x-y,t-\rho\tau)W(y,\rho\tau)dy|d\tau \le$$

$$\le \int\limits_0^\infty |k(\tau)| \left[\sup_{s>0} \int\limits_{\mathbb{R}^n} |f(x-y,t-\rho\tau)|W(y,s)dy \right] d\tau \le$$

$$\le c \int\limits_0^\infty |k(\tau)|\tilde{f}(x,t-\rho\tau)d\tau \le c_1 f^*(x,t),$$

c and c_1 being some constants independent of f. □

The proof of the analog of Theorem A for the weighted wavelet transforms (1.23) follows the same lines and is based on the equality

$$\int\limits_\varepsilon^\rho \mathcal{P}_\mu f(x,t;a) \frac{da}{a} = f * \tilde{K}_\varepsilon - f * \tilde{K}_\rho, \tag{2.6}$$

$\tilde{K}_\varepsilon(x,t) = \varepsilon^{-1}e^{-t}W(x,t)k(t/\varepsilon)$ (similarly for $\tilde{K}_\rho$; cf. (2.3)). Slight additional technicalities related to the extra factor e^{-t} are left to the reader.

3. Proof of Theorem B

Let $h_a^{\alpha/2}(x,t) = a^{\alpha/2-1}W(x,t)(I_{0+}^{\alpha/2}\mu)(t/a)$, where

$$(I_{0+}^{\alpha/2}\mu)(t) = \frac{1}{\Gamma(\alpha/2)} \int\limits_0^t (t-\tau)^{\alpha/2-1} d\mu(\tau)$$

is the Riemann-Liouville fractional integral of μ. We first show that

$$P_\mu H^\alpha f(x,t;a) = (f * h_a^{\alpha/2})(x,t). \tag{3.1}$$

Indeed, by (1.11), (1.19),

$$(P_\mu H^\alpha f)(x,t) = \int_0^\infty d\mu(\tau) \int_{\mathbb{R}^n} (H^\alpha f)(x - \sqrt{a\tau}z, t - a\tau) W(z,1) dz =$$

$$= \frac{1}{\Gamma(\alpha/2)} \int_0^\infty d\mu(\tau) \int_{\mathbb{R}^n} W(z,1) dz \int_{a\tau}^\infty (\zeta - a\tau)^{\alpha/2-1} d\zeta \times$$

$$\times \int_{\mathbb{R}^n} f(x-\xi, t-\zeta) W(\xi - \sqrt{a\tau}z, \zeta - a\tau) d\xi =$$

$$= \frac{1}{\Gamma(\alpha/2)} \int_{\mathbb{R}^{n+1}} f(x-\xi, t-\zeta) d\xi d\zeta \int_0^\infty (\zeta - a\tau)_+^{\alpha/2-1} I(\xi, \zeta, \tau) d\mu(\tau),$$

$$I(\xi,\zeta,\tau) = \int_{\mathbb{R}^n} W(z,1) W(\xi - \sqrt{a\tau}z, \zeta - a\tau) dz = W(\xi,\zeta) \qquad \forall \tau \geq 0, \ a > 0.$$

This implies (3.1). Furthermore, by (3.1),

$$T_\varepsilon^\alpha \varphi(x,t) \overset{\text{def}}{=} \int_\varepsilon^\infty \frac{(P_\mu \varphi)(x,t;a)}{a^{1+\alpha/2}} da = f * \psi_\varepsilon, \tag{3.2}$$

where (use 3.238(3) from [11])

$$\psi_\varepsilon(x,t) = \frac{W(x,t)}{\Gamma(\alpha/2)} \int_\varepsilon^\infty \frac{da}{a^{1+\alpha/2}} \int_0^\infty (t - a\tau)_+^{\alpha/2-1} d\mu(\tau) = \varepsilon^{-1} W(x,t) \lambda_\alpha(t/\varepsilon),$$

$\lambda_\alpha(t) = t^{-1}(I_{0+}^{\alpha/2+1}\mu)(t)$; cf. (2.3). By Lemma 1.3 from [24], the conditions (1.24), (1.25) imply that $\lambda_\alpha(s)$ has a decreasing integrable majorant and $\int_0^\infty \lambda_\alpha(s) ds = d_{\alpha,\mu}$ (see (1.27)). Hence (1.26) follows by the same argument as in the proof of Theorem A.

In the inhomogeneous case $\varphi = \mathcal{H}^\alpha f$ the proof is similar and relies on the equality

$$\int_\varepsilon^\infty \frac{\mathcal{P}_\mu \varphi(x,t;a)}{a^{1+\alpha/2}} da = f * \tilde{\psi}_\varepsilon, \quad \tilde{\psi}_\varepsilon(x,t) = \varepsilon^{-1} e^{-t} W(x,t) \lambda_\alpha(t/\varepsilon).$$

(cf. (2.6)). □

Remark 3.1. By keeping track of the proof of (3.2), one can readily see that the equality

$$T_\varepsilon^\alpha H^\alpha f = f * \psi_\varepsilon, \quad \psi_\varepsilon(x,t) = \varepsilon^{-1} W(x,t) \lambda_\alpha(t/\varepsilon), \tag{3.3}$$

holds for all $\alpha > 0$ provided, e.g., $f \in \Phi_0$. This remark will be used in the next section.

4. Proof of Theorem C

By Theorem 2.1 from [19], $\mathcal{L}^{\alpha}_{p,r} = L^r \cap H^{\alpha}(L^p)$. In other words, $\mathcal{L}^{\alpha}_{p,r}$ consists of functions $f \in L^r$ such that $f = H^{\alpha} g$ for some $g \in L^p$ in the Φ'_0-sense. Thus, it suffices to prove the equivalence

$$\sup_{\varepsilon>0} \| \int_{\varepsilon}^{\infty} \frac{P_{\mu} f(x,t;a)}{a^{1+\alpha/2}}\, da \|_p < \infty \Leftrightarrow f \overset{(\Phi'_0)}{=} H^{\alpha} g \tag{4.1}$$

for some $g \in L^p$. Assuming the right equality, we first show that

$$T^{\alpha}_{\varepsilon} f(x,t) = \int_{\varepsilon}^{\infty} \frac{P_{\mu} f(x,t;a)}{a^{1+\alpha/2}}\, da = (\psi_{\varepsilon} * g)(x,t) \tag{4.2}$$

(cf. (3.2), (3.3)). Let u and $v \in \Phi_0$ be such that $u = H^{\alpha}_{-} v$ (see Remark 1.4). Then

$$\langle T^{\alpha}_{\varepsilon} f, u \rangle = \langle f, U\overline{T^{\alpha}_{\varepsilon} U \overline{u}} \rangle = \langle f, U\overline{T^{\alpha}_{\varepsilon} H^{\alpha} U \overline{v}}\, \rangle \overset{(3.3)}{=} \langle f, U\overline{[\psi_{\varepsilon} * U\overline{v}]}\, \rangle.$$

Since $\psi_{\varepsilon} \in L^1$ there is a sequence $\{\psi_{\varepsilon,\ell}\} \subset C^{\infty}_c$ which converges to ψ_{ε} as $\ell \to \infty$ in the L^1-norm. Moreover, $\psi_{\varepsilon,\ell} * U\overline{v} \in \Phi_0 \;\; \forall \ell$, and

$$|\langle f, U\overline{[\psi_{\varepsilon} * U\overline{v}]} \rangle - \langle f, U\overline{[\psi_{\varepsilon,\ell} * U\overline{v}]}\, \rangle| \le \|f\|_r \|v\|_{r'} \|\psi_{\varepsilon} - \psi_{\varepsilon,\ell}\|_1 \to 0$$

$(1/r + 1/r' = 1)$, as $\ell \to \infty$. Hence

$$\langle T^{\alpha}_{\varepsilon} f, u \rangle = \lim_{\ell\to\infty} \langle f, U\overline{[\psi_{\varepsilon,\ell} * U\overline{v}]}\, \rangle =$$

$$= \lim_{\ell\to\infty} \langle g, H^{\alpha}_{-} U\overline{[\psi_{\varepsilon,\ell} * U\overline{v}]}\, \rangle = \lim_{\ell\to\infty} \langle g, U\overline{H^{\alpha}[\psi_{\varepsilon,\ell} * U\overline{v}]}\, \rangle =$$

$$= \lim_{\ell\to\infty} \langle g, U\overline{[\psi_{\varepsilon,\ell} * H^{\alpha} U\overline{v}]}\, \rangle = \lim_{\ell\to\infty} \langle g, U\overline{[\psi_{\varepsilon,\ell} * U\overline{u}]}\, \rangle =$$

$$= \langle g, U\overline{[\psi_{\varepsilon} * U\overline{u}]}\, \rangle = \langle g * \psi_{\varepsilon}, u \rangle.$$

We have proved that the functions $T^{\alpha}_{\varepsilon} f \in L^r$ and $g * \psi_{\varepsilon} \in L^p$ coincide as the Φ'_0-distributions. By Corollary 1.1 from [19] they coincide pointwise a.e. on $\mathbb{R}^{n+1}$, and (4.2) follows. The latter implies the left inequality in (4.1).

Conversely, if $\sup_{\varepsilon>0} \|T^{\alpha}_{\varepsilon} f\|_p < \infty$, then the set of functionals $\varphi \to \langle T^{\alpha}_{\varepsilon} f, \varphi \rangle, \;\; \varphi \in L^{p'}, \;\; 1/p + 1/p' = 1$, is bounded in $(L^{p'})^*$. Since the bounded set in the space which is dual to the reflexive Banach space is compact in the weak* topology, then there exist a function $g \in L^p$ and a sequence $\varepsilon_k \to 0$ such that $\langle T^\alpha_{\varepsilon_k} f, \varphi\rangle \to \langle g, \varphi\rangle$ as $\varepsilon_k \to 0$ $\forall \varphi \in L^{p'}$. For this g and any test function $\omega \in \Phi_0$ we have

$$\langle H^\alpha g, \omega\rangle = \langle g, H^\alpha_- \omega\rangle = \lim_{\varepsilon_k \to 0} \langle T^\alpha_{\varepsilon_k} f, H^\alpha_- \omega\rangle =$$

$$= \lim_{\varepsilon_k \to 0} \langle f, U\overline{T^\alpha_{\varepsilon_k} U H^\alpha_- \overline{\omega}}\,\rangle = \lim_{\varepsilon_k \to 0} \langle f, U\overline{T^\alpha_{\varepsilon_k} H^\alpha U \overline{\omega}}\,\rangle \overset{(3.3)}{=}$$

$$= \lim_{\varepsilon_k \to 0} \langle f, U\overline{[\psi_{\varepsilon_k} * U\overline{\omega}]}\,\rangle = \lim_{\varepsilon_k \to 0} \langle\, f * \psi_{\varepsilon_k}, \omega\rangle = \langle f, \omega\rangle,$$

i.e. $f = H^\alpha g$ in the Φ_0'-sense. This completes the proof. □

5. Concluding remarks

An implementation of wavelet-type integrals $\int_0^\infty P_\mu f(x,t;a) da/a^{1+\alpha/2}$ (and their inhomogeneous modifications including $\mathcal{P}_\mu f$) enables one to look with a "bird's-eye view" at the method of hypersingular integrals [16–19, 21, 25] and extract the essence of the latter.

This essence is represented by the orthogonality relations (1.24). Without going into technicalities, we note that the equality (3.1) can be extended to $\alpha \geq (n+2)/p$, thus demonstrating a regularizing effect of the wavelet transform $P_\mu \varphi$, when $\varphi = H^\alpha f$ is a Φ_0'-distribution. By Lemma 4.12 from [21] the conditions (1.24), (1.25) imply $h_a^{\alpha/2} \in L^1$, and the right-hand side of (3.1) is a usual function belonging to L^p for all $1 \leq p \leq \infty$.

By using the argument, which is similar to that in the proof of Theorems B and C, one can obtain the following characterization of potentials $H^\alpha f$, $f \in L^p$.

Theorem D. *Let $\alpha > 0$, $f \in L^p$, $\varphi \in L^r$; $1 \leq p, r < \infty$. Suppose that μ satisfies the conditions of Theorem B. If $\varphi = H^\alpha f$ pointwise a.e. for $\alpha < (n+2)/p$ or $\varphi = H^\alpha f$ in the Φ_0'-sense, then (1.26) holds with the limit interpreted in the L^p-norm. If $p > 1$ this limit can be understood also in the a.e. sense. Conversely, if $d_{\alpha,\mu} \neq 0$ (see (1.27)) and*

$$\overset{(L^p)}{\lim_{\varepsilon \to 0}} \frac{1}{d_{\alpha,\mu}} \int_\varepsilon^\infty P_\mu \varphi(x,t;a) \frac{da}{a^{1+\alpha/2}} = f,$$

then $\varphi = H^\alpha f$ *in the* Φ_0'*-sense (or pointwise a.e. for* $\alpha < (n+2)/p$*).*

A similar statement (which does not involve distributions) also holds for inhomogeneous potentials $\mathcal{H}^\alpha f$.

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