WEIGHTED SPHERICAL HARMONICS AND GENERALIZED SPHERICAL CONVOLUTIONS

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Abstract. *A theory of spherical harmonics associated with the Laplace-Bessel differential operator is developed. A notion of the generalized translation operator on the "upper" hemisphere S^{n-1}_+ in $\mathbb{R}^n$ and the relevant convolution operators are introduced. Natural analogs of the Laplace formula, the Funk-Hecke formula, the product formula and the addition theorem are obtained. It is shown that the weighted space $L^2(\mathbb{R}^n_+; x_n^{2\gamma})$ of functions on the half-space $\mathbb{R}^n_+$ can be decomposed in the direct sum of subspaces which are invariant under the action of the Fourier-Bessel transform. An explicit construction of the orthonormal basis of the generalized spherical harmonics is given in terms of the associated harmonics represented by the products of Jacobi polynomials and the ordinary spherical harmonics in lower dimensions. Convergence of the Fourier-Laplace-Bessel series is studied in the L^2-setting and for smooth functions defined on S^{n-1}_+. The paper provides the necessary foundations for forthcoming investigations of different problems of harmonic analysis related to the second order singular partial differential equations.*

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1. Introduction

The classical harmonic analysis on the unit sphere has proved to be a powerful tool for studying numerous problems related to partial differential equations, integral geometry and many other areas. It is natural to expect that various problems for singular PDE, initiated by A. Weinstein [45, 46], as well as geometrical problems for axially symmetric bodies, need the relevant harmonic analysis on the sphere reflecting the prescibed singularity or symmetry. Such an analysis might be useful in other occasions.

Another observation, which motivates our investigation, is the following. It has been known long ago that the usual exponential Fourier series (or the Fourier transform on $\mathbb{R}^1$) of an even function turns into the relevant cosine series (or the cosine transform on the half-line). Further generalization can be achieved by passing to orthogonal series of Jacobi polynomials and the Hankel transform. The basics of harmonic analysis for Jacobi polynomials and the Hankel transform in the context of the relevant convolution structures were developed in the fundamental papers by J. Delsarte, S. Bochner, B.M. Levitan, I.I. Hirschman Jr., B. Muckenhoupt and E.M. Stein, R. Askey and S. Wainger, T. Koornwinder, G. Gasper and W. Trebels and other authors (see references at the end of the paper, the list of which is certainly not complete). Additional information can be found in numerous papers by K. Trimeche (see, e.g., [40-42]) and his collaborators. Further generalizations in the framework of hypergroups are reflected in the books by W.R. Bloom and H. Heyer [10] and K. Trimeche [41].

In many dimensions for $x = (x', x_n)$, $x' \in \mathbb{R}^{n-1}$, the assumption of evenness with respect to the x_n-variable leads (after an obvious generalization) to the Fourier-Bessel transform

$$(F_\gamma f)(y) = \int_{\mathbb{R}^n_+} f(x)\, e^{-ix' \cdot y'}\, j_{\gamma-1/2}(x_n y_n)\, x_n^{2\gamma} dx, \tag{1.1}$$

where $y \in \mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$ and $j_{\gamma-1/2}(z) = 2^{\gamma-1/2}\Gamma(\gamma + 1/2) z^{1/2-\gamma} J_{\gamma-1/2}(z)$ is the normalized Bessel function. This transform is associated with the Laplace-

Bessel differential operator

$$\Delta_{n,\gamma} = \Delta_n + \frac{2\gamma}{x_n}\frac{\partial}{\partial x_n}, \qquad \Delta_n = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}. \tag{1.2}$$

The expression (1.1) combines the properties of the Hankel transform in the x_n-variable and the ordinary Fourier transform in $\mathbb{R}^{n-1}$. These transforms and related problems of the function theory were studied intensively in the Voronezh school (Russia) by I.A. Kipriyanov and his collaborators, in the Tunisian school of K. Trimeche, by J. Peetre, K. Stempak, I.A. Aliev and others. It would be natural to develop a parallel theory on the sphere.

First steps in this direction have been taken by L.N. Lyakhov [26, 27] and by Z. Ben Nahia and N. Ben Salem [8]. An alternative proof of the Laplace formula for the generalized spherical harmonics was given by I.A. Aliev [1] who applied Lyakhov's results to studying multipliers associated with singular integrals generated by the generalized translation operator in $\mathbb{R}^n$. Many details which are important for checking correctness of the results from [26, 27, 8] were skipped in these papers. Moreover, different versions of the aforementioned Laplace formula, given by these authors, are not consistent. In view of future developments and applications of these results to convolution operators and related problems, it was desirable to revise the works [26, 27, 8].

The paper is self-contained. We complete all gaps in [26, 27, 8] by giving independent simple proofs of the statements announced there and adding new results in order to make the theory developed as much as the classical theory of spherical harmonics. Instead of trying to reproduce the classical argument (see, e.g., [29-31, 36]) in the more general set-up corresponding to the Laplace-Bessel operator (1.2), we separate variables and introduce *associated B-harmonics* which combine properties of Jacobi polynomials and ordinary spherical harmonics in lower dimensions. This idea goes up to N.Ja. Vilenkin [43, 44]) and to the close results of F. Zernike and H.C. Brinkman [47], B.L.J. Braaksma and B. Meulenbeld [14] which were extended substantially by A. Dijksma and T. Koornwinder [15, 21-23].

This approach seems natural and more informative than that of [26, 27, 8]. Furthermore, it is consistent with the parallel theory for (1.1) which was not created "from the zero level" rather than derived from known facts for Bessel functions and the usual Fourier transform. Of course, inevitably we had to mimic some definitions and propositions from [26, 27, 8] and followed the classical scheme sometimes. Otherwise our presentation would not be complete and well-composed.

The paper is organized as follows. In Section 2 the spaces of "B-harmonic" homogeneous polynomials corresponding to the Laplace-Bessel operator (1.2) are studied. We evaluate dimension of these spaces and establish relationship between B-harmonic polynomials and usual harmonic polynomials, which are even in the last variable. Then we define *the spherical B-harmonics* as the restrictions of B-harmonic homogeneous polynomials to the "upper" hemisphere S^{n-1}_+, and introduce *associated B-harmonics* in terms of which an orthonormal basis in the weighted space $L^2(S^{n-1}_+; \xi_n^{2\gamma})$ is constructed and the relevant Plancherel theory is developed.

In Section 3 we obtain natural analogs of the Laplace formula, the product formula and the addition theorem for B-harmonics. Our results resemble, on one hand, the relevant results for Jacobi polynomials and, on the other hand, the corresponding statements for usual spherical harmonics employing ultraspherical (Gegenbauer) polynomials. In Section 4 we introduce convolution operators on S^{n-1}_+ which can be served by the theory of B-harmonics, and prove the relevant Funk-Hecke formula. In particular, a natural definition of the generalized translation operator on the hemisphere is given and L^p-estimates for this operator are obtained. The contents of this section are in the spirit of the results of Bochner [11-13], Hirschman [19], Askey-Wainger [5] and Gasper [17, 18] related to the convolution structure of ultraspherical and Jacobi polynomials. Section 5 contains estimates of derivatives of B-harmonic polynomials and studies decompositions of infinitely differentiable functions on S^{n-1}_+ in the "Fourier-Laplace-Bessel" series of B-harmonics.

In Section 6 we demonstrate an intimate interrelation between the Fourier-Bessel transform (1.1) and B-harmonics. A classical result says that the space $L^2(\mathbb{R}^n)$ can be decomposed in a direct sum of subspaces which are invariant under the action of the Fourier transform [37]. By making use of the theory of B-harmonics, we obtain an analog of this result for the Fourier-Bessel transform on the weighted space $L^2(\mathbb{R}^n_+; x_n^{2\gamma})$. In Section 7 we obtain a Gauss-like decomposition of a homogeneous polynomial into a sum of B-harmonic homogeneous polynomials $Z_j(x)$, i.e. those satisfying the equation $\Delta_{n,\gamma} Z_j = 0$. Such a decomposition was announced by Lyakhov [27] with a brief remark that it can be proved by the method used in Sobolev's book [36]. In contrast to our approach, this decomposition played a key role in [27]. A careful examination of the relevant Theorem XI.1 from [36] shows that some places in this theorem are not so rigorous (see Remark 7.2). We complete these gaps and give a detailed proof of the generalized Gauss decomposition for all $\gamma \geq 0$ and $n \geq 2$.

The problems studied in sections 4-6 were not considered in [1, 8, 26, 27]. The present paper can be regarded as an introduction to the "B-harmonic analysis" on the sphere. In future publications we plan to concern some more refined problems which can be investigated in the framework of this analysis.

Notation. As usual, the Gegenbauer and Jacobi polynomials are denoted by $C_j^\lambda(t)$ and $P_j^{(\alpha,\beta)}(t)$ respectively [16]. In the following

$$\mathbb{R}^n_+ = \{x : x = (x_1, \dots, x_n) \in \mathbb{R}^n;\ \ x_n > 0\}; \qquad n \geq 2;$$

$$S^{n-1} = \{\xi : \xi \in \mathbb{R}^n;\ |\xi| = 1\}; \qquad S^{n-1}_+ = \{\xi : \xi \in S^{n-1};\ \xi_n > 0\}.$$

For $1 \leq p < \infty$ and $\gamma \in \mathbb{R}$ we define the weighted spaces

$$L^p_\gamma(S^{n-1}_+) = \{f(\xi) : \|f\|_{p,\gamma} = \Big(\int\limits_{S^{n-1}_+} |f(\xi)|^p\, \xi_n^{2\gamma} d\xi\Big)^{1/p} < \infty\};$$

$$L^p_\gamma(\mathbb{R}^n_+) = \{g(x) : \|g\|_{(p,\gamma)} = \Big(\int\limits_{\mathbb{R}^n_+} |g(x|^p\, x_n^{2\gamma} dx\Big)^{1/p} < \infty\};$$

$$L^p_\gamma(-1,1) = \{h(t) : \|h\|_{[p,\gamma]} = \Big(\int\limits_{-1}^{1} |h(t)|^p (1-t^2)^\gamma dt\Big)^{1/p} < \infty\}.$$

For $p = \infty$ these spaces are defined in a standard way as the spaces of essentially bounded functions on the corresponding sets. The notation $C^k(\mathbb{R}^n)$ for k times continuously differentiable functions on $\mathbb{R}^n$ is traditional. We denote by $C^k_*(\mathbb{R}^n)$ the subclass of functions $f \in C^k(\mathbb{R}^n)$ which are even in the x_n-variable.

Other notations and definitions will be given in due course.

2. B-harmonics. Preliminaries

Let $x = (x_1, \dots, x_{n-1}, x_n) = (x', x_n) \in \mathbb{R}^n$, $n \ge 2$. The Laplace-Bessel singular differential operator is defined by

$$\Delta_{n,\gamma} = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2} + \frac{2\gamma}{x_n}\frac{\partial}{\partial x_n} = \Delta_{n-1} + B_{x_n} \tag{2.1}$$

where $\gamma \ge 0$,

$$\Delta_{n-1} = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_{n-1}^2}, \qquad B_{x_n} = \frac{\partial^2}{\partial x_n^2} + \frac{2\gamma}{x_n}\frac{\partial}{\partial x_n}. \tag{2.2}$$

We call a function $f \in C^2_*(\mathbb{R}^n)$ B-harmonic if $\Delta_{n,\gamma} f \equiv 0$. Accordingly, a homogeneous polynomial $P_j(x) \equiv P_j(x', x_n)$ of degree j is B-harmonic if $P_j(x', -x_n) \equiv P_j(x', x_n)$ and $\Delta_{n,\gamma} P_j \equiv 0$. The linear space of all such polynomials will be denoted by $H^{n,\gamma}_{j,*}$.

An important tool for studying B-harmonic functions is the Poisson operator which is defined on sufficiently nice even in x_n functions by

$$\Pi^\gamma f(x) = \frac{2\Gamma(\gamma+1/2)}{\sqrt{\pi}\,\Gamma(\gamma)} |x_n|^{1-2\gamma} \int\limits_0^{|x_n|} (x_n^2 - y^2)^{\gamma-1} f(x', y)\, dy = \tag{2.3}$$

$$= \frac{\Gamma(\gamma+1/2)}{\sqrt{\pi}\,\Gamma(\gamma)} \int\limits_0^{\pi} f(x', x_n \cos\psi) \sin^{2\gamma-1}\psi\, d\psi \tag{2.4}$$

if $\gamma > 0$, and $\Pi^\gamma f(x) = f(x)$ if $\gamma = 0$ [24, p. 128]. Clearly, $\Pi^\gamma[1] = 1$ and, for sufficiently nice f, $\lim\limits_{\gamma \to 0} \Pi^\gamma f(x) = f(x)$. For fixed x' the operator (2.3) is a special case of the Erdélyi-Kober operator, numerous properties of which and intimate relation to singular PDE are discussed in [35] where one can find further references.

Theorem 2.1. (i) *For fixed $n \ge 2$ and $j \in \mathbb{Z}_+$, the spaces $H^{n,\gamma}_{j,*}$ corresponding to different $\gamma \ge 0$, are isomorphic. Moreover, $H^{n,\gamma}_{j,*} = \Pi^\gamma H^{n,0}_{j,*}$.*

(ii) *The dimension of $H^{n,\gamma}_{j,*}$ is*

$$\sigma_n(j) = \frac{(j+n-2)!}{j!(n-2)!}. \tag{2.5}$$

Proof. The first statement was obtained in [8]. It follows from the transmutation property

$$\Delta_{n,\gamma}\Pi^\gamma = \Pi^\gamma \Delta_n \tag{2.6}$$

and the observation that Π^γ is an automorphism of the space of all homogeneous polynomials even in x_n (see also [40, Section 5]).

The equality (2.5) was presented in [27] without proof provided by a brief explanation, that it can be derived from the relevant Gauss decomposition (the proof of which was also omitted). Below we give a direct proof of (2.5).

Let $\mathcal{P}_{j,*}$ be the space of all homogeneous polynomials of degree j even in the x_n-variable. We first prove the formula

$$\dim \mathcal{P}_{j,*} = \sum_{k=0}^{j} \frac{(-1)^{j-k}(n+k-1)!}{k!(n-1)!}. \tag{2.7}$$

For $t > 0,\ |x_i t| < 1,\ i = 1, \ldots, n$, we have

$$\begin{aligned}
&\left[\prod_{i=1}^{n-1} \frac{1}{1-x_i t}\right] \frac{1}{1-x_n^2 t^2} = \\
&= [1 + (x_1 + \ldots + x_{n-1})t + (x_1x_2 + \ldots + x_{n-2}x_{n-1})t^2 + \ldots] \times \\
&\times (1 + x_n^2 t^2 + x_n^4 t^4 + \ldots) = \\
&= 1 + (x_1 + \ldots + x_{n-1})t + (x_1x_2 + \ldots + x_{n-2}x_{n-1} + x_n^2)t^2 + \\
&+ (x_1x_2x_3 + \ldots + x_{n-3}x_{n-2}x_{n-1} + x_1x_n^2 + \ldots + x_{n-1}x_n^2)t^3 + \ldots .
\end{aligned}$$

The expression at t^j in this decomposition represents the sum of all monomials of the form $x_1^{j_1} \ldots x_{n-1}^{j_{n-1}} x_n^{2j_n}$ with $j_1 + \ldots + j_{n-1} + 2j_n = j$. By setting $x_1 = \ldots = x_n = 1$, we get

$$u(t) \equiv \frac{1}{(1-t)^{n-1}(1-t^2)} = 1 + (n-1)t + \ldots + (\dim \mathcal{P}_{j,*})t^j + \ldots ,$$

and therefore

$$\dim \mathcal{P}_{j,*} = \frac{u^{(j)}(0)}{j!} = \frac{1}{j!} \sum_{k=0}^{j} \binom{j}{k} \, [(1-t)^{-n}]^{(k)}|_{t=0} \, [(1+t)^{-1}]^{(j-k)}|_{t=0}$$

which coincides with (2.7).

Now we can prove (2.5). By (i) it suffices to assume $\gamma = 0$. For $\psi \in \mathcal{P}_{j,*}$ the standard Gauss decomposition (see, e.g., [36, p. 476]) yields

$$\psi(x) = \sum_{s=0}^{[j/2]} |x|^{2s} h_{j-2s}(x), \qquad h_{j-2s}(x) = \sum_{k}^{[j/2]-s} c_{k,j,s} |x|^{2k} \Delta_n^{k+s} \psi(x), \tag{2.8}$$

$h_{j-2s}(x)$ being harmonic polynomials of degree $j-2s$. Since ψ is even in x_n, then $h_{j-2s} \in H^{n,0}_{j-2s,*}$, and by (2.8), $\dim \mathcal{P}_{j,*} = \dim H^{n,0}_{j,*} + \dim H^{n,0}_{j-2,*} + \dots$. Hence $\dim H^{n,0}_{j,*} = \dim \mathcal{P}_{j,*} - \dim \mathcal{P}_{j-2,*}$, and the required result follows by (2.7). $\square$

Definition 2.2. *The restriction $Y^{n,\gamma}_j(\xi)$ of a B-harmonic polynomial $P^\gamma_j(x) \in H^{n,\gamma}_{j,*}$ to the hemisphere S^{n-1}_+ is called a spherical harmonic of degree j associated with the Laplace-Bessel operator* (2.1) (*or a B-harmonic for short*). *The linear space of all B-harmonics of degree j will be denoted by $\mathcal{H}^{n,\gamma}_j$.*

According to this definition, $P^\gamma_j(x) = |x|^j Y^{n,\gamma}_j(x/|x|)$, and

$$\dim \mathcal{H}^{n,\gamma}_j = \sigma_n(j) = \frac{(j+n-2)!}{j!(n-2)!}. \tag{2.9}$$

In order to characterize the structure of B-harmonics we introduce *associated B-harmonics* by analogy with the ordinary associated spherical harmonics on S^{n-1} (cf. [30, §11], [44, p. 210]). Let

$$\xi = (\cos\theta) e_n + (\sin\theta)\zeta, \quad e_n = (0, \dots, 0, 1), \quad 0 \le \theta \le \pi/2, \quad \zeta \in S^{n-2}, \tag{2.10}$$

so that $d\xi = (\sin\theta)^{n-2} d\theta d\zeta$, where $d\xi$ and $d\zeta$ designate the usual Lebesgue measures on S^{n-1} and S^{n-2} respectively. Denote by $\{Y^{n-1}_{\ell,\nu}(\zeta)\}$ an orthonormal system of spherical harmonics on S^{n-2}; $\ell = 0, 1, \dots$ and $\nu = 1, 2, \dots, d_{n-1}(\ell)$ where

$$d_{n-1}(\ell) = (n-3+2\ell)\,\frac{(n-4+\ell)!}{\ell!(n-3)!} \tag{2.11}$$

is the dimension of the subspace of spherical harmonics on S^{n-2} of degree ℓ [29].

Theorem 2.3. *Each B-harmonic $Y^{n,\gamma}_j(\xi), \gamma \ge 0$, is a finite linear combination of the associated B-harmonics defined by*

$$U^{j,\gamma}_{\ell,m,\nu}(\xi) = a_{\ell,m}(\gamma) \sin^\ell \theta P_m^{(\lambda-1/2,\gamma-1/2)}(\cos 2\theta) Y^{n-1}_{\ell,\nu}(\zeta), \tag{2.12}$$

$$\lambda = \ell - 1 + n/2; \qquad j = \ell + 2m,$$

$$a_{\ell,m}(\gamma) = \left[\frac{2(2m+\ell+\gamma-1+n/2)\,\Gamma(m+1)\,\Gamma(m+\ell+\gamma-1+n/2)}{\Gamma(m+\ell+(n-1)/2)\,\Gamma(m+\gamma+1/2)}\right]^{1/2}. \qquad (2.13)$$

The polynomials $P_j^\gamma(x) = |x|^j U^{j,\gamma}_{\ell,m,\nu}(x/|x|)$ *and* $P_j^0(x) = |x|^j U^{j,0}_{\ell,m,\nu}(x/|x|)$*, corresponding to* $\gamma \neq 0$ *and* $\gamma = 0$*, are connected by*

$$P_j^\gamma(x) = b^j_{\ell,m}(\gamma) \int_0^\pi P_j^0(x', x_n \cos\psi) \sin^{2\gamma-1}\psi d\psi \qquad (2.14)$$

where

$$b^j_{\ell,m}(\gamma) = \frac{a_{\ell,m}(\gamma)\Gamma(m+\gamma+1/2)}{a_{\ell,m}(0)\Gamma(m+1/2)\Gamma(\gamma)} =$$

$$= \frac{1}{\Gamma(\gamma)} \left[\frac{\ell+2m+\gamma-1+n/2}{\ell+2m-1+n/2} \cdot \frac{\Gamma(m+\ell+\gamma-1+n/2)}{\Gamma(m+\ell-1+n/2)} \cdot \frac{\Gamma(m+\gamma+1/2)}{\Gamma(m+1/2)}\right]^{1/2} =$$

$$= O(j^\gamma) \ \text{ as } \ j \to \infty. \qquad (2.15)$$

Proof. We first prove (2.14). Let $r = |x|$, $\quad s = |x'| = (x_1^2 + \ldots + x_{n-1}^2)^{1/2}$. Then

$$P_j^\gamma(x) = a_{\ell,m}(\gamma)(x_n^2 + s^2)^m P_m^{(\lambda-1/2,\gamma-1/2)}\left(\frac{x_n^2 - s^2}{x_n^2 + s^2}\right) A(x'), \quad \gamma \geq 0,$$

$A(x') = |x'|^\ell Y^{n-1}_{\ell,\nu}(x'/|x'|)$, and a simple calculation yields

$$I = \int_0^\pi P_j^0(x', x_n \cos\psi) \sin^{2\gamma-1}\psi d\psi = \frac{2^{m+1} s^{2m+1} r^{2\gamma-2} a_{\ell,m}(0)}{x_n^{2\gamma-1}} A(x') B(z),$$

$$B(z) = \int_{-1}^{z} \frac{(z-\tau)^{\gamma-1}(1+\tau)^{-1/2}}{(1-\tau)^{m+\gamma+1/2}} P_m^{(\lambda-1/2,-1/2)}(\tau) d\tau, \qquad z = \frac{x_n^2 - s^2}{x_n^2 + s^2}.$$

By (1.23) from [4],

$$B(z) = \frac{\Gamma(\gamma)\Gamma(m+1/2)}{2^\gamma \Gamma(m+\gamma+1/2)} \frac{(1+z)^{\gamma-1/2}}{(1-z)^{m+1/2}} P_m^{(\lambda-1/2,\gamma-1/2)}(z)$$

and therefore

$$I = \frac{\Gamma(\gamma)\Gamma(m+1/2)}{\Gamma(m+\gamma+1/2)} r^{2m} \left[a_{\ell,m}(0)|x'|^\ell Y^{n-1}_{\ell,\nu}(x'/|x'|)\right] P_m^{(\lambda-1/2,\gamma-1/2)}\left(\frac{x_n^2 - s^2}{x_n^2 + s^2}\right) =$$

$$= \frac{a_{\ell,m}(0)\Gamma(\gamma)\Gamma(m+1/2)}{a_{\ell,m}(\gamma)\Gamma(m+\gamma+1/2)} P_j^\gamma(x)$$

which implies (2.14).

In order to prove the first statement, given $Y_j^{n,\gamma} \in \mathcal{H}_j^{n,\gamma}$, by Theorem 2.1(i) we have $Q_j^\gamma(x) \equiv |x|^j Y_j^{n,\gamma}(x/|x|) = \Pi^\gamma g(x)$ for some $g \in H_{j,*}^{n,0}$, i.e. for $g(x) = |x|^j Y_j^{n,0}(x/|x|)$, where $Y_j^{n,0}(\xi)$ is an ordinary spherical harmonic on S^{n-1} even in ξ_n. By [30, §11], $Y_j^{n,0}(\xi)$ is a finite linear combination of the associated harmonics

$$Y_j^{n,0}(\xi) = \sum_{\ell=0}^{j} \sum_{\nu=1}^{d_{n-1}(\ell)} c_{\ell,\nu}^j V_{\ell,\nu}^j(\theta,\zeta), \qquad V_{\ell,\nu}^j(\theta,\zeta) = \sin^\ell \theta \, C_{j-\ell}^{\ell-1+n/2}(\cos\theta) Y_{\ell,\nu}^{n-1}(\zeta).$$

Since $Y_j^{n,0}(\xi)$ is even in ξ_n, then necessarily $j-\ell$ is even. Indeed, by writing $Y_j^{n,0}(\xi) \equiv Y_j^{n,0}(\zeta,\theta)$, we have

$$\int\limits_{S^{n-2}} Y_j^{n,0}(\zeta,\theta) Y_{k,\mu}^{n-1}(\zeta)\, d\zeta = c_{k,\mu}^j \sin^\ell \theta \, C_{j-k}^{k-1+n/2}(\cos\theta).$$

The left-hand side does not change if one replaces θ by $\pi-\theta$. Hence for $0<\theta\le\pi/2$,

$$c_{k,\mu}^j C_{j-k}^{k-1+n/2}(\cos\theta) = c_{k,\mu}^j C_{j-k}^{k-1+n/2}(\cos(\pi-\theta)) = c_{k,\mu}^j (-1)^{j-k} C_{j-k}^{k-1+n/2}(\cos\theta),$$

i.e. $c_{k,\mu}^j = 0$ if $j-k$ is odd. Resuming this argument, we get

$$|x|^j Y_j^{n,\gamma}(\xi) = \sum_{\substack{\ell=0 \\ j-\ell=2m}}^{j} c_{\ell,\nu}^j \, \Pi^\gamma[|x|^j V_{\ell,m,\nu}^j(\xi)], \qquad \xi = x/|x|,$$

where

$$V_{\ell,m,\nu}^j(\xi) = \sin^\ell \theta C_{2m}^\lambda(\cos\theta) Y_{\ell,\nu}^{n-1}(\zeta) = c_m \sin^\ell \theta P_m^{(\lambda-1/2,-1/2)}(\cos 2\theta) Y_{\ell,\nu}^{n-1}(\zeta),$$
$$c_m = \frac{2^{2m} m! \Gamma(\lambda+m)}{(2m)!\Gamma(\lambda)} \quad \text{(see [32, Section II.11])}.$$

By (2.14), $\Pi^\gamma[|x|^j V_{\ell,m,\nu}^j(\xi)] = \mathrm{const}|x|^j U_{\ell,m,\nu}^{j,\gamma}(x/|x|)$ which gives the required result. □

Corollary 2.4. *B-harmonics $Y_j^{n,\gamma}$ and $Y_{j'}^{n,\gamma}$, $j \neq j'$, are orthogonal with respect to the scalar product*

$$(f,g)_\gamma = \int\limits_{S_+^{n-1}} f(\xi)\overline{g(\xi)}\, \xi_n^{2\gamma} d\xi. \tag{2.16}$$

The validity of this statement follows immediately from Theorem 2.3 by passing in (2.16) to the "polar coordinates" (θ,ζ) (see (2.10)) due to orthogonality of Jacobi polynomials and spherical harmonics on S^{n-2}.

Denote by $\mathcal{Y}_j^{n,\gamma}$ the collection of all functions $U^{j,\gamma}_{\ell,m,\nu}(\xi)$ defined by (2.12) for

$$\ell, m = 0,1,2,\dots, \qquad j=\ell+2m, \qquad \nu = 1,2,\dots, d_{n-1}(\ell),$$

and let

$$L^2_\gamma(S^{n-1}_+) = \{f : \|f\|_{2,\gamma} = \Big(\int\limits_{S^{n-1}_+} |f(\xi)|^2\, \xi_n^{2\gamma} d\xi\Big)^{1/2} < \infty\}$$

be the Hilbert space associated with the scalar product (2.16).

Theorem 2.5. *The collection $\mathcal{Y}_j^{n,\gamma}$ is an orthonormal basis of the $\sigma_n(j)$-dimensional space $\mathcal{H}_j^{n,\gamma}$ of all B-harmonics of degree j (see (2.9)). The union $\mathcal{Y}^{n,\gamma} = \bigcup\limits_{j=0}^{\infty} \mathcal{Y}_j^{n,\gamma}$ constitutes an orthonormal basis of $L^2_\gamma(S^{n-1}_+)$.*

Proof. The first statement follows from Theorem 2.3. In order to check the second one it remains to prove the completeness of the system $\mathcal{Y}^{n,\gamma}$ in $L^2_\gamma(S^{n-1}_+)$ or, in other words, to show that equalities

$$(f, U^{j,\gamma}_{\ell,m,\nu})_\gamma = 0 \quad \forall\, j,\ell,m,\nu, \quad j=\ell+2m, \tag{2.17}$$

imply $f \overset{a.e.}{=} 0$. Fix ℓ and ν, and write (2.17) in polar coordinates. We get

$$\int\limits_0^{\pi/2} \sin^{\ell+n-2}\theta \cos^{2\gamma}\theta\, P_m^{(\ell+(n-3)/2,\gamma-1/2)}(\cos 2\theta)\, f_{\ell,\nu}(\theta) d\theta = 0 \quad \forall m\in\mathbb{Z}_+,$$

$f_{\ell,\nu}(\theta) = \int_{S^{n-2}} f(\theta,\zeta) Y^{n-1}_{\ell,\nu}(\zeta) d\zeta$ being the Fourier-Laplace coefficients of $f(\xi)\equiv f(\theta,\zeta)$ in the ζ-variable. Due to the completeness of the system of Jacobi polynomials [38] we obtain $f_{\ell,\nu}(\theta)=0$ for all θ belonging to a certain set $A_{\ell,\nu}\subset I=[0,\pi/2]$ the complement $I-A_{\ell,\nu}$ of which is of the Lebesgue measure zero: $m(I-A_{\ell,\nu})=0$. Since $\bigcup\limits_{\ell,\nu}(I-A_{\ell,\nu}) = I - \bigcap\limits_{\ell,\nu} A_{\ell,\nu}$, then $m(I-\bigcap\limits_{\ell,\nu} A_{\ell,\nu})=0$. Thus for $\theta\in\bigcap\limits_{\ell,\nu} A_{\ell,\nu}$ all coefficients $f_{\ell,\nu}(\theta)$ are zero and therefore, for such θ, $f(\theta,\zeta)=0$ a.e. on S^{n-2}.

Hence $f(\xi)=0$ a.e. on S^{n-1}_+, which implies completeness of the system $\mathcal{Y}^{n,\gamma}$. It remains to note that

$$\|U^{j,\gamma}_{\ell,m,\nu}\|^2_{2,\gamma}=(a^\gamma_{m,\ell})^2\int\limits_0^{\pi/2}\sin^{2\ell+n-2}\theta\cos^{2\gamma}\theta\,|P_m^{(\lambda-1/2,\gamma-1/2)}(\cos 2\theta)|^2d\theta=$$

$$=\frac{(a^\gamma_{m,\ell})^2}{2^{1+\gamma+\lambda}}\int\limits_{-1}^{1}(1+t)^{\gamma-1/2}(1-t)^{\lambda-1/2}\,|P_m^{(\lambda-1/2,\gamma-1/2)}(t)|^2dt=$$

(see, e.g., the formula (2.9) in [3])

$$=(a^\gamma_{m,\ell})^2\frac{\Gamma(m+\lambda+1/2)\,\Gamma(m+\gamma+1/2)}{2(2m+\lambda+\gamma)\,\Gamma(m+1)\,\Gamma(m+\lambda+\gamma)}=1.$$

□

For the sake of convenience let us enumerate the elements of $\mathcal{Y}^{n,\gamma}_j$ by writing them in the form $Y^{n,\gamma}_{j,k}(\xi), k=1,2,\dots,\sigma_n(j)$ (see (2.9)). Thus, according to Theorem 2.5, we have constructed an orthonormal basis $\{Y^{n,\gamma}_{j,k}\}$ of $L^2_\gamma(S^{n-1}_+)$ which is assumed to be fixed henceforth. Each function $f\in L^2_\gamma(S^{n-1}_+)$ can be decomposed in the "Fourier-Laplace-Bessel" series

$$f(\xi)=\sum_{j=0}^{\infty}\sum_{k=1}^{\sigma_n(j)}\hat f(j,k)Y^{n,\gamma}_{j,k}(\xi) \tag{2.18}$$

which, by the Riesz-Fisher theorem, converges in the $L^2_\gamma(S^{n-1}_+)$-norm. The coefficients $\hat f(j,k)$ are defined by

$$\hat f(j,k)=\int\limits_{S^{n-1}_+}f(\xi)Y^{n,\gamma}_{j,k}(\xi)\,\xi_n^{2\gamma}d\xi=(f,Y^{n,\gamma}_{j,k})_\gamma \tag{2.19}$$

and satisfy Parseval's equality

$$\sum_{j,k}|\hat f(j,k)|^2=\int\limits_{S^{n-1}_+}|f(\xi)|^2\,\xi_n^{2\gamma}d\xi. \tag{2.20}$$

When proving Theorem 2.5 we have established, in fact, the following statement:

Theorem 2.6. *If $f\in L^1_\gamma(S^{n-1}_+)$ is such that $\hat f(j,k)=0\ \forall j,k$, then $f\overset{a.e.}{=}0$.*

3. The Laplace formula, the product formula and the addition theorem

We proceed in a standard way by starting with the following lemma.

Lemma 3.1 (cf. [8]). *For each $j = 0, 1, \ldots$ and each $\xi \in S_+^{n-1}$ there is a so-called zonal B-harmonic $Z_\xi^\gamma(\eta) \in \mathcal{H}_j^{n,\gamma}$ such that*

$$\text{(i)} \qquad Y_j^{n,\gamma}(\xi) = (Y_j^{n,\gamma}, Z_\xi^\gamma)_\gamma \quad \textit{for each } Y_j^{n,\gamma}; \tag{3.1}$$

$$\text{(ii)} \qquad Z_\xi^\gamma(\eta) = \sum_{k=1}^{\sigma_n(j)} Y_{j,k}^{n,\gamma}(\xi) Y_{j,k}^{n,\gamma}(\eta). \tag{3.2}$$

Proof. For fixed $\xi \in S_+^{n-1}$, the map $\mathcal{H}_j^{n,\gamma} \ni Y \to Y(\xi) \ni \mathbb{C}$ is a linear functional on the $\sigma_n(j)$-dimensional space $\mathcal{H}_j^{n,\gamma}$. Hence (3.1) holds by the Riesz representation theorem. The equality (3.2) is a consequence of (3.1). Indeed, since for all $Y \in \mathcal{H}_j^{n,\gamma}$,

$$\begin{aligned}(Y, Z_\xi^\gamma)_\gamma = Y(\xi) &= \sum_{k=1}^{\sigma_n(j)} (Y, Y_{j,k}^{n,\gamma})_\gamma \, Y_{j,k}^{n,\gamma}(\xi) = \\ &= \Big(Y(\eta), \sum_{k=1}^{\sigma_n(j)} Y_{j,k}^{n,\gamma}(\xi) \, Y_{j,k}^{n,\gamma}(\eta)\Big)_\gamma,\end{aligned}$$

then $Z_\xi^\gamma(\eta) = \sum\limits_{k=1}^{\sigma_n(j)} Y_{j,k}^{n,\gamma}(\xi) Y_{j,k}^{n,\gamma}(\eta)$. □

Corollary 3.2. *$Z_\xi^\gamma(\eta) = Z_\eta^\gamma(\xi)$.*

Our next goal is to express $Z_\xi^\gamma(\eta)$ through classical special functions. Three different (and non-consistent!) expressions, obtained in different ways, are known. The proofs given by Lyakhov [27] and Aliev [1] are pretty complicated. The proof of Ben Nahia and Ben Salem [8] looks elegant (the same method was applied earlier by Dijksma and Koornwinder [15]), but all technical details (playing a decisive role in this problem) are omitted in [8]. Below we obtain the required result in a simple way by employing associated B-harmonics (2.12).

Lemma 3.3. *Let*

$$\xi = (\xi_1, \ldots, \xi_{n-1}, \xi_n) = (\xi', \xi_n) \in S^{n-1}, \quad \eta = (\eta_1, \ldots, \eta_{n-1}, \eta_n) = (\eta', \eta_n) \in S^{n-1}.$$

Then for all $j = 0, 1, 2, \dots,$

$$Z_\xi^\gamma(\eta) = c_j^\gamma \int\limits_0^\pi C_j^{\gamma-1+n/2}(\xi'\cdot\eta' + \xi_n\eta_n\cos\psi)\sin^{2\gamma-1}\psi d\psi, \tag{3.3}$$

$$c_j^\gamma = \frac{(\gamma+j-1+n/2)\Gamma(\gamma-1+n/2)}{\pi^{n/2}\Gamma(\gamma)}, \tag{3.4}$$

if $\gamma > 0$ (*cf.*(2.4)), *and for* $\gamma = 0$,

$$Z_\xi^0(\eta) = a_{n,j}\left[C_j^{n/2-1}(\xi'\cdot\eta' + \xi_n\eta_n) + C_j^{n/2-1}(\xi'\cdot\eta' - \xi_n\eta_n)\right], \tag{3.5}$$

$$a_{n,j} = \frac{(n+2j-2)\Gamma(n/2)}{2\pi^{n/2}(n-2)}.$$

Proof. By Theorem 2.3 it suffices to prove (3.1) for B-harmonics (2.12) and $Z_\xi^\gamma(\eta)$ defined by (3.3)–(3.5). We do this first for $\gamma = 0, 1/2, 1, 3/2, \dots,$ by working with axially symmetric harmonics in higher dimensions. Each spherical harmonic Y_j^d on $S^{d-1} \subset \mathbb{R}^d$, $d \ge 3$, is represented by

$$Y_j^d(\omega) = a_{d,j}\int\limits_{S^{d-1}} Y_j^d(\tilde\omega)C_j^{d/2-1}(\omega\cdot\tilde\omega)d\tilde\omega, \qquad a_{d,j} = \frac{(d+2j-2)\Gamma(d/2)}{2\pi^{d/2}(d-2)} \tag{3.6}$$

[29, p. 10]. Let $d = p+q$; $p, q \ge 1$. By passing to bispherical coordinates [44]

$$\begin{aligned} &\omega = \sigma\sin\theta + \zeta\cos\theta, && \sigma \in S^{p-1}, \quad \zeta \in S^{q-1}, \quad 0 \le \theta \le \pi/2,\\ &\tilde\omega = \tilde\sigma\sin\tilde\theta + \tilde\zeta\cos\tilde\theta, && \tilde\sigma \in S^{p-1}, \quad \tilde\zeta \in S^{q-1}, \quad 0 \le \tilde\theta \le \pi/2, \end{aligned} \tag{3.7}$$

one can write the integral in (3.6) as

$$\int\limits_{S^{d-1}} (\dots)\, d\tilde\omega = \int\limits_0^{\pi/2} \sin^{p-1}\tilde\theta\cos^{q-1}\tilde\theta\, d\tilde\theta \int\limits_{S^{p-1}} d\tilde\sigma \int\limits_{S^{q-1}} (\dots)\, d\tilde\zeta.$$

Suppose that $Y_j^d(\omega)$ is $SO(p)$-invariant, i.e. independent of σ and has the form (cf. [44, p. 209, formula (6)])

$$Y_j^d(\omega) = \cos^\ell\theta P_m^{(p/2-1,\ell-1+q/2)}(\cos 2\theta)Y_{\ell,\nu}^q(\zeta), \qquad j = 2m+\ell. \tag{3.8}$$

Then the integral in (3.6) can be written as

$$\int\limits_0^{\pi/2} \sin^{p-1}\tilde\theta \cos^{q+\ell-1}\tilde\theta\, P_m^{(p/2-1,\ell-1+q/2)}(\cos 2\tilde\theta)\, d\tilde\theta \int\limits_{S^{q-1}} Y^q_{\ell,\nu}(\tilde\zeta)\, d\tilde\zeta \quad\times$$

$$\times \int\limits_{S^{p-1}} C_j^{(p+q)/2-1}((\sigma\cdot\tilde\sigma)\sin\theta\sin\tilde\theta + (\zeta\cdot\tilde\zeta)\cos\theta cos\tilde\theta)\, d\tilde\sigma =$$

(replace θ and $\tilde\theta$ by $\dfrac{\pi}{2}-\theta$ and $\dfrac{\pi}{2}-\tilde\theta$ respectively)

$$= \frac{2\pi^{(p-1)/2}(-1)^m}{\Gamma((p-1)/2)} \int\limits_0^{\pi/2} \cos^{p-1}\tilde\theta \sin^{q+\ell-1}\tilde\theta\, P_m^{(\ell-1+q/2,p/2-1)}(\cos 2\tilde\theta)\, d\tilde\theta \quad\times \tag{3.9}$$

$$\times \int\limits_{S^{q-1}} Y^q_{\ell,\nu}(\tilde\zeta) d\tilde\zeta \int\limits_0^{\pi} C_j^{(p+q)/2-1}(\cos\theta\cos\tilde\theta\cos\psi + (\zeta\cdot\tilde\zeta)\sin\theta\sin\tilde\theta)\sin^{p-2}\psi\, d\psi.$$

By setting $p = 2\gamma+1$, $q = n-1$, $\xi = (\cos\theta)e_n + (\sin\theta)\zeta$, and making use of (2.12), from (3.6)–(3.9) we obtain

$$U^{j,\gamma}_{\ell,m,\nu}(\xi) = (U^{j,\gamma}_{\ell,m,\nu}, Z^\gamma_\xi)_\gamma \tag{3.10}$$

with Z^γ_ξ of the form (3.3)–(3.5) and $\gamma = 0, 1/2, 1, \ldots$.

In order to extend (3.10) to all $\gamma \geq 0$ we proceed as in [15, p. 193]. Let $\alpha = \ell + (n-3)/2$, $\beta = p/2-1$ and rewrite (3.10) as

$$f(\gamma) \equiv a(\xi) P_m^{(\alpha,\gamma-1/2)}(2\xi_n^2 - 1) -$$

$$- \frac{(\gamma+j-1+n/2)\Gamma(\gamma-1+n/2)}{\pi^{n/2}\Gamma(\gamma)} \int\limits_{S^{n-1}_+} a(\eta) P_m^{(\alpha,\gamma-1/2)}(2\eta_n^2-1)\, \eta_n^{2\gamma} d\eta \quad\times$$

$$\times \int\limits_0^{\pi} C_j^{\gamma-1+n/2}(\xi'\cdot\eta' + \xi_n\eta_n\cos\psi)\sin^{2\gamma-1}\psi\, d\psi = 0, \quad a(\xi) = \sin^\ell\theta\, Y^{n-1}_{\ell,\nu}(\zeta).$$

Clearly, $f(\gamma)$ is an analytic function for Re$\gamma > 0$. Furthermore, by [38, (7.32.2)],

$$|P_m^{(\alpha,\beta)}(x)| \leq \frac{\Gamma(m+s+1)}{m!\,\Gamma(s+1)} = O(s^m), \quad s = \max\big(\ell + \frac{n-3}{2}, \gamma - \frac{1}{2}\big).$$

Since $P_m^{(\alpha,\gamma-1/2)}(t)$ and $C_j^{\gamma-1+n/2}(t)$ are polynomials of γ (of degree m and j resp.), then a simple estimation yields $f(\gamma) = O(\gamma^{m+j}), \gamma \to \infty$. Now we apply a theorem of Carlson [39, p. 186, Theorem 5.81], which says that if $f(z)$ is regular and of the form $O(e^{k|z|})$ for $k < \pi$ and $\mathrm{Re} z \geq 0$, and if $f(z) = 0$ for $z = 0, 1, 2, \ldots$, then $f(z) \equiv 0$. Thus (3.10) is established for all $\gamma \geq 0$, and the theorem is proven. $\square$

The equality (3.1) with $Z_\xi^\gamma(\eta)$ defined by (3.3)–(3.5) is called *the Laplace formula for B-harmonics*. We note that the constant (3.4) coincides with that in [1] and differs from the constant in the same formula in [8, Theorem 2.3] and [27, Theorem 2.1].

Corollary 3.4. *For $\gamma > 0, j \in \mathbb{Z}_+$, the Laplace formula for B-harmonics can be written in the form*

$$Y_j^{n,\gamma}(\xi) = d_j^\gamma \int\limits_{-1}^{1} C_j^{\gamma-1+n/2}(t)(1-t^2)^{\gamma+(n-3)/2} dt \int\limits_{S_+^{n-1}} Y_j^{n,\gamma}(\eta) K_\gamma(\xi,\eta,t)\, \eta_n^{2\gamma} d\eta \quad (3.11)$$

where

$$d_j^\gamma = \frac{(\gamma + j - 1 + n/2)\,\Gamma(\gamma - 1 + n/2)}{\sqrt{\pi}\,\Gamma(\gamma + (n-1)/2)}, \quad (3.12)$$

$$K_\gamma(\xi,\eta,t) = \begin{cases} 0 \quad \text{if } |t - \xi' \cdot \eta'| > \xi_n \eta_n, \\ \dfrac{\Gamma(\gamma + \frac{n-1}{2})}{\pi^{(n-1)/2}\Gamma(\gamma)} \left(\dfrac{\xi_n^2 \eta_n^2 - (t - \xi' \cdot \eta')^2}{\xi_n^2 \eta_n^2 (1 - t^2)} \right)^{\gamma - 1} \dfrac{1}{\xi_n \eta_n (1 - t^2)^{(n-1)/2}} \\ \qquad \text{otherwise.} \end{cases} \quad (3.13)$$

The validity of (3.11) follows from (3.1), (3.3) by changing variables due to the equality

$$Z_\xi(\eta) = d_j^\gamma \int\limits_{-1}^{1} C_j^{\gamma-1+n/2}(t) K_\gamma(\xi,\eta,t)(1 - t^2)^{\gamma+(n-3)/2} dt. \quad (3.14)$$

Let us introduce the normalized Gegenbauer polynomials

$$P_j^\lambda(t) = \frac{C_j^\lambda(t)}{C_j^\lambda(1)}, \qquad C_j^\lambda(1) = \frac{\Gamma(2\lambda + j)}{j!\,\Gamma(2\lambda)}. \quad (3.15)$$

Theorem 3.5. *The following product formula holds:*

$$Y_j^{n,\gamma}(\xi) P_j^{\gamma-1+n/2}(t) = \int\limits_{S_+^{n-1}} Y_j^{n,\gamma}(\eta) K_\gamma(\xi,\eta,t)\, \eta_n^{2\gamma} d\eta, \quad \gamma > 0,\ j \in \mathbb{Z}_+. \quad (3.16)$$

Proof. It suffices to show the coincidence of Fourier-Gegenbauer coefficients of both sides of (3.16). We recall that for the orthogonal system of Gegenbauer polynomials $C_m^\lambda(t)$ the corresponding Fourier coefficients of a function f in the space $L^1([-1,1];(1-t^2)^{\lambda-1/2})$ are defined by

$$\hat{f}(m) = \frac{1}{h_m}\int\limits_{-1}^{1} f(t)C_m^\lambda(t)(1-t^2)^{\lambda-1/2}dt, \tag{3.17}$$

$$h_m = \int\limits_{-1}^{1}[C_m^\lambda(t)]^2(1-t^2)^{\lambda-1/2}dt = \frac{\sqrt{\pi}\,\Gamma(2\lambda+m)\Gamma(\lambda+1/2)}{\Gamma(2\lambda)\,\Gamma(\lambda)m!\,(m+\lambda)} \tag{3.18}$$

(see, e.g., [16, 10.19] and 2.21.18(10) from [32]). Then $f(t) \sim \sum\limits_{m=0}^{\infty}\hat{f}(m)C_m^\lambda(t)$. For $\lambda = \gamma - 1 + n/2$ the Fourier-Gegenbauer coefficients of the function

$$\Omega_j(\xi,t) = \int\limits_{S_+^{n-1}} Y_j^{n,\gamma}(\eta)K_\gamma(\xi,\eta,t)\,\eta_n^{2\gamma}d\eta,$$

which is the right-hand side of (3.16), are as follows:

$$\frac{1}{h_m}\int\limits_{-1}^{1}\Omega_j(\xi,t)C_m^{\gamma-1+n/2}(t)(1-t^2)^{\gamma+(n-3)/2}dt = \begin{cases} 0 \ \text{ if } \ m \neq j, \\ Y_j^{n,\gamma}(\xi)/C_j^{\gamma-1+n/2}(1) \ \text{ if } \ m = j. \end{cases} \tag{3.19}$$

Indeed, for a zonal B-harmonic $Z_\xi(\eta)$ of degree $m \neq j$, according to (3.14) we have

$$\begin{aligned} 0 &= (Y_j^{n,\gamma}, Z_\xi)_\gamma = \\ &= d_m^\gamma \int\limits_{S_+^{n-1}} Y_j^{n,\gamma}(\eta)\,\eta_n^{2\gamma}d\eta \int\limits_{-1}^{1} C_m^{\gamma-1+n/2}(t)K_\gamma(\xi,\eta,t)(1-t^2)^{\gamma+(n-3)/2}dt = \\ &= d_m^\gamma \int_{-1}^{1}\Omega_j(\xi,t)C_m^{\gamma-1+n/2}(t)(1-t^2)^{\gamma+(n-3)/2}dt. \end{aligned}$$

For $m = j$, the equality (3.19) coincides with (3.11). Since the Fourier-Gegenbauer coefficients of the left-hand side of (3.16) also have the form (3.19) the result follows.

□

Remark 3.6. The product formula (3.16) is a natural analog of the product formula for Jacobi polynomials (cf. [17, 18, 23, 15]). The non-negativity of the kernel

$K_\gamma(\xi,\eta,t)$, which is obvious from (3.13), will play an important role in the following. In the close situation the non-negativity of the similar kernel for ultraspherical and Jacobi polynomials was studied by Bochner [11–13], Hirschman [19], Gasper [17, 18], Bavinck [6] and other authors.

We conclude this section by stating the *addition theorem for B-harmonics* which follows from (3.2), (3.3), (3.14).

Theorem 3.7. *For $\gamma > 0$,*

$$\sum_{k=1}^{\sigma_n(j)} Y^{n,\gamma}_{j,k}(\xi)Y^{n,\gamma}_{j,k}(\eta) = c^\gamma_j \int_0^\pi C_j^{\gamma-1+n/2}(\xi'\cdot\eta' + \xi_n\eta_n\cos\psi)\sin^{2\gamma-1}\psi d\psi = \tag{3.20}$$

$$= d^\gamma_j \int_{-1}^1 C_j^{\gamma-1+n/2}(t) K_\gamma(\xi,\eta,t)(1-t^2)^{\gamma+(n-3)/2}dt, \tag{3.21}$$

c^γ_j, d^γ_j and $K_\gamma(\xi,\eta,t)$ being defined by (3.4), (3.12) *and* (3.13) *respectively.*

In connection with this theorem we address the reader to [2, 3, 21, 44 (p. 217)] for further information concerning additional formulas for Gegenbauer and Jacobi polynomials.

4. B-convolutions on S^{n-1}_+ and the Funk-Hecke formula

According to (3.3), (2.4), we introduce a notion of a B-convolution on S^{n-1}_+.

Definition 4.1. *For sufficiently nice functions $f : S^{n-1}_+ \to \mathbb{C}$ and $g : [-1,1] \to \mathbb{C}$, a B-convolution on S^{n-1}_+ is defined by*

$$(f * g)(\xi) = \int_{S^{n-1}_+} f(\eta) g_\gamma(\xi;\eta)\, \eta_n^{2\gamma} d\eta, \tag{4.1}$$

$$g_\gamma(\xi;\eta) = \begin{cases} \dfrac{2\pi^\gamma}{\Gamma(\gamma)} \displaystyle\int_0^\pi g(\xi'\cdot\eta' + \xi_n\eta_n\cos\psi)\sin^{2\gamma-1}\psi\, d\psi & \text{if} \quad \gamma > 0, \\ g(\xi'\cdot\eta' + \xi_n\eta_n) + g(\xi'\cdot\eta' - \xi_n\eta_n) & \text{if} \quad \gamma = 0 \end{cases} \tag{4.2}$$

(*so that* $\lim\limits_{\gamma\to 0} g_\gamma(\xi;\eta) = g_0(\xi;\eta)$).

By changing variables, the equality (4.2) for $\gamma > 0$ can be written as

$$g_\gamma(\xi;\eta) = |S^{2\gamma+n-2}| \int_{-1}^{1} g(t) K_\gamma(\xi,\eta,t)(1-t^2)^{\gamma+(n-3)/2} dt \tag{4.3}$$

where $K_\gamma(\xi,\eta,t) \geq 0$ is the kernel (3.13) and the quantity

$$|S^{2\gamma+n-2}| \overset{def}{=} \frac{2\pi^{\gamma+(n-1)/2}}{\Gamma(\gamma+(n-1)/2)} \tag{4.4}$$

can be regarded formally as a volume of the $(2\gamma+n-2)$-dimensional unit sphere.

Note that for $\gamma = 0$, (4.1) can be written as the usual spherical convolution

$$(f * g)(\xi) = \int_{S^{n-1}} f_e(\eta) g(\xi \cdot \eta) d\eta, \tag{4.5}$$

$f_e(\eta) = f(\eta' + |\eta_n| e_n)$ being an even (in η_n) extension of f onto S^{n-1}.

By setting $t = \xi' \cdot \eta' + \xi_n \eta_n \cos\psi$ for $\gamma > 0$ and passing to spherical polar coordinates for $\gamma = 0$, one can represent (4.1) and (4.5) as

$$(f * g)(\xi) = |S^{2\gamma+n-2}| \int_{-1}^{1} g(t)(M_t^\gamma f)(\xi)(1-t^2)^{\gamma+(n-3)/2} dt \tag{4.6}$$

where

$$(M_t^\gamma f)(\xi) = \begin{cases} \displaystyle\int_{S_+^{n-1}} f(\eta) K_\gamma(\xi,\eta,t)\, \eta_n^{2\gamma} d\eta & \text{if } \gamma > 0, \\ (M_t f_e)(\xi) & \text{if } \gamma = 0, \end{cases} \tag{4.7}$$

$$(M_t f_e)(\xi) = \frac{(1-t^2)^{1-n/2}}{|S^{n-2}|} \int_{\Gamma_\xi(t)} f_e(\eta) d\sigma(\eta) \tag{4.8}$$

being the mean value of f_e over the planar section $\Gamma_\xi(t) = \{\eta \in S^{n-1} : \xi \cdot \eta = t\}$.

The mean value $M_t f_e$ has the following geometrical interpretation:

$$(M_t f_e)(\xi) = (M_t^0 f)(\xi) = \frac{(1-t^2)^{1-n/2}}{|S^{n-2}|} \int_{\tilde{\Gamma}_\xi(t)} f(\eta) d\sigma(\eta) \tag{4.9}$$

where $\tilde{\Gamma}_\xi(t) = \Gamma_\xi(t)$ if $\Gamma_\xi(t) \subset S_+^{n-1}$. Otherwise, $\tilde{\Gamma}_\xi(t) = \Gamma_\xi^+(t) \cup \Gamma_\xi^r(t)$ where $\Gamma_\xi^+(t) = \Gamma_\xi(t) \cap S_+^{n-1}$ and $\Gamma_\xi^r(t)$ is the reflection of $\Gamma_\xi^+(t)$ from the hyperplane $x_n = 0$ (the reader is adviced to draw a picture).

Definition 4.2. *The function $f_\gamma(\xi;t) = (M_t^\gamma f)(\xi)$, $\xi \in S^{n-1}_+$, $t \in (-1,1)$, will be called a generalized translation (or B-translation) of f on the hemisphere S^{n-1}_+. Correspondingly, the operator M_t^γ will be called a generalized translation operator (GTO), or a B-translation operator, on S^{n-1}_+.*

The operator M_t^γ is a natural generalization of the mean value operator M_t^0 (see (4.9)) to the case $\gamma > 0$. Similar operators in the euclidean case are well-known in analysis due to the works of J. Delsarte, B.M. Levitan, I.A. Kipriyanov, K. Trimeche, J. Peetre, K. Stempak, I.A. Aliev and other authors. In the context of ultraspherical and Jacobi polynomials, GTO's were studied by S. Bochner [11–13], I.I. Hirschman, Jr. [19], R. Askey and S. Wainger [5], G. Gasper [17, 18], H. Bavinck [7]. An information about GTO in the more general settings for various singular differential operators and hypergroups can be found in [41, 42] where many other references are given. The equality (4.6) is well known for $\gamma = 0$, and the relevant translation operator (4.8) on S^{n-1} is widely used in harmonic analysis and approximations on the sphere (see, e.g., [9, 25, 33, 34]).

Lemma 4.3. *B-harmonics are eigenfunctions of M_t^γ. Namely, for $\gamma > 0$ and $j \in \mathbb{Z}_+$,*

$$(M_t^\gamma Y_j^{n,\gamma})(\xi) = Y_j^{n,\gamma}(\xi) P_j^{\gamma-1+n/2}(t). \tag{4.10}$$

Proof. By (4.7), this statement is a rephrase of the product formula (3.16). □

Lemma 4.4. *For $f \in L^1_\gamma(S^{n-1}_+)$, $\gamma > 0$,*

$$(M_t^\gamma f)^\wedge(j,k) = \hat f(j,k) P_j^{\gamma-1+n/2}(t). \tag{4.11}$$

Proof. The result follows from (3.16) if we change the order of integration in the left-hand side. □

Corollary 4.5. *Let $f \in L^1_\gamma(S^{n-1}_+)$, $g \in L^1_{\gamma+(n-3)/2}(-1,1)$, $\gamma > 0$. Denote*

$$\hat g_\gamma(j) = |S^{2\gamma+n-2}| \int\limits_{-1}^{1} g(t) P_j^{\gamma-1+n/2}(t)(1-t^2)^{\gamma+(n-3)/2} dt. \tag{4.12}$$

Then

$$(f * g)^\wedge(j,k) = \hat{f}(j,k)\hat{g}_\gamma(j). \tag{4.13}$$

Proof. The statement follows from (4.6), (4.11). $\square$

The following theorem gives an analog of the Funk-Hecke formula.

Theorem 4.6. *If $g \in L^1_{\gamma+(n-3)/2}(-1,1)$, $\gamma > 0$, then for each B-harmonic $Y_j^{n,\gamma}$,*

$$Y_j^{n,\gamma} * g = \hat{g}_\gamma(j) Y_j^{n,\gamma}, \tag{4.14}$$

$\hat{g}_\gamma(j)$ being defined by (4.12).

Proof. Put $f = Y_j^{n,\gamma}$ in (4.6) and make use of (4.10). $\square$

Remark 4.7. For $\gamma = 0$, the equalities (4.10)–(4.14) are classical.

Remark 4.8. A useful observation which motivates Definition 4.1 of a B-convolution on S_+^{n-1} is the following. A usual spherical convolution on $S^{p+q-1} \subset \mathbb{R}^{p+q} = \mathbb{R}^p \times \mathbb{R}^q$ for $SO(p)$-invariant function $\varphi : S^{p+q-1} \to \mathbb{C}$ can be represented in the bispherical coordinates (3.7) by

$$\int\limits_{S^{p+q-1}} \varphi(\tilde{\omega}) g(\omega \cdot \tilde{\omega}) d\tilde{\omega} = \int\limits_0^{\pi/2} \sin^{p-1}\tilde{\theta} \cos^{q-1}\tilde{\theta}\, d\tilde{\theta} \quad \times$$

$$\times \int\limits_{S^{p-1}} d\tilde{\sigma} \int\limits_{S^{q-1}} \varphi(\tilde{\sigma}\sin\tilde{\theta} + \tilde{\zeta}\cos\tilde{\theta}) g((\sigma\cdot\tilde{\sigma})\sin\theta\sin\ \tilde{\theta} + (\zeta\cdot\tilde{\zeta})\cos\theta\cos\tilde{\theta}) d\tilde{\zeta}.$$

If we replace θ and $\tilde{\theta}$ by $\frac{\pi}{2} - \theta$ and $\frac{\pi}{2} - \tilde{\theta}$ respectively, and then set $p = 2\gamma + 1,\ q = n - 1$,

$$\xi = (\cos\theta) e_n + (\sin\theta)\zeta, \quad \eta = (\cos\tilde{\theta}) e_n + (\sin\tilde{\theta})\tilde{\zeta}, \quad \varphi \equiv \varphi(\tilde{\theta}, \tilde{\zeta}) \overset{def}{=} f(\eta),$$

after simple calculations we obtain

$$\begin{aligned} &\int\limits_{S^{p+q-1}} \varphi(\tilde{\omega}) g(\omega\cdot\tilde{\omega}) d\tilde{\omega} = \\ &= \frac{2\pi^\gamma}{\Gamma(\gamma)} \int\limits_{S_+^{n-1}} f(\eta)\, \eta_n^{2\gamma} d\eta \int\limits_0^\pi g(\xi'\cdot\eta + \xi_n\eta_n\cos\psi)\sin^{2\gamma-1}\psi d\psi \end{aligned} \tag{4.15}$$

which coincides with (4.1). Thus *B-convolutions generalize (from $\gamma = 0, 1/2, 1, \ldots$ to all $\gamma \geq 0$) usual spherical convolutions for $SO(p)$-invariant functions on S^{p+q-1}.*

Lemma 4.9. *Let* $\gamma \geq 0, \ \ 1 \leq p \leq \infty$. *Then*

$$\text{(i)} \qquad \|M_t^\gamma f\|_{p,\gamma} \leq \|f\|_{p,\gamma}, \tag{4.16}$$

$$\text{(ii)} \qquad \|g_\gamma(\cdot;\eta)\|_{p,\gamma} \leq |S^{2\gamma+n-2}|^{1/p}|S^{2\gamma}|^{1/p'}\|g\|_{L^p_{\gamma+(n-3)/2}(-1,1)} \tag{4.17}$$

where $|S^{2\gamma}| = 2\pi^{\gamma+1/2}/\Gamma(\gamma+1/2)$ (*cf.* (4.4)).

Proof. (i) For $\gamma = 0$ this statement is known. Let $\gamma > 0$. For $p = 1$, owing to the non-negativity of $K_\gamma(\xi,\eta,t)$, from (4.7) we get

$$\|M_t^\gamma f\|_{1,\gamma} \leq \int\limits_{S^{n-1}_+} |f(\eta)|\, \eta^{2\gamma} d\eta \int\limits_{S^{n-1}_+} K_\gamma(\xi,\eta,t)\, \xi_n^{2\gamma} d\xi = \|f\|_{1,\gamma}$$

because

$$\int\limits_{S^{n-1}_+} K_\gamma(\xi,\eta,t)\, \xi_n^{2\gamma} d\xi = 1 \tag{4.18}$$

(this is a particular case of (3.16) for $j = 0$). For $p = \infty$ the result follows by the same reason. The case $1 < p < \infty$ then holds by interpolation.

(ii) Let $\gamma > 0$. By comparing (4.2) and (4.3) (with $g \equiv 1$), we have

$$|S^{2\gamma+n-2}| \int\limits_{-1}^{1} K_\gamma(\xi,\eta,t)(1-t^2)^{\gamma+(n-3)/2} dt = \frac{2\pi^\gamma}{\Gamma(\gamma)} \int\limits_0^\pi \sin^{2\gamma-1}\psi d\psi = \frac{2\pi^{\gamma+1/2}}{\Gamma(\gamma+1/2)}$$

which gives

$$\int\limits_{-1}^{1} K_\gamma(\xi,\eta,t)(1-t^2)^{\gamma+(n-3)/2} dt = \frac{|S^{2\gamma}|}{|S^{2\gamma+n-2}|}. \tag{4.19}$$

Hence, by Hölder's inequality, from (4.3) we obtain

$$|g_\gamma(\xi;\eta)| \leq |S^{2\gamma+n-2}|^{1/p}|S^{2\gamma}|^{1/p'} \Big(\int\limits_{-1}^{1} |g(t)|^p K_\gamma(\xi,\eta,t)(1-t^2)^{\gamma+(n-3)/2} dt \Big)^{1/p}.$$

Owing to (4.18), this implies (4.17). For $\gamma = 0$, the required estimate follows from (4.2) by direct calculation:

$$\|g_\gamma(\cdot,\eta)\|_p = \Big(\frac{1}{2} \int\limits_{S^{n-1}} |g(\xi\cdot\eta) + g(\xi'\cdot\eta' - \xi_n\eta_n)|^p d\xi \Big)^{1/p} \leq$$

$$\leq \frac{2}{2^{1/p}} \Big(\int\limits_{S^{n-1}} |g(\xi\cdot\eta)|^p d\xi \Big)^{1/p} = 2^{1/p'}|S^{n-2}|^{1/p}\|g\|_{L^p_{(n-3)/2}(-1,1)}.$$

□

Lemma 4.10 (Young's inequality)**.** *For* $1 \le p \le q \le \infty,\ 1 - p^{-1} + q^{-1} = r^{-1}$, $\gamma \ge 0$,

$$\|f * g\|_{q,\gamma} \le |S^{2\gamma+n-2}|^{1/r} |S^{2\gamma}|^{1/r'} \|f\|_{p,\gamma} \|g\|_{L^r_{\gamma+(n-3)/2}(-1,1)}. \tag{4.20}$$

Proof. It suffices to prove the estimates

$$\|f * g\|_{L^p_\gamma(S^{n-1}_+)} \le |S^{2\gamma+n-2}| \, \|f\|_{L^p_\gamma(S^{n-1}_+)} \|g\|_{L^1_{\gamma+(n-3)/2}(-1,1)}, \tag{4.21}$$

$$\|f * g\|_{L^\infty_\gamma(S^{n-1}_+)} \le |S^{2\gamma+n-2}|^{1/p'} |S^{2\gamma}|^{1/p} \|f\|_{L^p_\gamma(S^{n-1}_+)} \|g\|_{L^{p'}_{\gamma+(n-3)/2}(-1,1)}$$

from which the result follows by interpolation. The first estimate can be obtained from (4.6) by applying the generalized Minkowski inequality and making use of (4.16). The second one follows from (4.1) by Hölder's inequality owing to (4.17). □

5. An expansion of smooth functions on S^{n-1}_+ in B-harmonics

Before we state the main theorem of this section let us prove a series of lemmas which are important on their own. Denote

$$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n_+, \quad |\alpha| = \alpha_1 + \dots + \alpha_n.$$

Lemma 5.1. *For* $\gamma \ge 0$,

$$|D^\alpha [|x|^j Y^{n,\gamma}_{j,k}(x/|x|)]| \le c j^{\gamma + |\alpha| - 1 + n/2} |x|^{j - |\alpha|} \tag{5.1}$$

where c *depends only on* $|\alpha|, \gamma$ *and* n.

Proof. Let $P^\gamma_j(x) = |x|^j Y^{n,\gamma}_{j,k}(x/|x|)$, $P^0_j(x) = |x|^j Y^{n,0}_{j,k}(x/|x|)$. By (2.14), (2.15), owing to the analog of (5.1) for $\gamma = 0$ [31, p. 225], we have

$$\begin{aligned} |D^\alpha P^\gamma_j(x)| &\le c j^\gamma \int_0^\pi |(D^\alpha P^0_j)(x', x_n \cos\psi)| \, |\cos\psi|^{\alpha_n} \sin^{2\gamma-1}\psi d\psi \le \\ &\le c j^{\gamma + |\alpha| - 1 + n/2} u_{j,\alpha}(x), \end{aligned}$$

where

$$u_{j,\alpha}(x) = \int_0^\pi (|x'|^2 + x_n^2 \cos^2\psi)^{(j-|\alpha|)/2} \, |\cos\psi|^{\alpha_n} \sin^{2\gamma-1}\psi \, d\psi \le c_\gamma |x|^{j-|\alpha|}$$

for $j \ge |\alpha|$ (if $j < |\alpha|$ the left-hand side of (5.1) is zero). □

Corollary 5.2.

$$|D^\alpha[Y^{n,\gamma}_{j,k}(x/|x|)]| \le c j^{\gamma-1+n/2+|\alpha|}|x|^{-|\alpha|}, \quad c=c(\alpha,\gamma,n). \tag{5.2}$$

Proof. The estimate follows from (5.1) by the Leibniz rule. □

It is known, that ordinary spherical harmonics $Y^n_j(\xi)$ constitute a complete system of eigenfunctions of the Beltrami-Laplace operator δ_n on S^{n-1} with the eigenvalues $-j(j+n-2)$ (see, e.g., [29, 31, 36]). A similar statement holds for B-harmonics provided that δ_n is replaced by a certain singular second order differential operator on S^{n-1}_+.

Let us introduce the spherical coordinates $r, \theta_1, \dots, \theta_{n-1}$ by

$$\begin{cases} x_1 = r\sin\theta_{n-1}\dots\sin\theta_2\sin\theta_1 \\ x_2 = r\sin\theta_{n-1}\dots\sin\theta_2\cos\theta_1 \\ \dots\dots\dots\dots \\ x_{n-1} = r\sin\theta_{n-1}\cos\theta_{n-2} \\ x_n = r\cos\theta_{n-1}, \end{cases} \tag{5.3}$$

where $0\le r<\infty, \;\; 0\le\theta_1<2\pi, \;\; 0\le\theta_k<\pi$ for $k\ne 1$. By (5.3),

$$r=(x_1^2+\dots+x_n^2)^{1/2}, \qquad \cos\theta_k = x_{k+1}(x_1^2+\dots+x_{k+1}^2)^{-1/2},$$

so that if $f(x)\equiv g(r,\theta_1,\dots,\theta_{n-1})$, then

$$\frac{\partial f}{\partial x_n} = \frac{\partial g}{\partial r}\,\frac{x_n}{r} - \frac{\partial g}{\partial\theta_{n-1}}\,\frac{x_n}{r^2}\,\tan\theta_{n-1}.$$

This gives the following representation of the Laplace-Bessel operator (2.1) in spherical coordinates:

$$\Delta_{n,\gamma} \;=\; \frac{\partial^2}{\partial r^2} + \frac{n-1+2\gamma}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\delta_{n,\gamma} \tag{5.4}$$

where

$$\delta_{n,\gamma} = \delta_n - 2\gamma\tan\theta_{n-1}\frac{\partial}{\partial\theta_{n-1}} \tag{5.5}$$

δ_n being the usual Beltrami-Laplace operator on S^{n-1}.

Definition 5.3. *We say that $f(\xi) \in C^k(S^{n-1}_+)$ if the function $f_0(x) = f(\frac{x'}{|x|}, \frac{|x_n|}{|x|})$, which extends f by homogeneity and the x_n-evenness, belongs to $C^k_*(\mathbb{R}^n \setminus \{0\})$. We set $C^\infty(S^{n-1}_+) = \bigcap\limits_{k\geq 0} C^k(S^{n-1}_+)$ and*

$$D^\alpha f(\xi) = D^\alpha f_0(x)|_{x=\xi}. \tag{5.6}$$

Lemma 5.4. *If $f(\xi) \in C^{2k}(S^{n-1}_+)$ (or $f_0(x) \in C^{2k}_*(\mathbb{R}^n \setminus \{0\})$), then $\delta^\ell_{n,\gamma} f(\xi) \in C^{2k-2\ell}(S^{n-1}_+)$ (or $f_0(x) \in C^{2k-2\ell}_*(\mathbb{R}^n \setminus \{0\})$), $\ell = 0, 1, \dots, k$.*

Proof. Since, by (5.4),

$$\delta_{n,\gamma} f(\xi) = |x|^2 \Delta_{n,\gamma} f_0(x), \quad \xi = x/|x|, \tag{5.7}$$

it suffices to prove the lemma for f_0. Furthermore, since differentiation in x' causes no trouble, it suffices to examine the action of the operator $\partial = x^{-1}\partial/\partial x$ on even functions on $\mathbb{R}^1$. Let us show that for $\varphi \in C^{2k}_*(\mathbb{R}^1)$ and $\ell = 0, 1, \dots, k$,

$$\partial^\ell \varphi \in C^{2k-2\ell}_*(\mathbb{R}^1) \text{ and } \lim_{x\to 0} (\partial^\ell \varphi)(x) = const\, \varphi^{(2\ell)}(0). \tag{5.8}$$

Indeed, by Taylor's formula,

$$\varphi(x) = \varphi(0) + x\varphi'(0) + \frac{x^2}{2!}\varphi''(0) + \ldots + \frac{x^{2k-1}}{(2k-1)!}\varphi^{(2k-1)}(0) + \psi(x) \tag{5.9}$$

where for $x > 0$,

$$\psi(x) = (I^{2k}_{0+}\varphi^{(2k)})(x) \overset{def}{=} \frac{x^{2k}}{\Gamma(2k)} \int_0^1 \varphi^{(2k)}(tx)(1-t)^{2k-1} dt.$$

Since φ is even, then ψ is also even and by (5.9),

$$\varphi(x) = \varphi(0) + \frac{x^2}{2!}\varphi''(0) + \ldots + \frac{x^{2k-2}}{(2k-2)!} f^{(2k-2)}(0) + \psi(x)$$

which implies

$$\partial^\ell \varphi(x) = \sum_{j=\ell}^{k-1} c_{j,\ell}\, x^{2(j-\ell)} \varphi^{(2j)}(0) + \partial^\ell \psi(x) \text{ if } \ell \leq k-1 \tag{5.10}$$

and $\partial^k\varphi(x) = \partial^k\psi(x)$ if $\ell = k$ (here and on c with subscripts designates a constant which can be different at each occurrence). For $x > 0$, the expression $\partial^\ell\psi$ can be written as

$$\partial^\ell\psi(x) = \big(\frac{1}{x}\frac{d}{dx}\big)^\ell (I^{2k}_{0+}\varphi^{(2k)})(x) = \sum_{j=1}^{\ell} \frac{c_j}{x^{2\ell-j}} (I^{2k-j}_{0+}\varphi^{(2k)})(x) \tag{5.11}$$

and

$$\lim_{x\to+0} \partial^\ell\psi(x) = \sum_{j=1}^{\ell} c_j x^{2(k-\ell)} \int\limits_0^1 \varphi^{(2k)}(xt)(1-t)^{2k-j-1}dt = \begin{cases} 0 \text{ if } \ell \le k-1, \\ const\,\varphi^{(2\ell)}(0). \end{cases}$$

Hence, by (5.10), $\lim\limits_{x\to+0} \partial^\ell\varphi(x) = const\,\varphi^{(2\ell)}(0)$ for $\ell \le k-1$, and the same result holds for $\ell = k$. Since ∂^ℓ preserves the evenness, the same holds for $x \to -0$. The expression (5.11) belongs to $C^{2k-2\ell}_*(\mathbb{R}^1)$ and

$$(\partial^\ell\psi)^{(2k-2\ell)}(x) = \sum_{j=1}^{\ell}\sum_{m=0}^{2k-2\ell} c_{j,m} \int\limits_0^1 \varphi^{(2k)}(xt)(1-t)^{2k-j-m-1}dt \ (\in C_*(\mathbb{R}^1)).$$

This completes the proof of (5.8) and hence the proof of the lemma. □

Lemma 5.5. *If $u, v \in C^2(S^{n-1}_+)$, $\gamma \ge 0$, then*

$$(u, \delta_{n,\gamma}v)_\gamma = (\delta_{n,\gamma}u, v)_\gamma \tag{5.12}$$

(i.e. $\delta_{n,\gamma}$ is a self-adjoint operator).

Proof. For $\gamma > 0$ we will use the modified Green formula

$$\int\limits_G (\varphi\Delta_{n,\gamma}\psi - \psi\Delta_{n,\gamma}\varphi)\, x_n^{2\gamma}dx = \int\limits_{(\partial G)^+} \big(\varphi\frac{\partial\psi}{\partial\nu} - \psi\frac{\partial\varphi}{\partial\nu}\big)\, x_n^{2\gamma}d\sigma. \tag{5.13}$$

Here G is a finite domain in the half-space $x_n \ge 0$ adjacent to the hyperplane $x_n = 0$, with a piecewise smooth boundary ∂G; $(\partial G)^+ = \partial G \cap \{x\colon x_n > 0\}$; $\varphi, \psi \in C^2(G \cup \partial G)$; ν is the outer unit normal to ∂G. The validity of (5.13) follows by application of the relevant usual Green formula to the functions $\tilde\varphi(x) = x_n^\gamma\varphi(x)$, $\tilde\psi(x) = x_n^\gamma\psi(x)$ if one takes into account that

$$\tilde\varphi\Delta_n\tilde\psi - \tilde\psi\Delta_n\tilde\varphi = [\varphi\Delta_{n,\gamma}\psi - \psi\Delta_{n,\gamma}\varphi]x_n^{2\gamma},$$
$$\tilde\varphi\frac{\partial\tilde\psi}{\partial\nu} - \tilde\psi\frac{\partial\tilde\varphi}{\partial\nu} = \big(\varphi\frac{\partial\psi}{\partial\nu} - \psi\frac{\partial\varphi}{\partial\nu}\big)x_n^{2\gamma}.$$

Let $G = \{x : 0 < a < |x| < 1, \quad x_n > 0\}$, $\varphi(x) = u(x/|x|)$, $\quad \psi(x) = \overline{v}(x/|x|)$. By (5.4), (5.13),

$$\int\limits_a^1 r^{n-3+2\gamma} dr \int\limits_{S_+^{n-1}} (u\delta_{n,\gamma}\overline{v} - \overline{v}\delta_{n,\gamma}u)\, \xi_n^{2\gamma} d\,\xi = 0$$

which implies (5.12) for $\gamma > 0$. The validity of (5.12) for $\gamma = 0$ follows from the similar well known relation on S^{n-1} [31]. Indeed, if $\tilde{u}(\xi) = u(\xi'; |\xi_n|)$, $\tilde{v}(\xi) = v(\xi', |\xi_n|)$, $\xi \in S^{n-1}$, then

$$(u, \delta_{n,0}v)_0 = \frac{1}{2}(\tilde{u}, \delta_n \tilde{v}) = \frac{1}{2}(\delta_n \tilde{u}, \tilde{v}) = (\delta_{n,0}u, v)_0.$$

□

Lemma 5.6. *For any B-harmonic $Y_j^{n,\gamma}(\xi)$, $\gamma \geq 0$,*

$$\delta_{n,\gamma} Y_j^{n,\gamma} = \lambda_j Y_j^{n,\gamma}, \qquad \lambda_j = -j(j + n + 2\gamma - 2). \tag{5.14}$$

The collection of all B-harmonics $Y_j^{n,\gamma}(\xi)$ forms a complete system of eigenfunctions of $\delta_{n,\gamma}$.

Proof. Let $P_j(x) = r^j Y_j^{n,\gamma}(\xi)$, $\xi = x/r$. By (5.4),

$$0 = \Delta_{n,\gamma} P_j(x) = \Big[\frac{\partial^2}{\partial r^2} + \frac{n + 2\gamma - 1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\delta_{n,\gamma}\Big] r^j Y_j^{n,\gamma}(\xi) =$$
$$= [j(j-1)Y_j^{n,\gamma}(\xi) + j(n + 2\gamma - 1)Y_j^{n,\gamma}(\xi) + \delta_{n,\gamma} Y_j^{n,\gamma}(\xi)] r^{j-2} \quad \forall r > 0.$$

Hence $j(j + n + 2\gamma - 2)Y_j^{n,\gamma}(\xi) + \delta_{n,\gamma} Y_j^{n,\gamma}(\xi) = 0$ which gives (5.14). Conversely, let $\delta_{n,\gamma} f = \lambda f$ for some $\lambda \not\equiv 0$ and $f \in C^2(S_+^{n-1})$, $f \not\equiv 0$. Then, by (5.12), for any $Y_j^{n,\gamma}(\xi)$ we have

$$\lambda(f, Y_j^{n,\gamma})_\gamma = (\delta_{n,\gamma} f, Y_j^{n,\gamma})_\gamma = (f, \delta_{n,\gamma} Y_j^{n,\gamma})_\gamma = \lambda_j (f, Y_j^{n,\gamma})_\gamma,$$

i.e. $(\lambda - \lambda_j)(f, Y_j^{n,\gamma})_\gamma = 0$. If $\lambda \neq \lambda_j$ for any j, then $(f, Y_j^{n,\gamma})_\gamma = 0$ $\forall j$ and, by Theorem 2.6, $f \equiv 0$. If $\lambda = \lambda_{j_0}$ for some j_0, then $(f, Y_j^{n,\gamma})_\gamma = 0$ if $j \neq j_0$ and $(f, Y_{j_0}^{n,\gamma})_\gamma \neq 0$ (otherwise $f \equiv 0$). It follows that

$$f(\xi) = \sum_{k=1}^{\sigma_n(j_0)} \hat{f}(j_0, k) Y_{j_0,k}^{n,\gamma}(\xi) \in \mathcal{H}_{j_0}^{n,\gamma}.$$

□

Lemma 5.7. *If $f \in C^{2m}(S^{n-1}_+)$, $\gamma \ge 0$, $m \in \mathbb{Z}_+$, then*

$$|\hat f(j,k)| \le c j^{-2m}, \qquad c = \|\delta^m_{n,\gamma} f\|_{2,\gamma}. \tag{5.15}$$

Proof. Owing to (5.14), (5.12), and taking into account Lemma 5.4, we have

$$\begin{aligned}|\hat f(j,k)| &= |(f, Y^{n,\gamma}_{j,k})_\gamma| = |\lambda_j^{-m}(f, \delta^m_{n,\gamma} Y^{n,\gamma}_{j,k})_\gamma| \le j^{-2m}|(\delta^m_{n,\gamma} f, Y^{n,\gamma}_{j,k})_\gamma| \le \\ &\le j^{-2m}\|\delta^m_{n,\gamma} f\|_{2,\gamma}\|Y^{n,\gamma}_{j,k}\|_{2,\gamma} = j^{-2m}\|\delta^m_{n,\gamma} f\|_{2,\gamma}.\end{aligned}$$

□

Now we can state the main theorem of this section.

Theorem 5.8.

(i) *If $f \in C^\infty(S^{n-1}_+)$, then all series*

$$\sum_{j,k} \hat f(j,k) D^\alpha Y^{n,\gamma}_{j,k}(\xi), \qquad \alpha \in \mathbb{Z}^n_+, \tag{5.16}$$

converge absolutely and uniformly to $D^\alpha f$, and the series

$$\sum_{j,k} j^m |\hat f(j,k)| \tag{5.17}$$

converges for all $m \in \mathbb{Z}_+$.

(ii) *If the series (5.17) converges for all $m \in \mathbb{Z}_+$, then $f \in C^\infty(S^{n-1}_+)$.*

Proof. (i) follows from (5.15) and (5.2). Conversely, if (5.17) converges for all $m \in \mathbb{Z}_+$, then by (5.2) all series (5.16) converge absolutely and uniformly. It follows that $f(\xi) \equiv \sum\limits_{j,k} \hat f(j,k) Y^{n,\gamma}_{j,k}(\xi) \in C^\infty(S^{n-1}_+)$. □

6. Symmetry properties of the Fourier-Bessel transform.

Let $\mathbb{R}^n_+ = \{x = (x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n > 0\}$. We recall that the Fourier-Bessel transform of a sufficiently good function $\varphi : \mathbb{R}^n_+ \to \mathbb{C}$ is defined by

$$(F_\gamma f)(y) = \int\limits_{\mathbb{R}^n_+} f(x) e^{ix'\cdot y'} j_{\gamma-1/2}(x_n y_n)\, x_n^{2\gamma} dx \tag{6.1}$$

where $y \in \mathbb{R}^n_+$, $\gamma \ge 0$ and $j_\nu(z) = 2^\nu \Gamma(\nu+1) z^{-\nu} J_\nu(z)$ is the normalized Bessel function so that $j_\nu(0) = 1$. Properties of this transform were studied in [20]. In particular, F_γ is an automorphism of the space $L^2_\gamma(\mathbb{R}^n_+)$ and the following Parseval equality holds

$$\|F_\gamma f\|^2_{(2,\gamma)} = c_\gamma(n)\|f\|^2_{(2,\gamma)}, \quad c_\gamma(n) = (2\pi)^{n-1}2^{2\gamma-1}\Gamma^2(\gamma+1/2). \tag{6.2}$$

Our aim is to find a natural decomposition of $L^2_\gamma(\mathbb{R}^n_+)$ in the direct sum of subspaces which are invariant under the action of F_γ. As we shall see below, solution of this problem relies heavily on the properties of B-harmonics.

Theorem 6.1. *Let $f \in L^1_\gamma(\mathbb{R}^n_+)$ have the form $f(x) = f_0(|x|)Y^{n,\gamma}_k(x/|x|)$. Then $(F_\gamma f)(y) = \Phi_0(|y|)Y^{n,\gamma}_k(y/|y|)$ where*

$$\Phi_0(\rho) = \left(\frac{i}{2}\right)^k \frac{\Gamma(\gamma+n/2)}{\Gamma(k+\gamma+n/2)} \int\limits_0^\infty f_0(r)j_{k-1+\gamma+n/2}(r\rho)r^{n+2\gamma-1}dr. \tag{6.3}$$

Proof. By Theorem 2.3 it suffices to prove the theorem for B-harmonics of the form $Y^{n,\gamma}_k(x/|x|) = \sin^\ell\theta P^{(\lambda-1/2,\gamma-1/2)}_m(\cos 2\theta)Y^{n-1}_{\ell,\nu}(\zeta)$, where $\lambda = \ell-1+n/2$, $k = \ell+2m$, $x = r[(\cos\theta)e_n + (\sin\theta)\zeta]$, $\zeta \in S^{n-2}$. By passing to polar coordinates, for $y = \rho[(\cos\tilde\theta)e_n + (\sin\tilde\theta)\tilde\zeta]$, $\tilde\zeta \in S^{n-2}$, we have

$$\begin{aligned}(F_\gamma f)(y) = &\int\limits_0^\infty f_0(r)r^{n+2\gamma-1}dr \quad \times \\ &\times \int\limits_0^{\pi/2} \sin^{n-2+\ell}\theta\cos^{2\gamma}\theta P^{(\lambda-1/2,\gamma-1/2)}_m(\cos 2\theta)j_{\gamma-1/2}(r\rho\cos\theta\cos\tilde\theta)d\theta \quad \times \\ &\times \int\limits_{S^{n-2}} e^{is\zeta\cdot\tilde\zeta}Y^{n-1}_{\ell,\nu}(\zeta)d\zeta, \qquad s = r\rho\sin\theta\sin\tilde\theta.\end{aligned} \tag{6.4}$$

Since

$$\int\limits_{S^{n-2}} e^{is\zeta\cdot\tilde\zeta}Y^{n-1}_{\ell,\nu}(\zeta)d\zeta = \left(\frac{i}{2}\right)^\ell \Gamma\left(\frac{n-1}{2}\right)\left(\frac{s}{2}\right)^{-\ell-(n-3)/2} J_{\ell+(n-3)/2}(s)Y^{n-1}_{\ell,\nu}(\tilde\zeta) \tag{6.5}$$

(see [44, p. 181, formula (2)]), then (6.4) can be written as

$$(F_\gamma f)(y) = A(\rho,\tilde\theta,\tilde\zeta)\int\limits_0^\infty f_0(r)B(r,\rho,\theta,\tilde\theta)r^{n+\gamma-\lambda}dr, \tag{6.6}$$

$$A = 2^{\lambda+\gamma-1}\rho^{1-\lambda-\gamma}\Big(\frac{i}{2}\Big)^{\ell}\Gamma\Big(\frac{n-1}{2}\Big)\Gamma\Big(\gamma+\frac{1}{2}\Big)\sin^{1/2-\lambda}\tilde{\theta}\cos^{1/2-\gamma}\tilde{\theta}\, Y^{n-1}_{\ell,\nu}(\tilde{\zeta}),$$

$$B = \int_0^{\pi/2} P_m^{(\lambda-1/2,\gamma-1/2)}(\cos 2\theta) J_{\lambda-1/2}(r\rho\sin\theta\sin\tilde{\theta}) \quad\times$$
$$\times\, J_{\gamma-1/2}(r\rho\cos\theta\cos\tilde{\theta})\sin^{(n-1)/2}\theta\cos^{\gamma+1/2}\theta d\theta. \tag{6.7}$$

As we shall see, the last integral is

$$B = \Big(\frac{i}{2}\Big)^{2m}\frac{\Gamma(\gamma+n/2)}{\Gamma(\gamma+1/2)\Gamma((n-1)/2)}\,\Big(\frac{r\rho}{2}\Big)^{-2m-1}\sin^{2\lambda-(n-1)/2}\tilde{\theta}\cos^{\gamma-1/2}\tilde{\theta} \quad\times \tag{6.8}$$
$$\times\, J_{2m+\lambda+\gamma}(r\rho)P_m^{(\lambda-1/2,\gamma-1/2)}(\cos 2\tilde{\theta}).$$

By putting (6.8) into (6.6) we obtain (6.3).

The equality (6.8) will be proven by the "method of descent" which was used in the proof of Lemma 3.3. By using the same notation as in that proof, for any spherical harmonic $Y^d_k(\omega)$ on S^{d-1}, $d=p+q$, and any $a>0$ we have (cf. (6.5))

$$\int_{S^{d-1}} e^{ia\omega\cdot\tilde{\omega}}Y^d_k(\tilde{\omega})d\tilde{\omega} = \Big(\frac{i}{2}\Big)^k\Gamma\Big(\frac{d}{2}\Big)\Big(\frac{a}{2}\Big)^{-k+1-d/2}J_{k-1+d/2}(a)Y^d_k(\omega). \tag{6.9}$$

Suppose that $Y^d_k(\omega) = \cos^{\ell}\theta P_m^{(p/2-1,\ell-1+q/2)}(\cos 2\theta)Y^q_{\ell,\nu}(\zeta)$, $k=2m+\ell$ (cf. (3.8)), and pass in (6.9) to the bispherical coordinates (3.7). After simple calculation using (6.9) twice (with $d\sim q$ and $d\sim p$), and after setting $p=2\gamma+1$, $q=n-1$, we arrive at (6.8) with $\gamma=0,1/2,1,\dots$. An extension to all $\gamma>0$ can be made by analyticity, owing to the Carlson theorem as in the proof of Lemma 3.3. □

According to the L^2-theory for the Fourier-Bessel and the Hankel transform, Theorem 6.1 extends to the functions $f(x)=f_0(|x|)Y^{n,\gamma}_k(x/|x|)\in L^2_\gamma(\mathbb{R}^n_+)$. By (2.18), this gives the following theorem.

Theorem 6.2. *The space $L^2_\gamma(\mathbb{R}^n_+)$ admits a decomposition into orthogonal direct sum of F_γ-invariant subspaces $\mathcal{B}_k$ generated by functions of the form $f_0(|x|)Y^{n,\gamma}_k(x/|x|)$. For $f\in L^2_\gamma(\mathbb{R}^n_+)$, its projection onto $\mathcal{B}_k$ is defined by*

$$f\to\int_{S^{n-1}_+} f(r\eta)Z^\gamma_\xi(\eta)d\eta = \sum_{\nu=1}^{\sigma_n(j)}\hat{f}_{k,\nu}(r)Y^{n-1}_{k,\nu}(\xi) \tag{6.10}$$

where $Z_\xi^\gamma(\eta)$ *is the zonal* B*-harmonic* (3.3) *and* $\hat{f}_{k,\nu}(r) = (f(r\eta), Y_{k,\nu}^{n-1}(\eta))_\gamma$.

7. The Gauss decomposition.

Denote by $\mathcal{P}_k^n$ the space of all homogeneous polynomials $\psi(x) = \sum\limits_{|j|=k} c_j x^j$, $x \in \mathbb{R}^n$, of degree k; $H_k^{n,\gamma} = \{\psi \in \mathcal{P}_k^n \colon \Delta_{n,\gamma}\psi \equiv 0\}$. In the following [a] designates the integer part of a number a, $\{a\} = a - [a]$.

Theorem 7.1. *Each homogeneous polynomial* $\psi \in \mathcal{P}_k^n$ *has a unique representation of the form*

$$\psi(x) = \sum_{s=0}^{[k/2]} \left(\frac{|x|}{2}\right)^{2s} Z_{k-2s,\gamma}(x) \tag{7.1}$$

where $Z_{k-2s,\gamma}(x)$ *is a* B*-harmonic homogeneous polynomial of degree* $k-2s$ *defined by*

$$Z_{k-2s,\gamma}(x) = \sum_{j=0}^{[k/2]-s} a_{j,k,s}^{n,\gamma} \left(\frac{|x|}{2}\right)^{2j} \Delta_{n,\gamma}^{s+j}\psi(x), \tag{7.2}$$

$$a_{j,k,s}^{n,\gamma} = \begin{cases} (-1)^j \dfrac{(k-2s+n/2-1+\gamma)\,\Gamma(k-2s-j+n/2-1+\gamma)}{s!\,j!\,\Gamma(k-s+n/2+\gamma)} \\ \qquad\qquad \textit{if } s \neq k/2, \quad \textit{or } n \neq 2, \quad \textit{or } \gamma > 0; \\ [(k/2)!]^{-2} \quad \textit{otherwise.} \end{cases}$$

Proof. We follow the scheme from [36, p. 476] for $\gamma = 0$. Given $\psi \in \mathcal{P}_k^n$, let us evaluate $A = \Delta_{n,\gamma}\Big[\left(\frac{|x|}{2}\right)^{2j}\psi\Big]$. By (XI.2.3) from [36, p. 477] we have

$$A = j\Big(j + k + \frac{n}{2} - 1\Big)\left(\frac{|x|}{2}\right)^{2j-2}\psi + \left(\frac{|x|}{2}\right)^{2j}\Delta_n\psi + 2\gamma B,$$

$$B = \frac{1}{x_n}\frac{\partial}{\partial x_n}\Big[\left(\frac{|x|}{2}\right)^{2j}\psi\Big] = \left(\frac{|x|}{2}\right)^{2j}\frac{1}{x_n}\frac{\partial\psi}{\partial x_n} + \left(\frac{|x|}{2}\right)^{2j-2}\frac{j\psi}{2},$$

or

$$\Delta_{n,\gamma}\Big[\left(\frac{|x|}{2}\right)^{2j}\psi\Big] = j\Big(j + k + \frac{n}{2} - 1 + \gamma\Big)\left(\frac{|x|}{2}\right)^{2j-2}\psi + \left(\frac{|x|}{2}\right)^{2j}\Delta_{n,\gamma}\psi. \tag{7.3}$$

Now we introduce a differential operator

$$Q_0\psi = \sum_{j=0}^{[k/2]} c_{j,k}\left(\frac{|x|}{2}\right)^{2j}\Delta_{n,\gamma}^j\psi, \qquad \psi \in \mathcal{P}_k, \quad k \geq 0, \tag{7.4}$$

$$c_{j,k} = \begin{cases} (-1)^j\,\dfrac{\Gamma(k-j+n/2-1+\gamma)}{\Gamma(j+1)} & \text{if } k-j+n/2-1+\gamma \neq 0, \\ 1 \text{ otherwise} \end{cases}$$

(the second case holds, e.g., for $n=2,\ \gamma=0,\ k=j=0$). Clearly, $Q_0\psi \in \mathcal{P}^n_k$. Let us show that $\Delta_{n,\gamma}Q_0\psi=0$, i.e. $Q_0\psi\in H^{n,\gamma}_k$. By (7.3) (with ψ replaced by $\Delta^j_{n,\gamma}\psi\in\mathcal{P}^n_{k-2j}$),

$$\Delta_{n,\gamma}Q_0\psi=\sum_{j=0}^{[k/2]} c_{j,k}\Delta_{n,\gamma}\Big[\left(\frac{|x|}{2}\right)^{2j}\Delta^j_{n,\gamma}\psi\Big]=$$
$$=\sum_{j=0}^{[k/2]-1} c_{j,k}\left(\frac{|x|}{2}\right)^{2j}\Delta^{j+1}_{n,\gamma}\psi+$$
$$+\sum_{j=0}^{[k/2-1]} c_{j+1,k}(j+1)(k-j-2+\frac{n}{2}+\gamma)\left(\frac{|x|}{2}\right)^{2j}\Delta^{j+1}\Delta^{j+1}_{n,\gamma}\psi.$$

This expression is zero. Indeed, for $k-j+\frac{n}{2}-1+\gamma\neq 0$ we have

$$c_{j,k}+c_{j+1,k}(j+1)(k-j-2+\frac{n}{2}+\gamma)=0.$$

If

$$k-j+\frac{n}{2}-1+\gamma\equiv\left(\left[\frac{k}{2}\right]-j\right)+\left[\frac{k}{2}\right]+2\left\{\frac{k}{2}\right\}+\frac{n}{2}-1+\gamma=0$$

(this is possible only for $n=2,\ \gamma=0,\ k=j=0$), then $\Delta_{n,\gamma}Q_0\psi=\Delta_{n,\gamma}\psi\equiv 0$ because in this case $\psi(\in\mathcal{P}^n_0)\equiv const.$

Consider now a sequence of polynomial $\{\Delta^s_{n,\gamma}\psi\}$, $s=0,1,\dots,[k/2]$. We have $\Delta^s_{n,\gamma}\psi\in\mathcal{P}^n_{k-2s}$, and by (7.4),

$$Q_0\Delta^s_{n,\gamma}\psi=\sum_{j=0}^{[k/2]-s} c_{j,k-2s}\left(\frac{|x|}{2}\right)^{2j}\Delta^{s+j}_{n,\gamma}\psi\overset{def}{=}Q_s\psi. \tag{7.5}$$

By the above, $Q_s\psi\in H^{n,\gamma}_{k-2s}$. Multiplying (7.5) by $\left(\frac{|x|}{2}\right)^{2s}$ and setting $m=j+s$, we obtain a system of $[k/2]+1$ equations

$$\sum_{m=s}^{[k/2]}\lambda_{m,s}\left(\frac{|x|}{2}\right)^{2m}\Delta^m_{n,\gamma}\psi=\left(\frac{|x|}{2}\right)^{2s}Q_s\psi, \tag{7.6}$$
$$s=0,1,\dots,[k/2],\qquad \lambda_{m,s}=c_{m-s,k-2s},$$

with "unknowns" $\psi,\left(\frac{|x|}{2}\right)^2\Delta_{n,\gamma}\psi,\dots,\left(\frac{|x|}{2}\right)^{2[k/2]}\Delta^{[k/2]}_{n,\gamma}\psi$. The matrix $\mathcal{A}$ of the system (7.6) is triangle and

$$\det\mathcal{A}=\lambda_{00}\lambda_{11\dots}\lambda_{[k/2][k/2]}\neq 0.$$

By solving (7.6), in particular we get

$$\psi = \sum_{s=0}^{[k/2]} \varkappa_s \left(\frac{|x|}{2}\right)^{2s} Q_s\psi, \tag{7.7}$$

$\varkappa_s$ being independent of ψ. This equality has the form (7.1).

Now let us prove the uniqueness of the representation (7.1). Suppose that

$$\psi = \sum_{s=0}^{[k/2]} \left(\frac{|x|}{2}\right)^{2s} Z_{k-2s,\gamma}(x) = \sum_{s=0}^{[k/2]} \left(\frac{|x|}{2}\right)^{2s} Z'_{k-2s,\gamma}(x).$$

Then

$$\sum_{s=0}^{[k/2]} \left(\frac{|x|}{2}\right)^{2s} Z^0_{k-2s,\gamma}(x) = 0, \qquad Z^0_{k-2s,\gamma} = Z_{k-2s,\gamma} - Z'_{k-2s,\gamma}. \tag{7.8}$$

We apply $\Delta_{n,\gamma}^{[k/2]}$ to this equality. By (7.3),

$$0 = \sum_{s=0}^{[k/2]} \Delta_{n,\gamma}^{[k/2]} \Big[\left(\frac{|x|}{2}\right)^{2s} Z^0_{k-2s,\gamma}(x)\Big] =$$
$$= \sum_{s=1}^{[k/2]} \Delta_{n,\gamma}^{[k/2]-1} \Big[s\Big(k-s+\frac{n}{2}-1+\gamma\Big)\left(\frac{|x|}{2}\right)^{2s-2} Z^0_{k-2s,\gamma} + \left(\frac{|x|}{2}\right)^{2s} \Delta_{n,\gamma} Z^0_{k-2s,\gamma}\Big].$$

Since $Z^0_{k-2s,\gamma}$ is B-harmonic the second term in square brackets is zero. Thus, by (7.3),

$$0 = \sum_{s=2}^{[k/2]} s\Big(k-s+\frac{n}{2}-1+\gamma\Big) \quad \times$$
$$\times\, \Delta_{n,\gamma}^{[k/2]-2} \Big[(s-1)\Big(k-s+\frac{n}{2}-2+\gamma\Big)\left(\frac{|x|}{2}\right)^{2(s-2)} Z^0_{k-2s,\gamma}\Big] =$$

$$= \frac{\Gamma(1+[k/2])\,\Gamma(k-[k/2]+n/2+\gamma)}{\Gamma(k-2[k/2]+n/2+\gamma)}\, Z^0_{k-2[k/2],\gamma} = 0.$$

It follows that $Z^0_{k-2[k/2],\gamma} = 0$ and (7.8) is written as

$$\sum_{s=0}^{[k/2]-1} \left(\frac{|x|}{2}\right)^{2s} Z^0_{k-2s,\gamma}(x) = 0.$$

Applying $\Delta_{n,\gamma}^{[k/2]-1}$ to this equality, as above we obtain $Z^0_{k-2s,\gamma}=0$ for $s=[k/2]-1$. Continuing this process, we get $Z^0_{k-2s,\gamma}=0$ for all $s=0,1,\dots,[k/2]$ which was required.

It remains to evaluate $\varkappa_s$ in (7.7). Fix $\ell\in\{0,1,\dots,[k/2]\}$ and replace ψ in (7.7) by $\left(\frac{|x|}{2}\right)^{2\ell} Q_\ell\psi\in\mathcal{P}^n_k$. We obtain

$$\left(\frac{|x|}{2}\right)^{2\ell} Q_\ell\psi=\sum_{s=0}^{[k/2]}\varkappa_s\left(\frac{|x|}{2}\right)^{2s} Q_s\Big[\left(\frac{|x|}{2}\right)^{2\ell} Q_\ell\psi\Big].$$

Due to the uniqueness of the decomposition (7.7), all terms in the right-hand side are zero rather than for $s=\ell$. Thus,

$$Q_\ell\psi=\varkappa_\ell Q_\ell\Big[\left(\frac{|x|}{2}\right)^{2\ell} Q_\ell\psi\Big].$$

Replace the first Q_ℓ in the right-hand side by (7.5). Then

$$Q_\ell\psi=\varkappa_\ell\sum_{j=0}^{[k/2]-\ell}c_{j,k-2\ell}\left(\frac{|x|}{2}\right)^{2j}\Delta_{n,\gamma}^{\ell+j}\Big[\left(\frac{|x|}{2}\right)^{2\ell} Q_\ell\psi\Big]. \tag{7.9}$$

By (7.3),

$$\begin{aligned}\Delta^\ell_{n,\gamma}\Big[\left(\frac{|x|}{2}\right)^{2\ell} Q_\ell\psi\Big]&=\Delta^{\ell-1}_{n,\gamma}\Big[\ell(k-\ell+\frac{n}{2}-1+\gamma)\left(\frac{|x|}{2}\right)^{2(\ell-1)} Q_\ell\psi\Big]=\dots=\\&=\frac{\ell!\,\Gamma(k-\ell+n/2+\gamma)}{\Gamma(k-2\ell+n/2+\gamma)}\,Q_\ell\psi\end{aligned}$$

and (7.9) is written as

$$\begin{aligned}Q_\ell\psi&=\varkappa_\ell\frac{\ell!\,\Gamma(k-\ell+n/2+\gamma)}{\Gamma(k-2\ell+n/2+\gamma)}\sum_{j=0}^{[k/2]-\ell}c_{j,k-2\ell}\left(\frac{|x|}{2}\right)^{2j}\Delta^j_{n,\gamma}Q_\ell\psi=\\&=\varkappa_\ell\,\frac{\ell!\,\Gamma(k-\ell+n/2+\gamma)}{\Gamma(k-2\ell+n/2+\gamma)}\,c_{0,k-2\ell}Q_\ell\psi.\end{aligned}$$

Hence, if $Q_\ell\psi\neq 0$, then

$$\varkappa_\ell=\frac{\Gamma(k-2\ell+n/2+\gamma)}{\ell!\,\Gamma(k-\ell+n/2+\gamma)\,c_{0,k-2\ell}}$$

and therefore, by (7.7), (7.5),

$$\psi(x)=\sum_{s=0}^{[k/2]}\left(\frac{|x|}{2}\right)^{2s} Z_{k-2s,\gamma}(x)$$

where

$$Z_{k-2s,\gamma}(x)=\frac{\Gamma(k-2s+n/2+\gamma)}{s!\,\Gamma(k-s+n/2+\gamma)\,c_{0,k-2s}}\sum_{j=0}^{[k/2]-s} c_{j,k-2s}\left(\frac{|x|}{2}\right)^{2j}\Delta_{n,\gamma}^{s+j}\psi=$$

$$=\sum_{j=0}^{[k/2]-s} a_{j,k,s}\left(\frac{|x|}{2}\right)^{2j}\Delta_{n,\gamma}^{s+j}\psi.$$

If $s\neq k/2$ or $n\neq 2$ or $\gamma>0$, then

$$a_{j,k,s}=(-1)^j\frac{(k-2s+n/2-1+\gamma)\,\Gamma(k-2s-j+n/2-1+\gamma)}{s!\,j!\,\Gamma(k-s+n/2+\gamma)}\equiv a_{j,k,s}(n,\gamma).$$

If $s=k/2$, $n=2$ and $\gamma=0$, then $j=0$, $c_{0,k-2s}=c_{00}=1$, and we have $a_{j,k,s}=1/[(k/2)!]^2$. Note that this coefficient coincides with

$$\lim_{\gamma\to+0} a_{0,k,k/2}(2,\gamma). \qquad \square$$

Remark 7.2. Our proof differs from Sobolev's by the following. We did not use the "method of projectors" when evaluating $\varkappa_s$. Sobolev's proof, employing projectors, assumes in advance that $\varkappa_s\neq 0$. But this inequality is not obvious. Furthermore, in order to cover the special case $n=2$, k even, we had to modify some places in Sobolev's proof leading to expressions like $\Gamma(0)$.

Remark 7.3. If $\psi(x)\in\mathcal{P}_k^n$ is even in the x_n-variable, then all B-harmonic polynomials $Z_{k-2s,\gamma}$ are also even in the x_n-variable. This follows from (7.2) by taking into account that $\Delta_{n,\gamma}$ preserves the evenness in x_n.

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