Problems for M.Sc. Workshop no.7, December 2, 2012

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39. A finite number of closed intervals whose total length is 2 are drawn in the interior of a disk of diameter $1/\pi$. Prove that there exists a line which intersects at least 5 intervals. Prove that for any rectifiable (having length) arc of length 2 inside of such disk there is a line which intersects it in 5 points.

40. For each prime $p > 3$ denote $A_p = \prod_{k=1}^{p}(k^2+1)$. Prove that either p divides A_p or $A_p \equiv 4 (\mathrm{mod}\ p)$.

41. Let X, Y and Z be identically distributed random variables with variance σ^2. Assume that X and Y are independent and $X + Y = aZ$ for some $a > 0$. Show that the random variables are normal $N(0, \sigma^2)$.

42. Prove that on any circle centered at $(\sqrt{2}, \sqrt{3})$ in the plane there is at most one point with integral coordinates.

43. Each trajectory of a billiard ball motion in a domain consists of stright line segments inside of the domain joined at boundary points according to the rule that the angle of incidence equals the angle of reflection (when they are defined). Prove that any billiard trajectory in a rectangle R which never hits a vertex (only those are well defined) is either periodic or everywhere dense in R.

44. Prove that any billiard trajectory in an equilateral triangle T which never hits a vertex (only those are well defined) is either periodic or everywhere dense in T.

45. Show that a billiard ball in an infinite wedge with the wedge angle $0 < \alpha < \pi$ can collide with its sides no more than $-[-\pi/\alpha]$ times (where $[a]$ is the biggest integer not exceeding a).