Problems for M.Sc. Workshop no.4, November 11, 2012

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21. Let $X_x(t)$ be the solution of an autonomous system of ordinary differential equations on the plane $\mathbb{R}^2$,

$$\frac{dX_x(t)}{dt} = A(X_x(t), Y_y(t)), \quad X_x(0) = x,$$

$$\frac{dY_y(t)}{dt} = B(X_x(t), Y_y(t)), \quad Y_y(0) = y,$$

where A and B are bounded twice differentiable functions on $\mathbb{R}^2$. For any open set U let U_t be the set of pairs $(X_x(t), Y_y(t))$ for all $(x, y) \in U$. Denote by ℓ the Lebesgue measure on $\mathbb{R}^2$. Prove that $\ell(U_t) = \ell(U)$ for any t and any open U if and only if

$$\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} \equiv 0.$$

The analogous result holds true for systems of n equations in $\mathbb{R}^n$.

22. Given a matrix A we are allowed for one step to change all signs in one column or in one row. Prove that for any initial matrix A we can arrive for a finite number of steps to a matrix which has nonnegative sums along each column and along each row.

23. Show that there exists a natural number n such that both 2^n and 3^n begin with the digit 7.

24. All edges of the complete graph with n vertices are colored by one of three colors. Prove that there exists a monochromatic connected subgraph whose number of vertices is not less than $n/2$.

25. Show that any bounded function $f : \mathbb{Z}^2 \to \mathbb{R}$ which satisfies

$$f(x, y) = \frac{1}{4}(f(x+1, y) + f(x-1, y) + f(x, y+1) + f(x, y-1))$$

is constant.

26. A particle performs a random walk on integers so that if $i \neq 0$ then it jumps from i to either $i+1$ or $i-1$ with probability $1/2$ but from 0 it jumps to 1 with probability $3/5$, to -1 with probability $1/5$ and stays at 0 with probability $1/5$. Find probability that the particle arrives at 1000000 before it arrives at -1000000.