Problems for M.Sc. Workshop no.3, November 4, 2012

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15. The set $\mathcal{L}_{\varphi,\psi} = \{(x_\varphi(t), y_\psi(t)) : -\infty < t < \infty\}$, where $x_\varphi(t) = A\sin(t+\varphi)$, $y_\psi(t) = B\sin(\omega t+\psi)$, is called a Lissajous figure (or a Bowditch curve). Prove that $\mathcal{L}_{\varphi,\psi}$ is dense in the rectangle $[-A, A] \times [-B, B]$ for each pair φ, ψ if and only if ω is irrational.

16. Find necessary and sufficient condition on two probability vectors $p = (p_1, ..., p_n)$ and $q = (q_1, ..., q_n)$ such that there exists a lower triangular $n \times n$ probability matrix (i.e. rows are probability vectors) $A = (a_{kl})$ so that $pA = q$ (i.e. $a_{kl} = 0$ for $k < l$ and $\sum_k p_k a_{kl} = q_l \, \forall l$).

17. Extend Problem 15 to infinite matrices and vectors, i.e. characterize infinite probability vectors $p = (p_1, p_2, ...)$ and $q = (q_1, q_2, ...)$ such that there exists a lower triangular infinite probability matrix $A = (a_{kl})$ so that $pA = q$.

18. Let $\{x_k, \, k = 1, 2, ...\}$ be an infinite sequence of vectors in $\mathbb{R}^n$ with integer coordinates. Show that there exists N such that any x_k is a linear combination of $x_1, x_2, ..., x_N$ with integer coefficients.

19. Show that in any given sequence of $n^2 + 1$ numbers there is a monotonic subsequence of length $n + 1$.

20. Let $X_1, X_2, X_3, ...$ be independent identically distributed bounded random variables. Prove that with probability one,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} X_n X_{2n} = (EX_1)^2.$$