Problems for M.Sc. Workshop no.2, October 28, 2012

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7. How many elements are there in the sequence $a_n = \ln\ln ... \ln n$ (where in a_n we take ln n times)?

8. Two villages A and B are connected by two very narrow paths so that two cyclists tied by a rope of length a and going by different paths can pass from A to B. Two thick men of radius a go one from A to B and the other from B to A. Can they pass without collision?

9. A rectangle R on the plane is partitioned into finite number of smaller rectangles each of which has a side of integer length. Prove that R itself has a side of integer length.

10. Prove that decimal representations of 2^n, $n = 1, 2, ...$ can start with any given string of digits $a_1, ..., a_k$, $a_1 \neq 0$, $a_i \in \{0, 1, ..., 9\}$.

11. Let $N_a(n)$ be the number of times $a = (a_1, ..., a_k)$ appears as the starting string in $1, 2, 2^2, ..., 2^n$. Prove that

$$\lim_{n\to\infty} \frac{1}{n} N_a(n)$$

exists and find it.

12. n points are thrown independently with uniform distribution on a circle of length 1. Find the probability that some half circle contains all n points.

13. Let f be a continuous function on the plane $\mathbb{R}^2$ such that its integral is zero over each rectangle of area 1 with sides parallel to coordinate axes. Prove that $f \equiv 0$.

14. Let $f(x, y)$ be a 1-periodic continuous function on $\mathbb{R}^2$, i.e. $f(x + 1, y) = f(x, y + 1) = f(x, y)$ for all x and y. Prove that if α is an irrational number then uniformly in x and y,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(x + \alpha s, y + s) ds = \int_0^1 \int_0^1 f(x, y) dx dy.$$